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# Multiple Solutions to a Boundary Value Problem for an n-th Order Nonlinear Difference Equation *

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**Abstract**

We seek multiple solutions to the n-th order nonlinear difference equation

$$\Delta^n x(t) = (-1)^{n-k} f(t, x(t)), \quad t \in [0, T]$$

satisfying the boundary conditions

$$x(0) = x(1) = \cdots = x(k-1) = x(T+k+1) = \cdots = x(T+n) = 0\,.$$

Guo's fixed point theorem is applied multiple times to an operator defined on annular regions in a cone. In addition, the hypotheses invoked to obtain multiple solutions to this problem involves the condition (A) $f : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous in $x$, as well as one of the following: (B) $f$ is sublinear at 0 and superlinear at $\infty$, or (C) $f$ is superlinear at 0 and sublinear at $\infty$.

## 1 Introduction

Define the operator $\Delta$ to be the forward difference

$$\Delta u(t) = u(t+1) - u(t),$$

and then for $i \geq 1$ define

$$\Delta^i u(t) = \Delta(\Delta^{i-1} u(t)).$$

For $a \leq b$ integers define $[a, b] = \{a, a+1, \ldots, b-1, b\}$. Let the integers $n, T \geq 2$ be given, and choose $k \in \{1, 2, \ldots, n-1\}$. Consider the nth order nonlinear difference equation

$$\Delta^n x(t) = (-1)^{n-k} f(t, x(t)), \quad t \in [0, T], \tag{1}$$

satisfying the boundary conditions

$$x(0) = x(1) = \cdots = x(k-1) = x(T+k+1) = \cdots = x(T+n) = 0. \tag{2}$$

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To simplify the discussion of the desired properties of the function $f$ define the following four functions:

$$f_{0,m} = \lim_{u\to 0^+} \min_{t\in[k,T+k]} \tfrac{f(t,u)}{u}, \qquad f_{\infty,m} = \lim_{u\to +\infty} \min_{t\in[k,T+k]} \tfrac{f(t,u)}{u},$$
$$f_{0,M} = \lim_{u\to 0^+} \max_{t\in[k,T+k]} \tfrac{f(t,u)}{u}, \quad \text{and} \quad f_{\infty,M} = \lim_{u\to +\infty} \max_{t\in[k,T+k]} \tfrac{f(t,u)}{u}.$$

We seek to prove the existence of multiple positive solutions to (1) and (2) where

(A) $f : [0,T]\times\mathbb{R}^+ \to \mathbb{R}^+$ is continuous in $x$ , where $\mathbb{R}^+$ denotes the nonnegative reals.

We also require that one of the following sublinearity and superlinearity conditions on the function $f$ holds:

(B) $f_{0,m} = +\infty$ and $f_{\infty,m} = +\infty$, or

(C) $f_{0,M} = 0$ and $f_{\infty,M} = 0$.

We apply Guo's Fixed point theorem, Guo and Lakshmikantham [5], using cone methods to accomplish this. This technique was first applied to differential equations in the landmark paper by Erbe and Wang [4] using Krasnosel'skiĭ's fixed point theorem, Krasnosel'skiĭ [9]. A key to applying this fixed point theorem involves discrete concavity of solutions of the boundary value problem in conjunction with a lower bound on an appropriate Green's function.

This work constitutes a complete generalization of the paper by Eloe, Henderson and Kaufmann [3] which we use extensively. We also utilize techniques from Hartman [6], Merdivenci [11], and Peterson [12]. Extensive use of the results by Eloe [2] concerning a lower bound for the Green's function will be made.

## 2 Preliminaries

Let $G(t,s)$ be the Green's function for the disconjugate boundary value problem

$$Lx(t) \equiv \Delta^n x(t) = 0, \quad t \in [0,T] \tag{3}$$

and satisfying (2). The characterization of the Green's function can be found in Kelley and Peterson [8]. We will use $G(t,s)$ as the kernel of an integral operator preserving a cone in a Banach space, the setting for our fixed point theorem.

A closed, non-empty subset $\mathcal{P}$ of a Banach space $\mathcal{B}$ is said to be a *cone* provided (i) $au + bv \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and for all $a, b \geq 0$, and (ii) $u, -u \in \mathcal{P}$ implies $u = 0$.

