Differential Equations and Computational Simulations III
J. Graef, R. Shivaji, B. Soni, & J. Zhu (Editors)
Electronic Journal of Differential Equations, Conference 01, 1997, pp. 223–235.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp 147.26.103.110 or 129.120.3.113 (login: ftp)

PERSISTENCE OF CRANDALL–RABINOWITZ TYPE BIFURCATIONS UNDER SMALL PERTURBATIONS

BETTINA E. SCHMIDT

Abstract

We discuss a class of nonlinear operator equations in a Banach space setting and present a generalization of the Crandall-Rabinowitz bifurcation theorem that describes the effect of small perturbations of the operators involved on the local structure of the solution set in the vicinity of a bifurcation point of the unperturbed equation. The result is applied to a parameter-dependent Neumann boundary-value problem with spatially homogeneous source terms that exhibits infinitely many bifurcation points. We obtain conditions for the persistence or nonpersistence of these bifurcations under small, spatially inhomogeneous perturbations of the source terms.

INTRODUCTION

One of the most frequently quoted works in local bifurcation theory is the 1971 paper [2], by M. G. Crandall and P. H. Rabinowitz, on bifurcation from simple eigenvalues. Roughly speaking, the paper's main result asserts that if H is a C^2-mapping between Banach spaces and if w_0 is a *regular-singular point* of H (see Section 1 for the definition), then there exists a neighborhood U of w_0 such that the set $\{w \in U \mid H(w) = H(w_0)\}$ is the union of two simple C^1-arcs that intersect transversally at w_0.

In countless applications, this result has been used, in one way or other, to establish the existence of bifurcation points on given solution curves of so-called nonlinear eigenvalue problems. Continuing the tradition, we use it here to prove the occurrence of infinitely many bifurcations from a branch of trivial (that is, constant) solutions of a parameter-dependent Neumann boundary-value problem (P) with spatially homogeneous source terms. Our main concern, however, is the effect of small, spatially inhomogeneous perturbations of the source terms on the local structure of the solution set of Problem (P).

If the perturbation is "small," the solution set of the perturbed problem (P_ϵ) should be somehow "close" to the solution set of the unperturbed problem (P), although (P_ϵ) has no *trivial* solutions, due to the spatial inhomogeneity. It is rather obvious that near *regular points* on a given solution curve of (P), the effect of the perturbation is simply a continuous deformation of the curve. What happens near

---

1991 *Subject Classification:* 34B15, 34C23, 46N20.
Key words and phrases: Neumann problem, nonlinear eigenvalue problem, bifurcation from simple eigenvalues, Crandall-Rabinowitz theorem, regular-singular points, perturbed bifurcation theory.

Published November 12, 1998.

bifurcation points (or more specifically, near *regular-singular points*) is much less obvious: Will the bifurcations persist or will they "unfold"? Does the letter X, under small perturbations, become a slightly distorted letter X, or will it turn into something like the union of two letters V (not intersecting and facing each other tip to tip)?

We will argue that in the abstract setting of the Crandall-Rabinowitz theorem, either scenario is possible. Nonpersistence (or "unfolding") of bifurcations is, in a sense, the generic case, but at least if the operators involved exhibit certain symmetries, bifurcations may very well persist. In our Neumann problem (P), for example, every other one of infinitely many bifurcations is persistent under small spatially inhomogeneous perturbations with a certain symmetry property.

The paper is organized as follows. In Section 1 we present a general functional-analytic framework for "perturbed Crandall–Rabinowitz bifurcation." We do not claim the two theorems in this section to be entirely new, but we are not aware of any directly quotable reference. In Sections 2 and 3 we study the trivial solution curve of our Neumann problem and, using the Crandall–Rabinowitz theorem, prove the existence of infinitely many bifurcation points. In Section 4 we apply the abstract results of Section 1 to describe possible effects of small, spatially inhomogeneous perturbations on the local structure of the solution set of (P) in the vicinity of a bifurcation point.

1. PERTURBED CRANDALL–RABINOWITZ BIFURCATION

In this section, we briefly discuss some general results on the local structure of the solution sets of certain types of nonlinear operator equations in Banach spaces. Our main concern is the effect of a small perturbation of such an equation on the local structure of the solution set in the vicinity of a Crandall–Rabinowitz type bifurcation point. The proofs of our main results, which rely on the Ljapunov–Schmidt method, the implicit function theorem, and the Morse lemma, are rather straightforward, but quite technical and lengthy. For details, the reader is referred to [7].

By way of motivation, consider a simple, algebraic equation in two variables,

$$H(x,y) = 0, \tag{1.1}$$

where $H \in C^2(\mathbb{R}^2, \mathbb{R})$. Suppose that (x_0, y_0) is a solution of Equation (1.1). We call (x_0, y_0) a *regular point* of H if the gradient of H does not vanish at (x_0, y_0), that is, if at least one of the two partial derivatives is nonzero. By the implicit function theorem, it is then possible to solve Equation (1.1) for either x or y. In any case, the solution set is locally a simple C^1-arc.

We call (x_0, y_0) a *regular-singular point* of H if the gradient of H vanishes at (x_0, y_0), while the Hessian of H at (x_0, y_0) has one positive and one negative eigenvalue. In this case, the Morse Lemma implies that (x_0, y_0) is a *saddle point* of H. That is, the solution set of Equation (1.1) is locally, near (x_0, y_0), the union of two simple C^1-arcs, transversally intersecting at (x_0, y_0).

Both scenarios have natural generalizations in a Banach space setting. Consider a C^2-mapping H from an open set Ω in a Banach space W into a Banach space Y. Let w_0 be a point in Ω and suppose that the Fréchet derivative $H'(w_0)$ is a Fredholm operator of index 1 (that is, the range $R(H'(w_0))$ of $H'(w_0)$ is closed and of finite codimension in Y, the nullspace $N(H'(w_0))$ of $H'(w_0)$ is finite-dimensional, and $\dim N(H'(w_0)) - \dim Y/R(H'(w_0)) = 1$). The point w_0 is called a *regular point* of H if $\dim N(H'(w_0)) = 1$ (or equivalently, if $R(H'(w_0)) = Y$). Otherwise, w_0 is called a *singular point* of H. Specifically, w_0 is called a *regular-singular point* of H if $\dim N(H'(w_0)) = 2$ and there exists a basis $\{w_1, w_2\}$ of $N(H'(w_0))$ such that the quadratic derivative $H''(w_0)w_1w_1$ belongs to $R(H'(w_0))$, while the mixed derivative $H''(w_0)w_1w_2$ does not. (Note that if $N(H'(w_0))$ is two-dimensional, then $R(H'(w_0))$ has codimension 1 in Y.) We mention that whenever w_0 is regular-singular in the above sense, there exists in fact a basis $\{\tilde{w}_1, \tilde{w}_2\}$ of $N(H'(w_0))$ such that *both* quadratic derivatives, $H''(w_0)\tilde{w}_1\tilde{w}_1$ and $H''(w_0)\tilde{w}_2\tilde{w}_2$, belong to $R(H'(w_0))$, while the mixed derivative $H''(w_0)\tilde{w}_1\tilde{w}_2$ does not.

