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Two-sided Mullins-Sekerka flow does not preserve convexity *

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Abstract

The (two-sided) Mullins-Sekerka model is a nonlocal evolution model for closed hypersurfaces, which was originally proposed as a model for phase transitions of materials of negligible specific heat. Under this evolution the propagating interfaces maintain the enclosed volume while the area of the interfaces decreases. We will show by means of an example that the Mullins-Sekerka flow does not preserve convexity in two space dimensions, where we consider both the Mullins-Sekerka model on a bounded domain, and the Mullins-Sekerka model defined on the whole plane.

Introduction

The (two-sided) Mullins-Sekerka flow is a nonlocal generalization of the mean curvature flow arising from physics [13], and was originally proposed as an isotropic model for solidification and liquidation of materials of negligible specific heat. It is also a sharp-interface model describing phase transition in quenched binary alloys. More precisely, it has been shown by Alikakos, Bates, and Chen that the Mullins-Sekerka model arises as a singular limit for the movement of level sets of solutions to the Cahn-Hilliard equation [1], the Cahn-Hilliard equation being a fourth-order nonlinear partial differential equation modeling the process of phase separation and coarsening in a melted alloy [3]. This relationship between the Mullins-Sekerka model and the Cahn-Hilliard equation was first formally derived by Pego [14], and has likewise been established by Stoth [15] for the radially symmetric case, but for somewhat more general boundary conditions. In the literature the Mullins-Sekerka model has sometimes been called the Hele-Shaw model (with surface tension), and it is also known as the quasi-stationary Stefan problem. The problem studied in this paper is the same as the one considered in [1, 4, 5, 6, 7, 8].

There are two independent proofs for existence of smooth solutions for the two-sided version of the Mullins-Sekerka model in a bounded domain, due to

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Chen, Hong, and Yi [6], and due to Escher and Simonett [7]; the existence of weak solutions had been established earlier by Chen for the two-dimensional case [4]. However, the author is unaware of any explicit result about the existence of smooth solutions of the Mullins-Sekerka model for unbounded domains.

As for the geometry of solutions, only little is known. It has been shown that smooth initial configurations that are close to being spherical (in a $C^{2+\alpha}$ sense) will converge exponentially fast to a sphere [8]; an analogous result had been obtained before for weak solutions in two space dimensions [4]. In the Mullins-Sekerka model, the normal velocity of a moving interface depends on the jump of the normal derivative of a function across the interface, the function being the harmonic extension of the mean curvature of the propagating interface. One can ask whether the Mullins-Sekerka flow shares properties with the mean curvature flow, since both flows are in some way driven by surface tension. Not all results can be expected to generalize, due to the nonlocal character of the Mullins-Sekerka problem, in particular not those that rest on a local argument for the mean curvature flow. It is known that the mean curvature flow preserves convexity [9, 11]. It is therefore a natural question to ask whether this is also true for the Mullins-Sekerka flow. Under the assumption of short-term existence of sufficiently smooth solutions this question was answered negatively for the one-sided Mullins-Sekerka model in a previous paper by the author [12]. The current paper is an extension of the previous one to the two-sided model. The loss of convexity for the two-sided Mullins-Sekerka model in all of $\mathbb{R}^2$ has also been numerically evidenced by Bates, Chen, and Deng [2], for an improved algorithm see the paper by Zhou, Chen, and Hou [16].

The two-sided Mullins-Sekerka model

We look at a curve Γ_0 contained in a fixed domain $\Omega \subset \mathbb{R}^2$, and we consider the free boundary problem governed by the evolution law given by

$$\begin{aligned} \Delta u &= 0 && \text{on } \Omega \setminus \Gamma_t\,, \\ \tfrac{\partial u}{\partial n} &= 0 && \text{on } \partial\Omega\,, \\ u &= \kappa && \text{on } \Gamma_t\,, \\ V &= \left[\tfrac{\partial u}{\partial n}\right] && \text{on } \Gamma_t\,. \end{aligned} \tag{1}$$

