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An Inverse Problem in a Parabolic Equation *

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Abstract

In this paper, an inverse problem in a parabolic equation is studied. An unknown function in the equation is related to two integral equations in terms of heat kernel. One of the integral equations is well-posed while another is ill-posed. A regularization approach for constructing an approximate solution to the ill-posed integral equation is proposed. Theoretical analysis and numerical experiment are provided to support the method.

1 Introduction

Consider a parabolic equation of the form:

$$u_t = u_{xx} + a(t)u, \quad 0 < x, \quad 0 < t < T, \tag{1}$$

$$u(x,0) = 0, \quad 0 < x, \tag{2}$$

$$u(0,t) = f(t), \quad f(0) = 0, \quad 0 < t < T, \tag{3}$$

$$-u_x(0,t) = g(t), \quad g(t) > 0, \quad 0 < t < T, \tag{4}$$

where $f(t)$ and $g(t)$ are assumed to be known and strictly increasing functions. We want to find the unknown function $u(x,t)$ and a function $a(t)$ to satisfy the equations above. So this is an inverse problem.

Since the publication of the AMS monograph [7] in 1984, hundreds of research papers on inverse problems have been published. For the problem studied here, we refer the readers to the references in [1, 2, 3, 4, 5, 6, 8].

Assumed that (1)-(4) has a solution $u(x,t)$, then it can be shown that

$$u(x,t) = 2\int_0^t K(x, t-\tau)\, g(\tau)\, e^{\theta(t)-\theta(\tau)} d\tau, \tag{5}$$

$$\text{where} \quad K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \tag{6}$$

$$\text{and} \quad \theta(t) = \int_0^t a(\tau) d\tau. \tag{7}$$

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Let x approach to zero in (5). Using the boundary condition (3), we have an integral equation for $y(t)$:

$$\int_0^t \frac{g(\tau)}{\sqrt{t-\tau}}\, y(\tau)\, d\tau = \sqrt{\pi} f(t) y(t), \tag{8}$$

from which we can obtain $\theta(t)$ as follows,

$$y(t) = e^{-\theta(t)}, \quad y(0) = 1. \tag{9}$$

Ideally we can solve the Volterra's equation of the second type (8) to get $y(t)$, and then $\theta(t)$ from (9). Once we have $\theta(t)$, the inverse problem can be solved from the second integral equation (7) to get $a(t)$. With given data $f(t)$ and $g(t)$ in $C[0,T]$, the integral equation (8) for $y(t)$ is well-posed, so is $\theta(t)$ from (9). However, the other integral equation (7) for $a(t)$ in the space $C[0,T]$ is ill-posed because $a(t)$ does not depend on the data $\theta(t)$ continuously.

2 Regularization approach

From now on we shall focus our attention to the integral equation (7). The idea of regularization method for ill-posed problem can be found in [8]. Let

$$A[a] = \int_0^t a(\tau)\, d\tau = \theta(t),$$

where

$$\theta(t) = -\ln y(t). \tag{10}$$

We start with the so-called smoothing functional

$$M^\alpha[a,\theta] = ||A[a] - \theta||^2_{L^2[0,T]} + \alpha\, ||a||^2_{W_2^1[0,T]}. \tag{11}$$

The solution of the minimization problem of the functional above will serve as an an approximate solution to the ill-posed integral equation (7) with appropriate choice of α. We have the following theorem.

Theorem 1 *For every element $\theta(t)$ in $L^2[0,T]$ and every parameter $\alpha > 0$, there exists a unique element $a_\alpha(t) \in C[0,T]$ for which the functional (11) attains its greatest lower bound:*

$$M^\alpha[a_\alpha, \theta] = \inf_{a \in C[0,T]} M^\alpha[a,\theta].$$

Proof: Take the first variation of the functional (11) and set it to zero to obtain the Euler integro-differential equation

$$\alpha\,(a'' - a) = \int_\tau^T dt \int_0^t a(\xi)\, d\xi - \int_\tau^T \theta(t)\, dt \tag{12}$$

with the boundary conditions:

$$a(0) = a_0^*, \quad a(T) = a_T^*, \quad a_0^*,\ a_T^* \quad \text{are given.} \tag{13}$$

Under conditions (13), the associated homogeneous equation

$$\alpha\,(a'' - a) = \int_\tau^T dt \int_0^t a(\xi)\,d\xi$$

can not possess a non-trivial solution. In fact, if $a(\tau)$ were such a solution, then multiplying both sides above by $a(\tau)$ and integrating with respect to τ from 0 to T, one should get the following equality

$$-\alpha||a||^2_{W_2^1[0,T]} = ||\,A[a]\,||^2_{L^2[0,T]},$$

which would contradict the hypothesis that $\alpha > 0$. Therefore, the inhomogeneous equation (12) has one and only one solution a_α. Thus the theorem is proved.

The above theorem means that an operator $R(\theta,\alpha)$ into $C[0,T]$ is defined on the set of pairs (θ,α), where $\theta \in L^2[0,T]$ and $\alpha > 0$, so that the element $a_\alpha = R(\theta,\alpha)$ minimizes the functional $M^\alpha[a,\theta]$. We also need to show that the operator $R(\theta,\alpha)$ is a regularizing one for equation (7) by selecting an appropriate parameter α.