Just as in the case of Equation (1.1), the implicit function theorem guarantees that if w_0 is a regular point of H, then the solution set of the equation $H(w) = H(w_0)$ is locally, near w_0, a simple C^1-arc. On the other hand, if w_0 is a regular-singular point of H, then the Crandall-Rabinowitz bifurcation theorem (see [2, Theorem 1]) applies and asserts that the solution set of $H(w) = H(w_0)$ is locally, near w_0, the union of two simple C^1-arcs that intersect transversally at w_0. In the sequel, a point w_0 with this latter property will be referred to as a *Crandall-Rabinowitz point* of H.

A question of interest is how the local structure of the solution set changes when the mapping H is perturbed. To study this question, let us now consider a *family* of mappings $H(\epsilon, \cdot)$ from an open subset Ω of a Banach space W into a Banach space Y, where ϵ varies over an open interval J containing 0. We think of $H(\epsilon, \cdot)$, for $\epsilon \in J$, as a perturbation of the mapping $H(0, \cdot)$ and wish to describe the structure of the set $\Sigma_\epsilon := \{w \in \Omega \mid H(\epsilon, w) = 0\}$, for ϵ close to 0, in the vicinity of a point $w_0 \in \Omega$ with $H(0, w_0) = 0$. Throughout we assume that $H \in C^2(J \times \Omega, Y)$ and that the (partial) Fréchet derivative $H_w(0, w_0) = \frac{\partial H}{\partial w}(0, w_0)$ is a Fredholm operator of index 1.

A straightforward generalization of the implicit function theorem shows that if w_0 is a *regular point* of $H(0, \cdot)$, then not only for $\epsilon = 0$, but for every ϵ sufficiently close to 0, the set Σ_ϵ is locally, near w_0, a simple C^1-arc (which varies continuously with ϵ). For the case where w_0 is a *regular-singular point* of H, we have the following generalization of the Crandall-Rabinowitz theorem (see [7, Theorem 4.6]).

Theorem 1.1. Suppose $w_0 \in \Omega$ is a regular-singular point of $H(0, \cdot)$. Then there exist

- an open interval $I \subset J$ with $0 \in I$,
- an open neighborhood $U \subset \Omega$ of w_0,
- a continuous mapping $\overline{w} : D \to U$, defined on an open set $D \subset \mathbb{R} \times \mathbb{R}^2$ that contains $(\epsilon, 0, 0)$ for every $\epsilon \in I$, with $\overline{w}(0, 0, 0) = w_0$, and
- a continuously differentiable mapping $\rho : I \to \mathbb{R}$ with $\rho(0) = 0$,

such that for every $\epsilon \in I$, the mapping $\overline{w}(\epsilon, \cdot, \cdot)$ is one-to-one and continuously differentiable, with linearly independent derivatives $\overline{w}_\sigma(\epsilon, \sigma, \tau)$ and $\overline{w}_\tau(\epsilon, \sigma, \tau)$ for all $(\sigma, \tau) \in \mathbb{R}^2$ with $(\epsilon, \sigma, \tau) \in D$, and we have

$$\{w \in U \mid H(\epsilon, w) = 0\} = \{\overline{w}(\epsilon, \sigma, \tau) \mid \sigma\tau + \rho(\epsilon) = 0 \text{ and } (\epsilon, \sigma, \tau) \in D\}.$$

Roughly speaking, the assertion of Theorem 1.1 is that for every ϵ sufficiently close to 0, the solution set Σ_ϵ of the equation $H(\epsilon, w) = 0$ is locally, near w_0, a homeomorphic image of the solution set, near $(0,0)$, of a simple, algebraic equation in $\mathbb{R}^2$, namely, $\sigma\tau + \rho(\epsilon) = 0$. Thus, whenever we have $\rho(\epsilon) = 0$, the set Σ_ϵ is locally, near w_0, the union of two simple C^1-arcs that intersect transversally at the point $\overline{w}(\epsilon, 0, 0)$. In particular, we recover the Crandall-Rabinowitz theorem, since $\rho(0) = 0$ and $\overline{w}(0,0,0) = w_0$. But whenever $\rho(\epsilon) \neq 0$, the set Σ_ϵ is locally, near w_0, the union of two disjoint, simple C^1-arcs.

Intuitively, this result is not very surprising. Recall that in the case of an algebraic equation in the plane, w_0 would be a saddle point of the unperturbed mapping $H(0,\cdot)$, located on the level-0 set of $H(0,\cdot)$. For ϵ close to 0, the perturbed mapping $H(\epsilon,\cdot)$ would still have a (unique) saddle point w_ϵ close to w_0, but at a level $\rho(\epsilon)$ in general different from 0. Only if $\rho(\epsilon) = 0$, would the level-0 set of $H(\epsilon,\cdot)$ be "cross-shaped" near w_0.

These last observations, too, can be naturally extended to our abstract Banach space setting (see [7, Theorem 4.16]).

Theorem 1.2. Suppose $w_0 \in \Omega$ is a regular-singular point of $H(0,\cdot)$ and $y_0 \in Y$ is a vector that does not belong to the range of $H_w(0, w_0)$. Then there exist an open interval $I \subset J$ with $0 \in I$ and an open neighborhood $U \subset \Omega$ of w_0 such that for every $\epsilon \in I$, the mapping $H(\epsilon,\cdot)$ has a unique singular point $w_\epsilon \in U$ whose image $H(\epsilon, w_\epsilon)$ is a constant multiple of y_0. The mapping $\epsilon \mapsto w_\epsilon$ is continuously differentiable, and for every $\epsilon \in I$, the point w_ϵ is a regular-singular point (and thus, a Crandall-Rabinowitz point) of $H(\epsilon,\cdot)$. Only if $H(\epsilon, w_\epsilon) = 0$, does the set $\{w \in U \mid H(\epsilon, w) = 0\}$ contain a singular point of $H(\epsilon,\cdot)$.

Now let $w_0 \in \Omega$ be a regular-singular point (and thus, a Crandall-Rabinowitz point) of $H(0,\cdot)$. We say that the bifurcation at w_0 is *persistent* (or *nonpersistent*, respectively) if there exists a neighborhood $U \subset \Omega$ of w_0 such that for every $\epsilon \in J \setminus \{0\}$ sufficiently close to 0, the set $\{w \in U \mid H(\epsilon, w) = 0\}$ contains a regular-singular point (or does not contain a singular point, respectively) of $H(\epsilon,\cdot)$.