The above assumes Ω to be bounded, in case $\Omega = \mathbb{R}^2$ one needs to replace the second line in this system with

$$|u| = O(1) \quad \text{as } |x| \to \infty\,.$$

Also in the equations above, n is the outer unit normal to Γ_t and to $\partial\Omega$, while V and κ are the normal velocity and the curvature of Γ_t, respectively. The signs are chosen in such a way that a circle has positive curvature and a shrinking curve has negative velocity. The expression on the right-hand side of the equation for V denotes the jump of the normal derivative of u across Γ_t, that is $\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n}$, where the superscripts $+$ and $-$ indicate the regions outside and inside of Γ_t. We define $n^+ = n$ and $n^- = -n$, which are the inner unit normals to the outside region Ω^+ and the inside region Ω^- of Γ_t, and then we rewrite the equation for the normal velocity as

$$V = \frac{\partial u^+}{\partial n^+} + \frac{\partial u^-}{\partial n^-}\,.$$

The principal idea is to consider an initial curve Γ_0 for which one can make qualitative statements about the initial velocity, and then to use continuity to forecast the shape of the evolving curves. We look at a shape given by a straight tube with two circular end caps. We will show that both $\frac{\partial u^+}{\partial n^+}$ and $\frac{\partial u^-}{\partial n^-}$ are positive quantities, but that they are smaller in the center of the straight part of the figure, whence the normal velocity will be smaller in the center, and the figure will become nonconvex. Notice that by the strong maximum principle one has $\frac{\partial u^+}{\partial n^+} > 0$ and $\frac{\partial u^-}{\partial n^-} > 0$ at any point at which the curvature attains its minimum, provided the curve is not a circle. Thus any piece of a convex curve at which the curve loses its convexity must necessarily move outwards, and convexity can therefore only be lost when some parts of the curve move out slower than others, just as for the example given herein. However, one may construct examples that lose their convexity at one place only, as compared to the example herein, which simultaneously loses convexity at both straight line pieces.

The inside of the curve

First we consider only the region Ω^- inside of the curve. The statement made above about $\frac{\partial u^-}{\partial n^-}$ follows from repeated applications of the maximum principle, the precise argument was presented in a previous paper [12]. We will repeat here the flow of the argument for the sake of completeness, and we write u for u^- in this section. Let us place the figure into a standard Cartesian coordinate system, the straight parts parallel to the x-axis, and the figure being centered about the origin. We restrict our attention to the right half of the figure. As u has a maximum on the circular part this implies $\frac{\partial u}{\partial n} > 0$, and hence $u_x > 0$. On the straight part we have $u_x \equiv 0$, as u is identically zero there. We also have $u_x \equiv 0$ on the y-axis by the symmetry of u. Of course, one has to take care of the non-smoothness of the curve where the circular arc connects to the straight parts. This is done in the usual fashion by inserting a small C^∞ transition piece, and it can be shown that even on this transition piece one can maintain the condition $u_x \geq 0$, for the technical details see [12]. We conclude $u_x > 0$ in the interior of the right half of Ω^- by the maximum principle.

As $u_x \equiv 0$ on the upper straight part, we must have $\frac{\partial}{\partial n} u_x = u_{xy} < 0$ on the right half of it by another application of the maximum principle. Yet another application of the maximum principle for the function u itself tells us that $\frac{\partial u}{\partial n} = u_y < 0$ on the upper straight line. Therefore on the right half of the upper line the quantity $\frac{\partial u}{\partial n^-} = |u_y|$ decreases towards the center. By symmetry the same effect happens on the left half of the curve.

The outside of the curve for the case $\Omega = \mathbb{R}^2$

We start out with this case because it illustrates the main idea, and it is technically easier than the case of a bounded domain. We will place the curve Γ_0 again into a Cartesian coordinate system, but shifted down from the position chosen in the previous section, so that the upper straight part falls onto the x-axis, see Figure 1. This way we can use Poisson's formula to represent the harmonic function $u = u^+$ in the upper half plane if we know its restriction to the x-axis,

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2}\, u(\xi,0)\, d\xi\,.$$

As the upper straight part of the curve is on the x-axis we know that $u \equiv 0$ on this part of the x-axis, say on the interval $[-2\varepsilon, 2\varepsilon]\times\{0\}$. Using the symmetry of u we obtain

$$u(x,y) = \frac{1}{\pi}\int_{2\varepsilon}^{\infty} \left(\frac{y}{(x-\xi)^2+y^2} + \frac{y}{(x+\xi)^2+y^2}\right) u(\xi,0)\, d\xi\,.$$