Repeated application of the following fixed point theorem from Guo, Guo and Lakshmikantham [5], will yield two solutions to (1) and (2).

**Theorem 2.1** *Let $\mathcal{B}$ be a Banach space and $\mathcal{P} \subset \mathcal{B}$ be a cone. Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Let*

$$\mathcal{H}: \quad \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

*be a completely continuous operator satisfying either*

*(i)* $\|Hx\| \leq \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Hx\| \geq \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_2$, *or*

*(ii)* $\|Hx\| \geq \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Hx\| \leq \|x\|$, $x \in \mathcal{P} \cap \partial\Omega_2$.

*Then $\mathcal{H}$ has a fixed point in $\mathcal{P} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.*

Two applications of 2.1 to the problem (1) and (2) following along the lines of methods incorporated by Eloe, Henderson and Kaufmann [3] will be performed.

Note that $x(t)$ is a solution of (1) and (2) if and only if

$$x(t) = (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s, x(s)), \qquad t \in [0, T+n]\,.$$

Hartman [6] extensively studied the boundary value problem (1) and (2) with $(-1)^{n-k} f(t,u) \geq 0$. Eloe [2] employed lemmas from Hartman to arrive at the following theorem that gives a lower bound for the solution to the class of boundary value problems studied by Hartman.

**Theorem 2.2** *Assume that $u$ satisfies the difference inequality $(-1)^{n-k}\Delta^n u(t) \geq 0$, $t \in [0,T]$, and the homogeneous boundary conditions, (2). Then for $t \in [k, T+k]$,*

$$(-1)^{n-k}\Delta^n u(t) \geq \frac{T!\ \nu!}{(T+\nu)!}\|u\|,$$

*where $\|u\| = \max\limits_{t\in[k,T+k]} |u(t)|$ and $\nu = \max\{k, n-k\}$.*

We remark that Agarwal and Wong [1] have recently sharpened the inequality of Theorem 2.2. This sharper inequality is of little consequence for this work.

Eloe also contributed the following corollary.

**Corollary 2.3** *Let $G(t,s)$ denote the Green's function for the boundary value problem, (3) and (2). Then for all $s \in [0,T]$, $t \in [k, T+k]$,*

$$(-1)^{n-k}\Delta^n G(t,s) \geq \frac{T!\ \nu!}{(T+\nu)!}\|G(\cdot,s)\|,$$

*where $\|G(\cdot,s)\| = \max\limits_{t\in[k,T+k]} |G(t,s)|$ and $\nu = \max\{k, n-k\}$.*

To fulfill the hypotheses of Theorem 2.1 let $\mathcal{B} = \{u : [0, T+n] \to \mathbb{R} |$ $u(0) = u(1) = \cdots = u(k-1) = u(T+k+1) = \cdots = u(T+n) = 0\}$ with $\|u\| = \max_{t\in[k,T+k]} |u(t)|$. Now $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

Let
$$\sigma = \frac{T!\ \nu!}{(T+\nu)!} \tag{4}$$
with $\nu = \max\{k, n-k\}$ and define a cone
$$\mathcal{P} = \{u \in \mathcal{B} \mid u(t) \geq 0 \ \texttt{on}\ [0, T+n] \ \texttt{and} \ \min_{t\in[k,T+k]} u(t) \geq \sigma\|u\|\}.$$

# 3 Main Results

We first seek two solutions to the case when $f$ is sublinear at 0 and superlinear at $\infty$. Define
$$\eta = \left(\sum_{s=0}^{T} \|G(\cdot, s\|\right)^{-1}. \tag{5}$$

**Theorem 3.1** *Assume $f(t,x)$ satisfies conditions (A) and (B). Suppose there exists $p > 0$ such that if $0 \leq u(t) \leq p$, $t \in [0,T]$, then $f(t,u) \leq \eta p$. Then the boundary value problem (1) and (2) has at least two positive solutions $u_1, u_2 \in \mathcal{P}$ satisfying $0 \leq \|u_1\| \leq p \leq \|u_2\|$.*

**proof** Define a summation operator $\mathcal{H} :\ \mathcal{P} \to \mathcal{B}$ by
$$\mathcal{H}x(t) = (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s, x(s)), \qquad x \in \mathcal{P} \tag{6}$$
Now $\mathcal{H} :\ \mathcal{P} \to \mathcal{P}$ and is completely continuous.