It is easy to derive a simple, sufficient condition for *nonpersistence*. To that end, pick a vector $y_0 \in Y \setminus R(H_w(0, w_0))$ and let $y_0^* \in Y^*$ denote the (unique) functional with $<y_0^*, y_0> = 1$ and $N(y_0^*) = R(H_w(0, w_0))$. Choose an interval I, a neighborhood U, and points w_ϵ according to Theorem 1.2 and define $\rho \in C^1(I)$ by $\rho(\epsilon) := <y_0^*, H(\epsilon, w_\epsilon)>$. Then we have $H(\epsilon, w_\epsilon) = \rho(\epsilon) y_0$, for all $\epsilon \in I$, and the set $\{w \in U \mid H(\epsilon, w) = 0\}$ contains a singular point of $H(\epsilon,\cdot)$ if and only if $\rho(\epsilon) = 0$. Since $\rho(0) = 0$, it follows that $\rho'(0) \neq 0$ is a sufficient condition for nonpersistence. Using the fact that $y_0 \notin R(H_w(0, w_0))$, we readily show that $\rho'(0) = <y_0^*, H_\epsilon(0, w_0)>$ and conclude that the bifurcation at w_0 is nonpersistent provided that

$$H_\epsilon(0, w_0) \notin R(H_w(0, w_0)), \tag{1.2}$$

that is, provided that the partial derivative $H_\epsilon(0, w_0)$ does not belong to the hyperplane $R(H_w(0, w_0))$. This is, in a sense, the generic case: In the absence of special symmetries, the condition (1.2) is likely to be satisfied.

We do not have an equally simple, sufficient condition for *persistence* (obviously, $H_\epsilon(0, w_0) \in R(H_w(0, w_0))$ is *necessary*, but not *sufficient*). However, in concrete applications it is often possible to show directly that $\rho(\epsilon)$ must vanish for all ϵ near 0. Not surprisingly, the argument is usually based on special symmetry properties of the mapping H. A specific example will be discussed in Section 4.

2. A SPATIALLY HOMOGENEOUS NEUMANN PROBLEM

Consider the boundary-value problem

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) = \mu f(u) - g(u) \quad \text{in } (-1,1), \qquad u'(\pm 1) = 0. \tag{P}$$

Here, μ is a nonnegative parameter, and we seek nonnegative classical solutions u in $C^2([-1,1])$. Our assumptions on the data are as follows: The coefficient k is a positive, continuously differentiable function on $[-1,1]$. The function f is continuous on $\mathbb{R}_+$, and there exists a number $c>0$ such that $f(y)=0$ for all $y \in [0,c]$, while f is twice continuously differentiable on (c,∞), with $f>0$, $f' \geq 0$, and $f'' \leq 0$. The function g is continuous on $\mathbb{R}_+$, with $g(0)=0$, and twice continuously differentiable on $(0,\infty)$, with $g>0$, $g' \geq 0$, and $g'' \geq 0$. Finally we assume that $\frac{g(y)}{f(y)} \to \infty$ as $y \to \infty$. (Note that due to our earlier assumptions, $f(y)$ grows *at most* linearly, $g(y)$ *at least* linearly. Thus, the last assumption just excludes the case of both functions being asymptotically linear.)

We can interpret Problem (P) as a simple equilibrium model for heat conduction in a thin rod with insulated ends. In this model, k would be the rod's thermal conductivity. The term $\mu f(u)$ would represent a parameter-dependent heat source that kicks in as soon as the (absolute) temperature u exceeds a certain threshold c. The term $g(u)$ would represent radiative cooling ($g(u) \sim u^4$ if the process is governed by the Stefan-Boltzmann law).

Let Σ denote the set of all pairs (μ,u) with $\mu \in \mathbb{R}_+$ and $u \in C^2([-1,1])$ a nonnegative solution of (P). With slight abuse of language, we call the pairs $(\mu,u) \in \Sigma$ solutions of Problem (P). A solution (μ,u) is called *trivial* if the function u is *constant*. Clearly, Σ contains exactly two maximal continua of trivial solutions, namely, $\mathbb{R}_+ \times \{0\}$ and the trace Σ^* of the curve $\mu = \frac{g(y)}{f(y)}$ with $y \in (c,\infty)$.

To describe the trivial solution branch Σ^* in more detail, we need to analyze the function $\bar{\mu} := \frac{g}{f}\big|_{(c,\infty)}$. Clearly, we have $\bar{\mu}(c+) = \infty = \bar{\mu}(\infty)$. Moreover, $\bar{\mu}' = \frac{\sigma}{f^2}$ with $\sigma := g'f - gf'$. The function σ is nondecreasing on (c,∞) (in fact, $\sigma' = g''f - gf'' \geq 0$), negative near c (since $f(c)=0$ and $f'(c+) \in (0,\infty]$), and positive near ∞ (else, $\bar{\mu}$ would be nonincreasing throughout, contradicting the fact that $\bar{\mu}(\infty)=\infty$). This implies the existence of two numbers $\underline{y}_0$ and $\overline{y}_0$ with $c < \underline{y}_0 \leq \overline{y}_0 < \infty$ such that σ is negative on $(c,\underline{y}_0)$, zero on $[\underline{y}_0,\overline{y}_0]$, and positive on $(\overline{y}_0,\infty)$. The same then holds for $\bar{\mu}'$, and it follows that $\bar{\mu}$ is strictly decreasing on $(c,\underline{y}_0)$ (with values between ∞ and $\mu_0 := \bar{\mu}(\underline{y}_0)$), constant on $[\underline{y}_0,\overline{y}_0]$ (with value μ_0), and strictly increasing on $(\overline{y}_0,\infty)$ (with values between μ_0 and ∞). (Of course, we will usually have $\underline{y}_0 = \overline{y}_0$, except in degenerate cases where the graphs of f and g contain parallel line segments.)

We conclude that Σ^* consists of the graphs of two functions, namely, $\underline{u} := (\bar{\mu}|_{(c,\underline{y}_0)})^{-1}$ and $\overline{u} := (\bar{\mu}|_{(\overline{y}_0,\infty)})^{-1}$, connected by the vertical segment $\{\mu_0\} \times [\underline{y}_0,\overline{y}_0]$ (a turning point if $\underline{y}_0 = \overline{y}_0$). Both $\underline{u}$ and $\overline{u}$ are defined and twice continuously differentiable on (μ_0,∞); the former is strictly decreasing with range $(c,\underline{y}_0)$, the latter is strictly increasing with range $(\overline{y}_0,\infty)$. The figure at the end of the paper shows a typical example (see the discussion following Lemma 2.1 for details).