Theorem 2 *Let a_T be the solution of equation (7) corresponding to a given $\theta = \theta_T$, that is, $A[a_T] = \theta_T$. Then for any positive number ϵ, there exists a number $\delta(\epsilon) > 0$ such that the inequality*

$$||\bar\theta - \theta_T||_{L^2} \le \delta \le \delta(\epsilon)$$

implies the inequality

$$||a_\alpha - a_T||_C \le \epsilon,$$

where

$$a_\alpha = R(\bar\theta,\alpha), \quad \textit{with} \quad \alpha = \alpha(\delta) = \delta^\lambda,\ \ 0 < \lambda \le 2.$$

Proof: Since the functional $M^\alpha[a,\bar\theta]$ attains its minimum when $a = a_\alpha$, we have

$$M^\alpha[a_\alpha,\bar\theta] \le M^\alpha[a_T,\bar\theta].$$

Therefore

$$\alpha||a_\alpha||^2_{W_2^1[0,T]} \le M^\alpha[a_\alpha,\bar\theta] \le M^\alpha[a_T,\bar\theta] \le \delta^2 + \alpha||a_T||^2_{W_2^1[0,T]} \le \delta^\lambda d,$$

where

$$d = 1 + ||a_T||^2_{W_2^1[0,T]}.$$

Thus

$$||a_\alpha||^2_{W_2^1[0,T]} \leq d, \quad \text{and} \quad ||a_T||^2_{W_2^1[0,T]} \leq d.$$

Consequently, both a_α and a_T belong to the following compact subset of space $C[0,T]$

$$E = \left\{ a(t) : \ ||a||^2_{W_2^1[0,T]} \leq d \right\}.$$

By virtue of the continuity of the inverse A^{-1} defined on AE, for arbitrary $\epsilon > 0$, there exists a number $\eta(\epsilon) > 0$ such that the inequality

$$||\theta_\alpha - \theta_T||_{L^2[0,T]} \leq \eta(\epsilon), \quad \text{for} \quad \theta_\alpha = A[a_\alpha], \quad \theta_T = A[a_T] \in AE$$

implies the inequality

$$||a_\alpha - a_T||_{C[0,T]} \leq \epsilon.$$

Now for $\bar{\theta}$ and θ_α, we have

$$||\theta_\alpha - \bar{\theta}||_{L^2[0,T]} = ||A[a_\alpha] - \bar{\theta}||_{L^2[0,T]} \leq M^\alpha[a_\alpha, \bar{\theta}] \leq M^\alpha[a_T, \bar{\theta}] \leq \delta^\lambda d,$$

and thus

$$\begin{aligned} ||\theta_\alpha - \theta_T||_{L^2[0,T]} &\leq ||\theta_\alpha - \bar{\theta}||_{L^2[0,T]} + ||\bar{\theta} - \theta_T||_{L^2[0,T]} \\ &\leq \delta^{\lambda/2}\sqrt{d} + \delta \leq \delta^{\lambda/2}(1 + \sqrt{d}). \end{aligned}$$

If we set

$$\delta(\epsilon) = \left(\frac{\eta(\epsilon)}{1 + \sqrt{d}} \right)^{2/\lambda},$$

then the conclusion of the theorem follows. Therefore it is justified to take a_α as an approximate solution of equation (7) with an approximate left hand side $\theta = \bar{\theta}$.

Finally, the continuous dependence of the θ_T on y is almost clear from the following. If $||y_\delta||_C = ||\bar{y} - y_{\eta_\tau}||_C \leq \delta$, from (10), we can conclude:

$$||\bar{\theta} - \theta_T||^2_{L^2[0,T]} = \int_0^T (\ln \bar{y}(t) - \ln y_T(t))^2 \, dt \leq \int_0^T \frac{y_\delta^2}{y_T^2} \, dt \leq c^2 \delta^2, \tag{14}$$

where

$$c^2 = \int_0^T \frac{1}{y_T^2(t)} dt.$$

The following theorem shows that $y(t)$ depends on the initial data $f(t)$ and $g(t)$ continuously.

Theorem 3 *Suppose*

$$||f_\delta||_C = ||\bar{f} - f_T||_C \le \delta, \quad \textit{and} \quad ||g_\delta||_C = ||\bar{g} - g_T||_C \le \delta.$$

Define

$$D = \frac{\sqrt{\pi}\int_0^T y_T(t)\,dt + \int_0^T \frac{y_T(t)}{\sqrt{T-t}}\,dt}{\sqrt{\pi}\int_0^T g_T(t)\,dt + \int_0^T \frac{f_T(t)}{\sqrt{T-t}}\,dt}, \tag{15}$$

then

$$||y_\delta||_C = ||\bar{y} - y_T||_C \le 2D\delta. \tag{16}$$

Proof: From (8) we can write

$$\begin{aligned} \int_0^t \frac{\bar{g}(\tau)}{\sqrt{t-\tau}}\bar{y}(\tau)d\tau &= \sqrt{\pi}\bar{f}(t)\bar{y}(t), \\ \text{and} \quad \int_0^t \frac{g_T(\tau)}{\sqrt{t-\tau}}y_T(\tau)d\tau &= \sqrt{\pi}f_T(t)y_T(t). \end{aligned}$$