Choose $\alpha > 0$ such that
$$\alpha\sigma^2 \sum_{s=k}^{T} \|G(\cdot, s)\| \geq 1. \tag{7}$$
By the sublinearity of $f$ at 0 there exists $0 < r < p$ such that $f(t,u) \geq \alpha u$ for all $0 \leq u \leq r$, $t \in [0, T+n]$. For $x \in \mathcal{P}$ with $\|x\| = r$
$$\begin{aligned}\mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s,x(s)) \\ &\geq \sigma \sum_{s=0}^{T} \|G(\cdot,s)\| f(s,x(s)) \\ &\geq \alpha\sigma \sum_{s=0}^{T} \|G(\cdot,s)\| x(s)\end{aligned}$$

$$\begin{aligned} &\geq \alpha\sigma^2 \sum_{s=k}^{T} \|G(\cdot,s)\|\,\|x\| \\ &\geq \|x\|, \qquad t \in [k, T+k]. \end{aligned}$$

Therefore $\|\mathcal{H}x\| \geq \|x\|$. Hence if we set

$$\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < r\}$$

then

$$\|\mathcal{H}x\| \geq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_1. \tag{8}$$

Now for $x \in \mathcal{P}$ with $\|x\| = p$,

$$\begin{aligned} \mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s, x(s)) \\ &\leq \sum_{s=0}^{T} \|G(\cdot,s)\| f(s, x(s)) \\ &\leq \sum_{s=0}^{T} \|G(\cdot,s)\| \eta p \leq p = \|x\|, \qquad t \in [0, T+k]. \end{aligned}$$

Now if we take

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < p\}$$

then

$$\|\mathcal{H}x\| \leq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_2. \tag{9}$$

Thus with (8) and (9), we have shown that $\mathcal{H}$ satisfies the hypotheses to Theorem 2.1(ii). This yields a fixed point $u_1$ of $\mathcal{H}$ belonging to $\mathcal{P} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$. This fixed point is a solution of (1) and (2) satisfying $r \leq \|u_1\| \leq p$.

Next, choose $\omega > 0$ such that

$$\omega\sigma^2 \sum_{s=k}^{T} \|G(\cdot,s)\| \geq 1. \tag{10}$$

By the superlinearity of $f$ at infinity there exists $R_1 > 0$ such that $f(t,u) \geq \omega u$ for all $u \geq R_1$, $t \in [0, T+n]$. Let $R = \max\{2p, R_1\}$. Now for $x \in \mathcal{P}$ with $\|x\| = R$

$$\begin{aligned} \mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s, x(s)) \\ &\geq \sigma \sum_{s=0}^{T} \|G(\cdot,s)\| f(s, x(s)) \\ &\geq \omega\sigma \sum_{s=0}^{T} \|G(\cdot,s)\| x(s) \end{aligned}$$

$$\begin{aligned} &\geq \omega\sigma^2 \sum_{s=k}^{T} \|G(\cdot,s)\| \|x\| \\ &\geq \|x\|, \qquad t \in [k, T+k]. \end{aligned}$$

Therefore $\|\mathcal{H}x\| \geq \|x\|$. Hence if we set

$$\Omega_3 = \{u \in \mathcal{B} \mid \|u\| < R\}$$

then

$$\|\mathcal{H}x\| \geq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_3. \tag{11}$$

Thus with (9) and (11), we have shown that $\mathcal{H}$ satisfies the hypotheses to Theorem 2.1(i). This yields a fixed point $u_2$ of $\mathcal{H}$ belonging to $\mathcal{P} \cap \left(\overline{\Omega}_3 \setminus \Omega_2\right)$. This fixed point is a solution of (1) and (2) satisfying $p \leq \|u_2\| \leq R$.

Therefore, the boundary value problem (1) and (2) has at least two positive solutions $u_1, u_2 \in \mathcal{P}$ such that $0 \leq \|u_1\| \leq p \leq \|u_2\|$. ♢

We now seek two solutions for the case when $f$ is superlinear at 0 and sublinear at $\infty$.