To investigate the possibility of bifurcations of nontrivial solutions from the trivial solution branch Σ^*, we compute the eigenvalues of the linearization of Problem (P), with respect to u, at a point $(\mu, y) \in \Sigma^*$, that is, the eigenvalues of

$$-\frac{d}{dx}\Big(k(x)\frac{dv}{dx}\Big) = \big(\mu f'(y) - g'(y)\big)v + \lambda v \quad \text{in } (-1,1), \qquad v'(\pm 1) = 0. \tag{2.1}$$

If we enumerate the eigenvalues of $-\frac{d}{dx}\big(k\frac{d}{dx}\big)$ (under Neumann boundary conditions) as a strictly increasing sequence $0 = \ell_0 < \ell_1 < \ell_2 < \dots$, then the eigenvalues of (2.1) are simply given by $\lambda_j(\mu, y) = \ell_j + g'(y) - \mu f'(y)$, for $j \in \mathbb{Z}_+$. In particular, $\lambda_0(\mu, y) = g'(y) - \mu f'(y)$ and

$$\lambda_j(\mu, y) = \lambda_0(\mu, y) + \ell_j\,, \tag{2.2}$$

for $j \in \mathbb{N}$. Also, since $(\mu, y) \in \Sigma^*$, we have $\mu = \bar\mu(y) = \frac{g(y)}{f(y)}$ and thus,

$$\lambda_0(\mu, y) = g'(y) - \bar\mu(y) f'(y) = \bar\mu'(y) f(y)\,. \tag{2.3}$$

This implies that λ_0 is *positive* on the graph of $\overline{u}$ (the upper branch of Σ^*, where $\bar\mu' > 0$), *zero* on the vertical segment $\{\mu_0\} \times [\underline{y}_0, \overline{y}_0]$ (the possibly degenerate turning point of Σ^*, where $\bar\mu' = 0$), and *negative* on the graph of $\underline{u}$ (the lower branch of Σ^*, where $\bar\mu' < 0$). Thus, the solutions on the upper branch are *stable*, while those on the lower branch are *unstable*, and bifurcations can occur only on the lower branch, at points where one of the eigenvalues λ_j with $j \geq 1$ vanishes (that is, at points $(\mu, y) \in \Sigma^*$ where $\lambda_0(\mu, y) = -\ell_j$ for some $j \geq 1$).

To find out whether such points exist, we must trace the smallest eigenvalue, λ_0, along the lower branch of Σ^*. To that end, define $\bar\lambda : (c, \underline{y}_0) \to \mathbb{R}$ by $\bar\lambda(y) := \lambda_0(\bar\mu(y), y)$. By (2.3), $\bar\lambda = g' - \bar\mu f' = \bar\mu' f$, and we know that this is strictly negative on $(c, \underline{y}_0)$, with $\bar\lambda(\underline{y}_0 -) = 0$. Moreover, $\bar\lambda' = g'' - \bar\mu' f' - \bar\mu f'' \geq -\bar\mu' f'$, since $g'' \geq 0$, $f'' \leq 0$, and $\bar\mu > 0$. Also, $\bar\mu' < 0$ on $(c, \underline{y}_0)$, that is, $\sigma = g'f - gf' < 0$, and thus, $f' > \frac{g'f}{g} > 0$ (our assumptions on g imply $g' > 0$ on $(0, \infty)$). This proves that $\bar\lambda' > 0$ on $(c, \underline{y}_0)$. Finally, we observe that $\bar\lambda(c+) = -\infty$, since $\bar\lambda = g' - \bar\mu f' = g' - g\frac{f'}{f}$ and $\frac{f'(y)}{f(y)} \to \infty$ as $y \to c+$ (note that $g'(c) > 0$, $g(c) > 0$, $f'(c+) \in (0, \infty]$, and $f(c+) = 0+$).

Summarizing, we showed that $\bar\lambda$ is strictly increasing on $(c, \underline{y}_0)$, with range $(-\infty, 0)$ and with a strictly positive derivative. But this means that the function $\mu \mapsto \lambda_0(\mu, \underline{u}(\mu))$ is strictly decreasing on (μ_0, ∞), with range $(-\infty, 0)$ and with a strictly negative derivative. Because of (2.2), it follows that in fact *all* the eigenvalues λ_j are strictly decreasing and eventually negative along the lower branch of Σ^*, with $\frac{d}{d\mu}\lambda_j(\mu, \underline{u}(\mu)) < 0$ for all $\mu \in (\mu_0, \infty)$. In particular, each of the eigenvalues λ_j with $j \geq 1$ has a unique nondegenerate zero on the lower branch of Σ^*.

The following lemma gathers our findings about the trivial solutions of Problem (P).

Lemma 2.1. (a) There are exactly two maximal continua of trivial solutions of Problem (P), namely $\mathbb{R}_+ \times \{0\}$ and the trace Σ^* of the curve $\mu = \frac{g(y)}{f(y)}$ with $y \in (c, \infty)$. The set Σ^* consists of the graphs of two functions $\underline{u}$, $\overline{u} \in C^2((\mu_0, \infty))$ and a vertical segment $\{\mu_0\} \times [\underline{y}_0, \overline{y}_0]$, where $\mu_0 > 0$ and $c < \underline{y}_0 \leq \overline{y}_0 < \infty$. The function $\underline{u}$ is strictly decreasing with range $(c, \underline{y}_0)$, while the function $\overline{u}$ is strictly increasing with range $(\overline{y}_0, \infty)$.

(b) Denoting by $\big(\lambda_j(\mu,y)\big)_{j\in\mathbb{Z}_+}$ the strictly increasing enumeration of the eigenvalues of the linearization of Problem (P) at $(\mu,y)\in\Sigma^*$, we have $\lambda_0>0$ on the graph of $\overline{u}$, $\lambda_0=0$ on the vertical segment $\{\mu_0\}\times[\underline{y}_0,\overline{y}_0]$, and $\lambda_0<0$ on the graph of $\underline{u}$. All the eigenvalues λ_j are strictly decreasing and eventually negative along the graph of $\underline{u}$, with $\frac{d}{d\mu}\lambda_j(\mu,\underline{u}(\mu))<0$ for all $\mu\in(\mu_0,\infty)$. In particular, each of the eigenvalues λ_j with $j\geq 1$ has a unique zero (μ_j,u_j) on Σ^*, with $\mu_0<\mu_1<\mu_2<\dots$ and $u_j=\underline{u}(\mu_j)$ for $j\in\mathbb{N}$.

The figure at the end of the paper depicts the trivial solution branch Σ^* in a typical situation, where $f(y):=\sqrt{y-1}$, for $y>c:=1$, and $g(y)=y^4$, for $y>0$. Open circles mark the location of the first few of the potential bifurcation points (μ_j,u_j), $j=1,2,3\dots$; those were found by solving the equation $\bar{\lambda}(u_j)=-\ell_j$, under the assumption that $k\equiv 1$ (so that $\ell_j=\frac{\pi^2}{4}j^2$, for $j\in\mathbb{N}$).

Next we collect some a-priori information about possible nontrivial solutions of Problem (P).

Lemma 2.2. Let (μ,u) be a nontrivial solution of Problem (P). Then we have $\mu>\mu_0$ and $0<u(x)<\overline{u}(\mu)$ for all $x\in[-1,1]$. More precisely, every local minimum of u lies between 0 and $\underline{u}(\mu)$, while every local maximum of u lies between $\underline{u}(\mu)$ and $\overline{u}(\mu)$. The function $u-\underline{u}(\mu)$ has a positive finite number of zeros, all of which are simple and occur in the open interval $(-1,1)$.