This implies

$$\sqrt{\pi}\left(\bar{f}(t)y_\delta(t) + f_\delta(t)y_T(t)\right) = \int_0^t \frac{\bar{g}(\tau)y_\delta(\tau) + g_\delta(\tau)y_T(\tau)}{\sqrt{t-\tau}}d\tau.$$

Multiplying both sides by $1/\sqrt{T-t}$ and integrating with respect to t over $[0,T]$, we can obtain

$$\int_0^T \frac{\bar{f}(t)y_\delta(t) + f_\delta(t)y_T(t)}{\sqrt{T-t}}dt = \sqrt{\pi}\int_0^T \left(\bar{g}(t)y_\delta(t) + g_\delta(t)y_T(t)\right)dt.$$

Thus

$$\left|\int_0^T \left(\frac{\bar{f}(t)}{\sqrt{T-t}} - \sqrt{\pi}\bar{g}(t)\right) y_\delta(t)\,dt\right| \le \delta \int_0^T \left(\frac{y_T(t)}{\sqrt{T-t}} + \sqrt{\pi}y_T(t)\right)dt,$$

from which (16) follows. This completes the proof of the theorem.

3 A numerical example

In this section, we provide an example with exact solution to see how the regularization method proposed in this paper works. Take

$$f_T(t) = 2(t+1)\sqrt{t/\pi}, \quad \text{and} \quad g_T(t) = t+1. \tag{17}$$

From (7) and (8), it is easy to verify that

$$y_T = \frac{1}{t+1}, \quad \text{and} \quad a_T(t) = \frac{1}{t+1}. \tag{18}$$

For simplicity, we use a uniform grid with step size $h = T/(n+1)$. The first step in the numerics is to replace (8) and (12) with finite difference approximation on the grid to get the following recursive relations

$$
\begin{aligned}
y_1 &= 2g_0 \frac{\sqrt{h/\pi}}{f_1} \\
y_i &= \frac{2\left((\sqrt{i}-\sqrt{i-1})g_0 + \sum_{j=2}^{i}(\sqrt{i-j+1}-\sqrt{i-j})g_{j-1}y_{j-1}\right)\sqrt{h/\pi}}{f_i}, \\
& \quad i = 2, \cdots, n,
\end{aligned}
$$

and the system of linear equations

$$
\alpha\left(\frac{a_{j-1}-2a_j+a_{j+1}}{h^2} - a_j\right) = h^2 \sum_{i=j}^{n}\sum_{k=1}^{i} a_k - h\sum_{i=j}^{n}\theta_i, \quad j = 1, \cdots, n, \tag{19}
$$

with $a_0^* = 1$ and $a_T^* = 0.5$ corresponding to $T = 1$, see (13). Next, we take the regularization parameter α in the form

$$
\alpha = \alpha(\delta) = (2CD\delta)^{\lambda}, \quad 0 < \lambda \le 2. \tag{20}
$$

Table 1 shows the exact solution and the solution using the regularization approach with the perturbations $f_\delta(t) = g_\delta(t) = \delta \sin(2\pi t)$, $n = 79$, $\delta = 10^{-6}$ and $\lambda = 0.6$. The results agree with each other pretty well.

Table 1: Reconstruction of $a(t)$ using the regularization method. $a_T(t)$ is the exact solution. $a_\alpha(t)$ is the solution of the regularization method. The parameters are: $n = 79$, $T = 1.0$, $\delta = 10^{-6}$, and $\lambda = 0.6$.

t	0.05	0.1	0.15	0.2	0.25
$a_T(t)$	0.95238	0.90909	0.86956	0.83333	0.8
$a_\alpha(t)$	0.95614	0.91484	0.87606	0.83978	0.80595
t	0.3	0.35	0.4	0.45	0.5
$a_T(t)$	0.76923	0.74074	0.71429	0.68966	0.66667
$a_\alpha(t)$	0.77446	0.74520	0.71802	0.69274	0.66919
t	0.55	0.6	0.65	0.7	0.75
$a_T(t)$	0.64516	0.62500	0.60606	0.58824	0.57143
$a_\alpha(t)$	0.64720	0.62661	0.60726	0.58903	0.57181
t	0.8	0.85	0.9	0.95	
$a_T(t)$	0.55556	0.54054	0.52632	0.51282	
$a_\alpha(t)$	0.55555	0.54020	0.52578	0.51234	

In practice, the exact solution a_T is unknown. More work remains to be done to study the behavior, such as accuracy and stability, of a numerical method applied to the equation (19).

In a summary, the regularization method proposed in this paper seems to be an effcient way for solving the inverse problem described at the begining of this paper. We have proved some good theoretical results about this method. Naive numerical discretization gives reasonably accurate result for the example provided in the paper. More work needs to be done on numerical study of such problems especially for two or higher dimensional problems.

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