Also, since Γ_0 is convex, we know that u is nonnegative everywhere on Γ_0, hence positive everywhere outside of Γ_0, and so in particular on the rest of the x-axis. Using methods from elementary calculus it is easily seen that the kernel

$$k(x) = \frac{y}{(x-\xi)^2+y^2} + \frac{y}{(x+\xi)^2+y^2}$$

is increasing for $0 < x < \varepsilon$, where we restrict to $\xi > 2\varepsilon$ and $0 < y < \varepsilon$. Hence $u(x,y)$ is increasing in x on the rectangle $\mathcal{R}$ bounded by the x-axis, the y-axis, and the lines $x = \varepsilon$ and $y = \varepsilon$, see Figure 1. Thus we have $u_x > 0$ on the interior of $\mathcal{R}$, and, by the strong maximum principle, $u_{xy} > 0$ on the interval $(0,\varepsilon)\times\{0\}$, because u_x vanishes on this part of the boundary of the rectangle $\mathcal{R}$. For use in the next section we note that for any sufficiently small $\delta > 0$ there is a constant $c > 0$ such that

$$\frac{\partial^2 u}{\partial x \partial y}(x,0) > c \quad \text{for } \delta \le x \le \varepsilon - \delta\,. \tag{2}$$

We interpret this as the statement $\dfrac{\partial}{\partial x}\Big(\dfrac{\partial u}{\partial n^+}\Big) > 0$, and we see that the normal derivative grows towards the right. The analogous statement is true for the corresponding part of the figure on the other side of the y-axis.

The outside of the curve for the case Ω a disk

Just as in the previous section we will place the curve into a Cartesian coordinate system so that the upper straight part falls onto the x-axis. For simplicity we assume that $\Omega = \Omega_R$ is the disk of radius R centered at the origin, but this is not essential. We will now consider the family of harmonic functions $u_R = u_R^+$

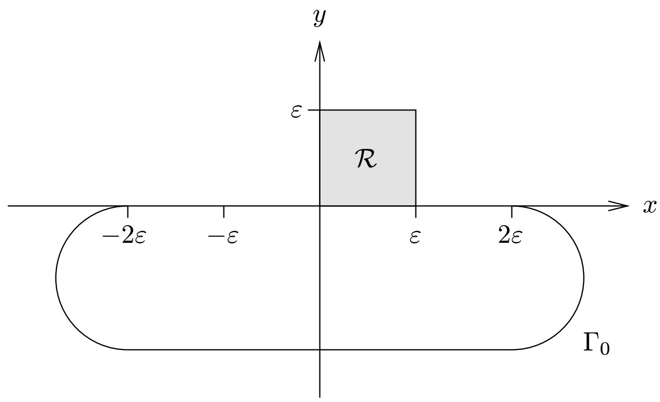

Figure 1: The rectangle $\mathcal{R}$ and the location of the curve Γ_0. Not drawn to scale.

obtained by varying the radius R of Ω_R. Let $\widetilde{u}$ be the function obtained in the previous section; $\widetilde{u}$ is the harmonic extension of the curvature of Γ_0 to the outside of Γ_0 in $\mathbb{R}^2$. Our objective is to show that the functions u_R do not differ much from $\widetilde{u}$, at least not for R sufficiently large. To do this we will show that the family $\{u_R, R \geq R_0\}$ is a normal family (R_0 is arbitrary). First we establish the necessary equicontinuity near Γ_0, and for this we will need the following boundary estimate from elliptic regularity theory.

Lemma 1 *Let W be a $C^{2+\alpha}$ domain in $\mathbb{R}^m$, and let Γ_0 be a compact component of ∂W with $\operatorname{dist}(\Gamma_0, \partial W \setminus \Gamma_0) > 0$ in case $\Gamma_0 \neq \partial W$. Let $v \in C^{2+\alpha}(\overline{W})$ be a solution to $\Delta v = f$, $v = 0$ on Γ_0, where $f \in C^{\alpha}(\overline{W})$. Then there is a $\delta > 0$ such that for all $x_0 \in \Gamma_0$ one has*

$$|v|_{2+\alpha; B_\delta(x_0)\cap W} \leq C(|v|_{0;W} + |f|_{\alpha;W}) \tag{3}$$

where the constants δ and C depend only on m, α, Γ_0, and W. The various norms are the usual Hölder norms.