Proof. Let $(\mu,u)\in\Sigma$ with $u\neq$ const. Suppose that u attains a local maximum at $x_0\in[-1,1]$. Then $u'(x_0)=0$ and $u''(x_0)\leq 0$ (even if x_0 is a boundary point). In fact, we must have $u''(x_0)<0$, for otherwise $-\frac{d}{dx}\big(k(x)\frac{du}{dx}\big)\big|_{x=x_0}=0$ and thus, $\mu f(u(x_0))-g(u(x_0))=0$, which would mean that $u(x_0)$ is a constant solution of Problem (P). But then, the uniqueness theorem for second-order ordinary differential equations would imply that $u(x)=u(x_0)$ for all $x\in[-1,1]$. (The fact that f may not be Lipschitz continuous near c does not interfere with this argument, since necessarily $u(x_0)\neq c$.) So we have indeed $u''(x_0)<0$, hence $-\frac{d}{dx}\big(k(x)\frac{du}{dx}\big)\big|_{x=x_0}>0$, and thus, $\mu f(u(x_0))-g(u(x_0))>0$. But this implies $u(x_0)>c$ and $\mu>\bar{\mu}(u(x_0))=\frac{g(u(x_0))}{f(u(x_0))}$, or equivalently, $\mu>\mu_0$ and $\underline{u}(\mu)<u(x_0)<\overline{u}(\mu)$.

Similarly, one shows that if u attains a local minimum at $x_1\in[-1,1]$, then $0<u(x_1)<\underline{u}(\mu)$. This proves the first two assertions of the lemma and the fact that $u-\underline{u}(\mu)$ has at least one zero. Another invocation of the uniqueness theorem yields that all zeros of $u-\underline{u}(\mu)$ are simple and located in the open interval $(-1,1)$.

3. BIFURCATION ANALYSIS OF PROBLEM (P)

To make the abstract theory of Section 1 applicable, we need to reformulate Problem (P) as an operator equation in a suitable Banach or Hilbert space setting. To that end, let L denote the selfadjoint linear operator in $Y:=L^2((-1,1))$ induced by the differential expression $-\frac{d}{dx}\big(k\frac{d}{dx}\big)$ under Neumann boundary conditions. Having a compact resolvent, L is a Fredholm operator of index 0 in Y. Let X denote the domain of L, endowed with the graph norm. Then X is a Banach space that embeds compactly into $C([-1,1])$.

The functions f and g induce substitution operators $u\mapsto f\circ u$ and $u\mapsto g\circ u$ in $C([-1,1])$, defined and continuous on the cone of nonnegative functions and twice continuously differentiable on the open set $\{u\in C([-1,1])\mid u(x)>c\;\forall x\in[-1,1]\}$.

Let X_+ denote the cone of nonnegative functions in X. Since X embeds compactly into $C([-1,1])$ and $C([-1,1])$ embeds continuously into Y, the nonlinear operator $N : \mathbb{R}_+ \times X_+ \to Y$, defined by

$$N(\mu, u) := \mu f \circ u - g \circ u,$$

is completely continuous (with respect to the norms of $\mathbb{R} \times X$ and Y) and twice continuously differentiable on $\Omega := (0, \infty) \times \{u \in X \mid u(x) > c \ \forall x \in [-1,1]\}$ (an open subset of $\mathbb{R} \times X$).

With these definitions, Problem (P) is equivalent to the equation $Lu = N(\mu, u)$, which we can also write as

$$H(\mu, u) = 0, \tag{$\tilde{\mathrm{P}}$}$$

where $H : \mathbb{R}_+ \times X_+ \to Y$ is defined by

$$H(\mu, u) := Lu - N(\mu, u).$$

More precisely, the solution set Σ of Problem (P) coincides with that of ($\tilde{\mathrm{P}}$):

$$\Sigma = \{(\mu, u) \in \mathbb{R}_+ \times X_+ \mid H(\mu, u) = 0\}.$$

We consider Σ a metric subspace of $\mathbb{R} \times X$; as such it is closed and locally compact. Moreover, routine arguments show that the topology of Σ as a metric subspace of $\mathbb{R} \times X$ coincides with the metric topologies it inherits from $\mathbb{R} \times C^m([-1,1])$ with $m = 0$, 1, or 2.

Finally, we observe that the eigenvalue problem (2.1), obtained by linearizing Problem (P) at a point $(\mu, y) \in \Sigma^*$, is equivalent to the abstract eigenvalue problem for the operator

$$H_u(\mu, y) = L - N_u(\mu, y) = L + \lambda_0(\mu, y) Id\,,$$

which is selfadjoint as an unbounded operator in the Hilbert space Y.

The following lemma provides the basis for applying the abstract results of Section 1.

Lemma 3.1. For every $(\mu, y) \in \Sigma^*$, the Fréchet derivative $H'(\mu, y)$ is a Fredholm operator of index 1. The only singular points of H on Σ^* are the zeros (μ_j, u_j) of the eigenvalues λ_j with $j \geq 1$, as given in Lemma 2.1(b). All of those points are in fact regular-singular points of H.

Proof. The Fréchet derivative of H at $(\mu, y) \in \Sigma^*$ is given by

$$H'(\mu, y)(\nu, v) = \nu H_\mu(\mu, y) + H_u(\mu, y) v\,,$$

for $(\nu, v) \in \mathbb{R} \times X$. Moreover, $H_\mu(\mu, y)$ is the constant function $-f(y)$, and $H_u(\mu, y)$ is the operator $L + \lambda_0(\mu, y) Id$, a Fredholm operator of index 0. It follows that $H'(\mu, y)$ is a Fredholm operator of index 1. Also, since the eigenvalues of $H_u(\mu, y)$ are simple, the nullspace of $H'(\mu, y)$ is at most two-dimensional. It is indeed two-dimensional if and only if 0 is an eigenvalue of $H_u(\mu, y)$ and $H_\mu(\mu, y)$ belongs to the range of $H_u(\mu, y)$. Consequently, (μ, y) is a singular point of H (in the sense of

Section 1) if and only if one of the eigenvalues $\lambda_j(\mu, y)$, with $j \in \mathbb{Z}_+$, vanishes and the nonzero constant $f(y)$ belongs to the range of $L+\lambda_0(\mu, y)Id = L+\big(\lambda_j(\mu, y)-\ell_j\big)Id = L-\ell_j Id$. But $R(L-\ell_j Id)$ is the orthogonal complement (in Y) of the j-th normalized eigenfunction ϕ_j of L and contains nonzero constants if and only if $j \geq 1$. It follows that the singular points of H on Σ^* coincide with the zeros (μ_j, u_j) of the eigenvalues λ_j with $j \geq 1$.