**Theorem 3.2** *Assume $f(t,x)$ satisfies conditions (A) and (C). Suppose there exists $q > 0$ such that if $\sigma q \leq u(t) \leq q$, $t \in [k, T+k]$, then $f(t,u) \geq \tau q$, where*

$$\tau = \left(\sigma \sum_{s=k}^{T} \|G(\cdot,s)\|\right)^{-1}. \tag{12}$$

*Then the boundary value problem (1) and (2) has at least two positive solutions $u_1, u_2 \in \mathcal{P}$ such that $0 \leq \|u_1\| \leq q \leq \|u_2\|$.*

**proof** Define the summation operator as in (6) and define $\eta$ as in (5). By the superlinearity of $f$ at 0 there exists $0 < r < q$ such that $f(t,u) \leq \eta u$ for all $0 \leq u \leq r$, $t \in [0,T]$. For $x \in \mathcal{P}$ with $\|x\| = r$,

$$\begin{aligned} \mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s, x(s)) \\ &\leq \sum_{s=0}^{T} \|G(\cdot,s)\| \eta x(s) \\ &\leq \left(\sum_{s=0}^{T} \|G(\cdot,s)\|\right) \eta \|x\| = \|x\|, \qquad t \in [0, T+k]. \end{aligned}$$

Therefore $\|\mathcal{H}x\| \leq \|x\|$. Hence if we set

$$\Omega_1 = \{u \in \mathcal{B} \mid \|u\| < r\}$$

then

$$\|\mathcal{H}x\| \leq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_1. \tag{13}$$

Next, for $x \in \mathcal{P}$ with $\|x\| = q$

$$\begin{aligned} \mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s,x(s)) \\ &\geq \sigma \sum_{s=k}^{T} \|G(\cdot,s)\| \tau q \geq q = \|x\| \qquad t \in [k, T+k]. \end{aligned}$$

Therefore $\|\mathcal{H}x\| \geq \|x\|$. Hence if we set

$$\Omega_2 = \{u \in \mathcal{B} \mid \|u\| < q\}$$

then

$$\|\mathcal{H}x\| \geq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_2. \tag{14}$$

Thus with (13) and (14), we have shown that $\mathcal{H}$ satisfies the hypotheses to Theorem 2.1(i) which yields a fixed point $u_1$ of $\mathcal{H}$ belonging to $\mathcal{P} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$. This fixed point is a solution of (1) and (2) satisfying $r \leq \|u_1\| \leq q$.

Next, by condition (C), for every $\varepsilon > 0$, there exists a $\xi > 0$ such that for all $u \geq 0$, $t \in [0, T+k]$, $f(t,u) \leq \xi + \varepsilon u$. Let $\varepsilon = \frac{\eta}{2}$, where $\eta$ is defined by (5) and select a corresponding $\xi$. Let $R = \max\{2q, 2\frac{\xi}{\eta}\}$. Then for $x \in \mathcal{P}$ with $\|x\| = R$

$$\begin{aligned} \mathcal{H}x(t) &= (-1)^{n-k} \sum_{s=0}^{T} G(t,s) f(s,x(s)) \\ &\leq \sum_{s=0}^{T} \|G(\cdot,s)\| [\xi + \varepsilon x(s)] \\ &\leq \xi \sum_{s=0}^{T} \|G(\cdot,s)\| \ + \ \varepsilon \sum_{s=0}^{T} \|G(\cdot,s)\| x(s) \\ &\leq \frac{\xi}{\eta} + \varepsilon \sum_{s=0}^{T} \|G(\cdot,s)\| \, \|x\| \\ &\leq \frac{R}{2} + \frac{\|x\|}{2} = \|x\|, \qquad t \in [0, T+k]. \end{aligned}$$

Therefore $\|\mathcal{H}x\| \leq \|x\|$. Hence if we set

$$\Omega_3 = \{u \in \mathcal{B} \mid \|u\| < R\}$$

then

$$\|\mathcal{H}x\| \leq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_3. \tag{15}$$

Thus with (14) and (15), we have shown that $\mathcal{H}$ satisfies the hypotheses to Theorem 2.1(i) which yields a fixed point of $\mathcal{H}$ belonging to $\mathcal{P} \cap \left(\overline{\Omega}_3 \setminus \Omega_2\right)$.

This fixed point, $u_2$, is a solution of (1) and (2) satisfying $q \leq \|u_2\| \leq R$. Therefore, the boundary value problem (1) and (2) has at least two positive solutions $u_1, u_2 \in \mathcal{P}$ such that $0 \leq \|u_1\| \leq q \leq \|u_2\|$.

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