The proof of this lemma is similar to the proof of Lemma 6.5 in [10], and is omitted here.

Recall that Ω_R^+ stands for the part of Ω_R outside of Γ_0. Let Ω_{R_0} be any fixed disk containing Γ_0 and set $W = \Omega_{R_0}^+$. Let $\phi \in C^{2+\alpha}(\overline{W})$ be any function with $\phi(x, y) = \kappa(x, y)$ for $(x, y) \in \Gamma_0$. Finally, let u_R be the harmonic function associated with Ω_R^+, $R \geq R_0$. Then we apply the lemma to $v := u_R - \phi$ and $f := \Delta\phi$, and we conclude in particular that there is a neighborhood of Γ_0 in $\overline{W}$ on which we have a uniform bound on the first derivatives of v, and hence on the first derivatives of u_R. Thus there is a neighborhood of Γ_0 on which the family $\{u_R, R \geq R_0\}$ is equicontinuous.

The equicontinuity in the interior of W follows from the well-known interior gradient bound for harmonic functions, see for example Theorem 2.10 in [10]. We quote it here in its entirety because we will need it twice.

Lemma 2 *Let u be any harmonic function in $W \subset \mathbb{R}^m$ and let K be any compact subset of W. Then for any multi-index α we have*

$$\sup_K |D^\alpha u| \leq (m|\alpha| / \operatorname{dist}(W, K))^{|\alpha|} \sup_W |u| \,, \tag{4}$$

where D stands for the derivative operator.

Finally, notice that the functions u_R are uniformly bounded by their maximum on Γ_0. Hence, fixing any ball Ω_{R_i} with $R_i > R_0$ we see that by the Arzela-Ascoli Theorem we get a subsequence of $\{u_R, R \geq R_0\}$ converging uniformly to a harmonic function on $\Omega^+_{R_i}$ as the radius R converges to infinity. Now we choose a sequence $R_i \to \infty$ and apply the usual diagonalization process, by which we obtain a subsequence of harmonic functions converging uniformly to a harmonic function $\bar{u}$ defined on the complete outside of Γ_0. In other words, if we index that subsequence with i again, we find R_i such that the sequence u_{R_i} converges uniformly to $\bar{u}$ on any fixed Ω^+_R. Of course, the function u_{R_i} is only defined on Ω^+_R provided $R_i \geq R$, but this is irrelevant for our argument. Now, the function $\bar{u}$ is bounded at infinity and has the same boundary values on Γ_0 as all the functions u_R. By the unique solvability of the exterior Dirichlet problem we conclude that $\bar{u} = \widetilde{u}$, which is the harmonic extension of the curvature of Γ_0 to all of $\mathbb{R}^2$. Note that the dimension 2 is important here, boundedness of a harmonic function is not enough to conclude harmonicity at infinity for dimensions greater than two. In fact, the argument shows that any sequence u_{R_k} with $R_k \to \infty$ has a subsequence converging to $\widetilde{u}$, and so we see that the complete family $\{u_R, R \geq R_0\}$ converges to $\bar{u}$ as $R \to \infty$. In summary, we have shown the following.

Proposition 1 *Let W be the exterior of Γ_0 in all of $\mathbb{R}^2$, and set $\Omega^+_R = W \cap \Omega_R$. Let $\widetilde{u}$ be the harmonic extension to W of a smooth function κ defined on Γ_0, and let u_R denote the harmonic extension of κ to Ω^+_R, where we constitute a zero Neumann condition on $\partial\Omega_R$. Then as $R \to \infty$ the functions u_R converge uniformly to $\widetilde{u}$ on any compact subset of $W \cup \Gamma_0$.*