We claim that all those points are in fact regular-singular points of H. To verify this, let $j \in \mathbb{N}$ and let ϕ_j, as before, denote the j-th normalized eigenfunction of L. We know already that $N(H'(\mu_j, u_j))$ is two-dimensional, and it clearly contains the vector $(0, \phi_j)$. To find a second, linearly independent member of $N(H'(\mu_j, u_j))$, we differentiate the equation $H(\mu, \underline{u}(\mu)) = 0$, valid for all $\mu > \mu_0$, with respect to μ and obtain

$$H'(\mu, \underline{u}(\mu))(1, \underline{u}'(\mu)) = 0\,, \tag{3.1}$$

for all $\mu > \mu_0$. Setting $\mu = \mu_j$, we see that $(1, \underline{u}'(\mu_j)) \in N(H'(\mu_j, u_j))$. Differentiating (3.1) once again, we obtain

$$H''(\mu, \underline{u}(\mu))(1, \underline{u}'(\mu))(1, \underline{u}'(\mu)) + H'(\mu, \underline{u}(\mu))(0, \underline{u}''(\mu)) = 0\,,$$

for $\mu > \mu_0$. With $\mu = \mu_j$, it follows that $H''(\mu_j, u_j)(1, \underline{u}'(\mu_j))(1, \underline{u}'(\mu_j))$ belongs to $R(H'(\mu_j, u_j))$. Now it just remains to be shown that the mixed derivative $H''(\mu_j, u_j)(1, \underline{u}'(\mu_j))(0, \phi_j)$ does *not* belong to $R(H'(\mu_j, u_j))$. To that end, observe that for all $\mu > \mu_0$, we have $H'(\mu, \underline{u}(\mu))(0, \phi_j) = H_u(\mu, \underline{u}(\mu))\phi_j = \lambda_j(\mu, \underline{u}(\mu))\phi_j$. Differentiating this with respect to μ, we get

$$H''(\mu, \underline{u}(\mu))(1, \underline{u}'(\mu))(0, \phi_j) = \frac{d}{d\mu}\lambda_j(\mu, \underline{u}(\mu))\phi_j\,,$$

for $\mu > \mu_0$, and thus

$$H''(\mu_j, u_j)(1, \underline{u}'(\mu_j))(0, \phi_j) = \Big(\frac{d}{d\mu}\lambda_j(\mu, \underline{u}(\mu))\Big|_{\mu=\mu_j}\Big)\phi_j\,.$$

Since $\frac{d}{d\mu}\lambda_j(\mu, \underline{u}(\mu))\big|_{\mu=\mu_j} \neq 0$ (see Lemma 2.1(b)) and ϕ_j is not in $R(H'(\mu_j, u_j)) = R(H_u(\mu_j, u_j)) = R(L - \ell_j Id)$, neither is $H''(\mu_j, u_j)(1, \underline{u}'(\mu_j))(0, \phi_j)$. This finishes the proof.

The following is an immediate consequence of Lemma 3.1 and the fact that regular-singular points of H are Crandall-Rabinowitz points of H (see Section 1).

Corollary 3.2. The only bifurcation points on the trivial solution branch Σ^* of Problem (P) are the zeros (μ_j, u_j) of the eigenvalues λ_j with $j \geq 1$, as given in Lemma 2.1(b). Locally, near any of the points (μ_j, u_j), the solution set Σ of Problem (P) is the union of two simple C^1-arcs that intersect transversally at (μ_j, u_j).

Following Rabinowitz [4], we can show that the bifurcations occurring at the points (μ_j, u_j) are in fact *global* rather than *local* phenomena. Since this is not our primary concern here, we sketch the argument only briefly.

For $j \in \mathbb{N}$, let C_j denote the connected component of (μ_j, u_j) in $\overline{\Sigma \setminus \Sigma^*}$, which we consider a metric subspace of $\mathbb{R} \times X$ (or equivalently, of $\mathbb{R} \times C([-1, 1])$ or of $\mathbb{R} \times C^1([-1, 1])$). The continuum C_j obeys the "Rabinowitz alternative": Either C_j

is a compact subset of the interior of $\mathbb{R}_+ \times X_+$ (the domain of H) and contains a point of Σ^* different from (μ_j, u_j). Or C_j is unbounded or intersects the boundary of $\mathbb{R}_+ \times X_+$. (See [7, Theorem 3.5] for a version of the Rabinowitz bifurcation theorem that is directly applicable in the situation at hand.) In view of our a-priori bounds for nontrivial solutions of Problem (P) (see Lemma 2.2), C_j cannot reach the boundary of $\mathbb{R}_+ \times X_+$. Moreover, C_j can intersect Σ^* only at a zero of the eigenvalue λ_j, that is, only at the point (μ_j, u_j). The latter is a consequence of the nodal properties of the nontrivial solutions of Problem (P) (see Lemma 2.2 again) and of analogous nodal properties of the eigenfunctions of L. (We refer to [7, Chapter 3] or [1, 3, 4, 5, 6, 8] for details of the argument, which is by now routine.) It follows that C_j is unbounded.

Moreover, $C_j \setminus \{(\mu_j, u_j)\}$ is entirely contained in the set Σ_j consisting of all nontrivial solutions (μ, u) of Problem (P) for which $u - \underline{u}(\mu)$ has exactly j zeros (necessarily simple and located in the open interval $(-1, 1)$). We can decompose Σ_j into subsets Σ_j^+ and Σ_j^- consisting of those pairs $(\mu, u) \in \Sigma_j$ for which $u - \underline{u}(\mu)$ is positive or negative, respectively, at $x = 1$. The bifurcation point (μ_j, u_j) belongs to $\overline{\Sigma_j^+} \cap \overline{\Sigma_j^-}$, and the Rabinowitz alternative holds, separately, for the connected components $C_j^\pm$ of (μ_j, u_j) in $\overline{\Sigma_j^\pm}$ (see the references quoted above for similar reasoning). It follows that both C_j^+ and C_j^- are unbounded with $C_j^+ \cap C_j^- = \{(\mu_j, u_j)\}$ and $C_j^+ \cup C_j^- = C_j$.

Finally, we note that thanks to the a-priori bounds of Lemma 2.2, every unbounded set of nontrivial solutions of Problem (P) must contain solutions (μ, u) with arbitrarily large μ. Summarizing all our results, we have the following theorem.

Theorem 3.3. There is a sequence $((\mu_j, u_j))_{j \in \mathbb{N}}$, with $(\mu_j)_{j \in \mathbb{N}}$ strictly increasing, of Crandall-Rabinowitz type bifurcation points on the lower part of the trivial solution branch Σ^* of Problem (P), and these are the only points where nontrivial solutions bifurcate from the trivial solution set $\Sigma^* \cup \big(\mathbb{R}_+ \times \{0\}\big)$. At each of the points (μ_j, u_j), an unbounded continuum C_j of solutions bifurcates from Σ^*, which does not contain any trivial solution other than (μ_j, u_j). This continuum C_j is the union of two subcontinua C_j^+ and C_j^- with the following properties: (a) Both C_j^+ and C_j^- contain solutions (μ, u) with arbitrarily large μ. (b) If $(\mu, u) \in C_j^+$ (C_j^-) and $(\mu, u) \neq (\mu_j, u_j)$, then $\mu > \mu_0$ and the function $u - \underline{u}(\mu)$ has exactly j zeros and is positive (negative) at $x = 1$.

4. SPATIALLY INHOMOGENEOUS PERTURBATIONS OF PROBLEM (P)

We will now embed Problem (P) into a family (P_ϵ) of problems with spatially inhomogeneous source terms. Due to the spatial inhomogeneity of the right-hand side, Problem (P_ϵ) will not have trivial solutions (μ, u) with $u \neq 0$. Still, the abstract results of Section 1 will allow us to prove the existence of solutions of Problem (P_ϵ) close to the trivial solution branch Σ^* of Problem (P)=(P_0) and to describe the local structure of this set of solutions.

Let J be an open interval containing 0 and let q denote a C^2-function on $\bar{J} \times [-1, 1]$ with $q(0, x) = 1$ for all $x \in [-1, 1]$. For $\epsilon \in \bar{J}$, consider the boundary-value problem

$$-\frac{d}{dx}\Big(k(x)\frac{du}{dx}\Big) = \mu q(\epsilon, x) f(u) - g(u) \quad \text{in } (-1, 1), \qquad u'(\pm 1) = 0, \qquad (\mathrm{P}_\epsilon)$$

under the same assumptions on k, f, and g as in Section 2. Clearly, (P_0) coincides with our earlier problem, (P). Using the same functional-analytic setting as in Section 3, we can write Problem (P_ϵ) as an operator equation of the form $Lu = N(\epsilon, \mu, u)$, with $N : \bar{J} \times \mathbb{R}_+ \times X_+ \to Y$ defined by

$$N(\epsilon, \mu, u) := \mu q(\epsilon, \cdot) f \circ u - g \circ u,$$

or equivalently, as

$$H(\epsilon, \mu, u) = 0, \qquad (\tilde{P}_\epsilon)$$

with $H : \bar{J} \times \mathbb{R}_+ \times X_+ \to Y$ defined by

$$H(\epsilon, \mu, u) := Lu - N(\epsilon, \mu, u).$$

The mapping N is completely continuous (with respect to the norms of $\mathbb{R} \times \mathbb{R} \times X$ and Y) and twice continuously differentiable on the open set $J \times \Omega$ (where, as before, $\Omega = (0, \infty) \times \{u \in X \mid u(x) > c \; \forall x \in [-1, 1]\}$). In conjunction with the well-known properties of L, this implies that for every $(\epsilon, \mu, u) \in \Omega$, the partial Fréchet derivative $H_{(\mu,u)}(\epsilon, \mu, u)$ is a Fredholm operator of index 1.

For $\epsilon \in \bar{J}$, let Σ_ϵ denote the solution set of Problem (P_ϵ), or equivalently, of Problem ($\tilde{P}_\epsilon$):

$$\Sigma_\epsilon = \{(\mu, u) \in \mathbb{R}_+ \times X_+ \mid H(\epsilon, \mu, u) = 0\}.$$

For any $\epsilon \in \bar{J}$, the set Σ_ϵ contains the trivial solutions $(\mu, 0)$ with $\mu \in \mathbb{R}_+$, but no further trivial solutions unless $q(\epsilon, \cdot)$ is constant (as in the case $\epsilon = 0$). Of course, Σ_0 coincides with the solution set Σ of Problem (P), which we analyzed in Sections 2 and 3. We are interested in how the spatially inhomogeneous perturbations affect the trivial solution branch Σ^* of Problem (P).

Specifically, what is the local structure of Σ_ϵ, for ϵ close to 0, near a point (μ^*, u^*) on Σ^*? Applying the results of Section 1, we first note that if (μ^*, u^*) is a regular point of $H(0, \cdot, \cdot)$, then Σ_ϵ is locally, near (μ^*, u^*), just a continuous deformation of the curve Σ^* and in particular, a simple C^1-arc. If (μ^*, u^*) is one of the bifurcation points (μ_j, u_j) (and thus, a regular-singular point of $H(0, \cdot, \cdot)$), then Σ_ϵ is locally, near (μ^*, u^*), a homeomorphic image of the set $\{(\sigma, \tau) \in \mathbb{R}^2 \mid \sigma\tau + \rho(\epsilon) = 0\}$, where ρ is a C^1-function with $\rho(0) = 0$ (see Theorem 1.1 for a much more precise statement). In particular, Σ_ϵ contains a bifurcation point near (μ^*, u^*) if and only if $\rho(\epsilon) = 0$.

At the end of Section 1, we derived a simple criterion for *nonpersistence* of the bifurcations occurring at regular-singular points. Applying this criterion in the present situation, we infer the following: If $H_\epsilon(0, \mu^*, u^*)$ does not belong to the range of $H_{(\mu,u)}(0, \mu^*, u^*)$, then $\rho(\epsilon) \neq 0$ for all $\epsilon \neq 0$ close to 0, and then, none of the corresponding sets Σ_ϵ contains a bifurcation point near (μ^*, u^*). Now, if $(\mu^*, u^*) = (\mu_j, u_j)$ for some $j \in \mathbb{N}$, then $H_\epsilon(0, \mu^*, u^*) = -\mu_j f(u_j) \frac{\partial q}{\partial \epsilon}(0, \cdot)$, which is a nonzero constant multiple of $\frac{\partial q}{\partial \epsilon}(0, \cdot)$. Moreover, $R(H_{(\mu,u)}(0, \mu^*, u^*)) = R(H_u(0, \mu^*, u^*))$ and $H_u(0, \mu^*, u^*) = L - \ell_j Id$ (see the proof of Lemma 3.1), so that $R(H_{(\mu,u)}(0, \mu^*, u^*))$ is nothing but the orthogonal complement, in $Y = L^2((-1, 1))$, of the j-th normalized eigenfunction ϕ_j of L. We conclude that the bifurcation at (μ^*, u^*) is *nonpersistent* provided that

$$\int_{-1}^{1} \frac{\partial q}{\partial \epsilon}(0, x) \phi_j(x) \, dx \neq 0 \, .$$

In the absence of special symmetries, this condition is generic. But let us now assume that the diffusion coefficient k as well as the functions $q(\epsilon,\cdot)$ are *even*:

$$k(x) = k(-x) \quad \text{and} \quad q(\epsilon,x) = q(\epsilon,-x)$$

for all $x \in [-1,1]$ and $\epsilon \in \bar{J}$. For every function $u : [-1,1] \to \mathbb{R}$, let $\bar{u}$ denote its reflection at $x = 0$, that is, $\bar{u}(x) := u(-x)$ for $x \in [-1,1]$. A simple calculation shows that $L\bar{u} = \overline{Lu}$ for every $u \in X$. As a consequence, the eigenfunctions of L are either even or odd; in fact $\overline{\phi}_j = (-1)^j \phi_j$, for $j \in \mathbb{Z}_+$. It follows that if $(\mu,u) \in \mathbb{R}_+ \times X_+$ satisfies $H(\epsilon,\mu,u) = r\phi_j$, for some $\epsilon \in \bar{J}$ and $r \in \mathbb{R}$, then $(\mu,\bar{u})$ satisfies $H(\epsilon,\mu,\bar{u}) = (-1)^j r\phi_j$.