Now consider the interval $I = [\delta, \varepsilon - \delta] \times \{0\}$ from the previous section, and set $u = \widetilde{u} - u_R$ in Lemma 2. We want to use $\Omega^+_{R_0}$ as the set W in Lemma 2, however, the interval I is not in the interior of $\Omega^+_{R_0}$. We will therefore enlarge all the domains Ω^+_R of the harmonic functions u_R slightly by reflecting near the interval I across the x-axis. This extension is possible since $\widetilde{u} \equiv u_R \equiv 0$ on this part of Γ_0. The convergence statement made in the proposition above is of course also true for these slightly bigger regions, and hence we can use the interior derivative estimate (4) from Lemma 2 for the interval I. Finally, making the radius R large enough, we see that then

$$\sup_I \left| \frac{\partial^2(\widetilde{u} - u_R)}{\partial x \partial y} \right| \leq (4/\delta)^2 \sup_{\Omega^+_{R_0}} |\widetilde{u} - u_R| \leq c/2 \,,$$

where c is the lower bound from (2). Hence, using (2), we get

$$\frac{\partial^2 u_R}{\partial x \partial y}(x,0) > c/2 \quad \text{for } \delta \leq x \leq \varepsilon - \delta\,.$$

As in the previous section, we interpret this as the statement $\frac{\partial}{\partial x}\Big(\frac{\partial u_R}{\partial n^+}\Big) > 0$, whence we see the normal derivative grows towards the right.

The conclusion

We have shown in the previous sections that the two normal derivatives making up the normal velocity grow as we go away from the center (where we start a distance $\delta > 0$ away from the center), and hence so does the normal velocity, which is their sum. Hence the straight part of the curve moves out, but the center moves slower than the ends, and thus by continuity of the evolution the curve will become non-convex, see Figure 2.

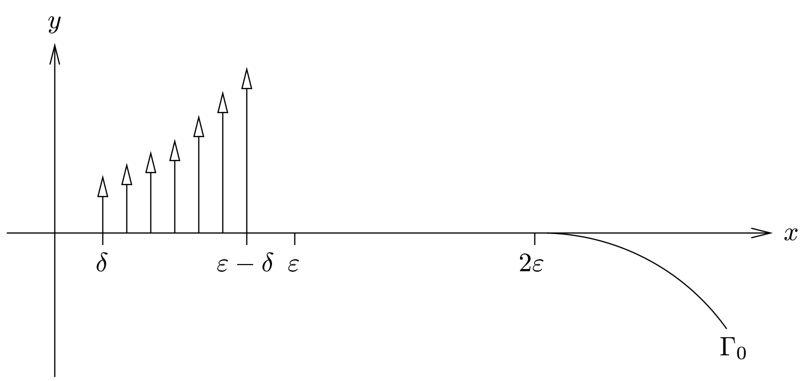

Figure 2: A schematic sketch of the initial velocity of the interval I belonging to the straight part of Γ_0.

As mentioned in the introduction, the existence of smooth solutions for the Mullins-Sekerka flow on a bounded domain has been established recently [6, 7]. In fact, as follows from the proofs in [7], the solution constitutes a semi-flow on a space of curves (in two dimensions) parameterized over a reference curve, and one has continuous dependence of the solution on the initial data, measured in the $C^{2+\alpha}$ norm. The example presented herein that leads to a loss of convexity can therefore be slightly perturbed, and it will still evolve into a nonconvex shape. In particular we can perturb it in such a fashion that the resulting initial curve is strictly convex.

Theorem 1 *There are strictly convex smooth initial configurations that will evolve into nonconvex curves under the two-sided Mullins-Sekerka flow on a large disk. In particular one can choose a small smooth perturbation of a curve consisting of a straight tube with two end caps, where the flat tube can be arbitrarily short, so that these initial curves can be chosen arbitrarily close in the C^1 norm to a circle.*

As noted above, the example herein also leads to a loss of convexity in case of the Mullins-Sekerka model on all of $\mathbb{R}^2$. In fact, the analytic argument for the loss of convexity was easier for the unbounded case than for the bounded case. It is clear that if one has continuous dependence of the solution on the initial data that one can also go over to a strictly convex example.

Theorem 2 *Assume that the two-sided Mullins-Sekerka flow on all of $\mathbb{R}^2$ allows a smooth solution provided the initial configuration is C^∞. Then there are convex smooth initial configurations consisting of a straight tube with two end caps that will evolve into nonconvex curves. The flat tube can be arbitrarily short. In particular these initial curves can be chosen arbitrarily close in the C^1 norm to a circle.*

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