Again, let (μ^*,u^*) be one of the bifurcation points of Problem (P) on Σ^*, that is, $(\mu^*,u^*) = (\mu_j,u_j)$ for some $j \in \mathbb{N}$. Applying Theorem 1.2, with the eigenfunction ϕ_j playing the role of the vector $y_0 \in Y \setminus R(H_{(\mu,u)}(0,\mu^*,u^*))$, we obtain a neighborhood $U \subset \Omega$ of (μ^*,u^*) and an interval $I \subset J$ containing 0 such that for every $\epsilon \in I$, the mapping $H(\epsilon,\cdot,\cdot)$ has a unique singular point $(\mu^*_\epsilon,u^*_\epsilon) \in U$ whose image $H(\epsilon,\mu^*_\epsilon,u^*_\epsilon)$ is a constant multiple of ϕ_j. The point $(\mu^*_\epsilon,u^*_\epsilon)$ is a regular-singular point of $H(\epsilon,\cdot,\cdot)$, and there is a function $\rho \in C^1(I)$ such that $H(\epsilon,\mu^*_\epsilon,u^*_\epsilon) = \rho(\epsilon)\phi_j$.

Being a regular-singular point of $H(\epsilon,\cdot,\cdot)$, the point $(\mu^*_\epsilon,u^*_\epsilon)$ is a bifurcation point for the equation

$$H(\epsilon,\mu,u) = \rho(\epsilon)\phi_j \,,$$

and with our earlier observation regarding the symmetries of H and ϕ_j, it follows that $(\mu^*_\epsilon,\overline{u}^*_\epsilon)$ is a bifurcation point for

$$H(\epsilon,\mu,u) = (-1)^j \rho(\epsilon)\phi_j \,.$$

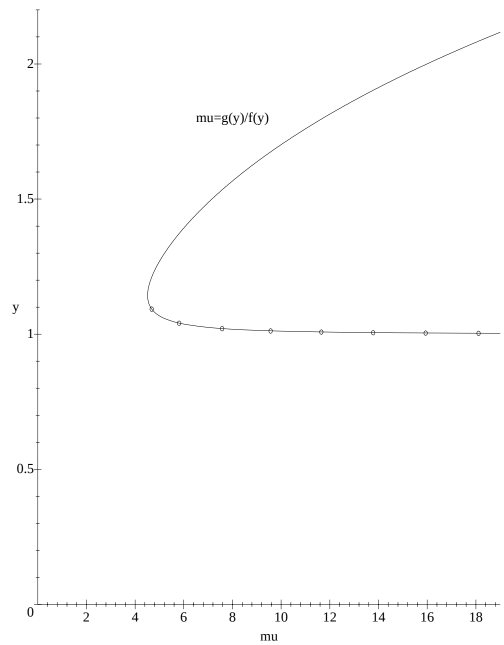
Also, since $(\mu^*_\epsilon,u^*_\epsilon)$ is close to (μ^*,u^*), so is $(\mu^*_\epsilon,\overline{u}^*_\epsilon)$ (recall that u^* is a constant). That is, the neighborhood U of (μ^*,u^*) provided by Theorem 1.2 can be chosen so that both points, $(\mu^*_\epsilon,u^*_\epsilon)$ and $(\mu^*_\epsilon,\overline{u}^*_\epsilon)$, belong to U. But U contains only *one* singular point of $H(\epsilon,\cdot,\cdot)$ whose image is a constant multiple of ϕ_j. We conclude that $\overline{u}^*_\epsilon = u^*_\epsilon$ and $(-1)^j\rho(\epsilon) = \rho(\epsilon)$. If j is *odd*, we must have $\rho(\epsilon) = 0$, and then $(\mu^*_\epsilon,u^*_\epsilon)$ is a bifurcation point for Problem (P_ϵ). In other words, at least every other one of the bifurcations occurring on the trivial solution branch of Problem (P) is persistent!

The following theorem summarizes our results.

Theorem 4.1. Let $((\mu_j,u_j))_{j\in\mathbb{N}}$ denote the sequence of bifurcation points on the trivial solution branch Σ^* of Problem (P), as described in Theorem 3.3, and let $(\phi_j)_{j\in\mathbb{Z}_+}$ be the sequence of normalized eigenfunctions of the operator L.

(a) For every $j \in \mathbb{N}$, if $\int_{-1}^1 \frac{\partial q}{\partial\epsilon}(0,x)\phi_j(x)\,dx \neq 0$, then the bifurcation at (μ_j,u_j) is *nonpersistent*: None of the problems (P_ϵ) with ϵ close to but different from 0 has a bifurcation point near (μ_j,u_j). In fact, the solution set of (P_ϵ), for any such ϵ, is locally, near (μ_j,u_j), the union of two disjoint, simple C^1-arcs.

(b) If the coefficient k and the functions $q(\epsilon,\cdot)$, for $\epsilon \in \bar{J}$, are *even*, then the bifurcations occurring at *odd-numbered* points (μ_j,u_j) are *persistent*: Each of the problems (P_ϵ) with ϵ close to 0 has a unique bifurcation point $(\mu_{j,\epsilon},u_{j,\epsilon})$ near (μ_j,u_j). In fact, the solution set of (P_ϵ), for any such ϵ, is locally, near (μ_j,u_j), the union of two simple C^1-arcs that intersect transversally at $(\mu_{j,\epsilon},u_{j,\epsilon})$.



Example of a trivial solution branch Σ^* with bifurcation points (μ_j, u_j).

REFERENCES

1. M. G. CRANDALL & P. H. RABINOWITZ, Nonlinear Sturm-Liouville eigenvalue problems and topological degree, *J. Math. Mech.* 19 (1970), 1083–1102.
2. M. G. CRANDALL & P. H. RABINOWITZ, Bifurcation from simple eigenvalues, *J. Funct. Anal.* 8 (1971), 321–340.
3. P. H. RABINOWITZ, Nonlinear Sturm-Liouville problems for second order ordinary differential equations, *Comm. Pure Appl. Math.* 23 (1970), 939–962.
4. P. H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* 7 (1971), 487–513.
5. R. SCHAAF, Global behaviour of solution branches for some Neumann problems depending on one or several parameters, *J. Reine Angew. Math.* 346 (1984), 1–31.
6. R. SCHAAF, Global solution branches of two-point boundary value problems, Lecture Notes in Mathematics, Vol. 1458, Springer-Verlag, Berlin, 1990.
7. B. E. SCHMIDT, Bifurcation of stationary solutions for Legendre-type boundary value problems arising from energy balance climate models, Ph.D. thesis, Auburn University, 1994.
8. B. E. SCHMIDT, On a nonlinear eigenvalue problem arising from climate modeling, *Nonlin. Anal.* 30 (1997), 3645–3656.

Bettina E. Schmidt
Department of Mathematics
Auburn University at Montgomery
Montgomery, AL 36124-4023, USA
Email address: bes@strudel.aum.edu