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Sorting and cost analysis of reworking items in rejected lots based on non-destructive variable sampling plan *

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Abstract

A mathematical model for a decision criterion for disposing an inspection lot is developed. An expression of the posterior cost is formulated in terms of the quality characteristics X of the items manufactured, the sample size n, the lot size N, the upper and lower limits of X (U, L) of the individual items, the sample mean $\overline{x}$, the mean μ and variance σ^2 of X, also in terms of the economical cost parameters, Optimizing the posterior cost equation leads to the estimation of the decision points. A procedure to accept, reject, screen or scrap the entire lot based on the values of the decision points is developed. Mathematical expressions are derived for the expected cost of lot acceptance, screening and scrapping. In developing the model, the distribution of X and μ are normal. The tested items can be used for their intended purposes after testing. The defective items can be repaired or reworked. Rejected lots are either screened or scrapped. The decision to accept or reject a lot depends on the upper and lower limits of the sample mean, which constitutes the decision points.

1 Introduction

In this work, different sample sizes are selected to compute the cost for each tested sample. By comparing the costs, it is possible to discover the fluctuations, if any, in the model selected due to computational errors. It is logical to assume that the sample mean can take four different values depending on its upper and lower limits. These limits are relative to the acceptance and rejection values of X. The cost in this case is a function of the number of defective units in the accepted lot, the cost of replacing these items, and the cost of inspecting the lot. The lot is screened to isolate the defective units. The cost relative to this case consists of the cost of inspecting each item in the uninspected portion of the lot and the cost of replacing the defective items produced by the manufacturing

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facility. The cost of scrapping consists of the cost of each unit scrapped. The cost of scrapping items produced by the production facility is reduced by the revenue of the salvaged material. After reaching a decision on the rejected items that can be reworked, the cost of reworking these items is derived. In this process of screening, the expected value of the fraction of items that can be reworked is evaluated and used as a standard for future production. The work is concentrated on finding a set of upper and lower limits of the quality characteristic X, namely L_A, U_A, L_{sn}, U_{sn}. A control chart is constructed based on these values to test the manufactured lots. Values of the sample mean above U_A and below U_{sn}, or below L_A and above L_{sn} are screened.

After screening the items produced, a decision can be made to scrap or rework the defective items found. The main advantage of this procedure are to: (1) reduce the cost due to penalty of producing defective items, (2) satisfy the needs of both the producers and consumers, who are seeking good products with a reasonable cost, (3) keep the quality of the items produced at a very high standard at any stage of production.

2 Mathematical development of the model

In estimating the expected posterior cost of rejecting and reworking defective items the following assumptions are made: (1) The probability that an individual measurement is above or below the upper and lower specification limit when both the lot and the sample are considered. (2) The costs of accepting and repairing items with dimensions above or below the specification limits for both the lot and the sample are considered. (3) The screening errors of types I and II are negligible. (4) The process can exist in one statistical state. The components of the cost are:

(A) Cost of Items Worked With or Without Success

The cost per lot resulting from defective items found during inspection and reworked with and without success. $K_{w1}(\overline{x}, \mu)$, is then:

$$\begin{aligned} K_{w1}(\overline{x}, \mu) &= K_{c1} n P_{3_s} + K_{c2} n P_{4_s} + [(K_R - K_J)(1 - K_Y)] P_{3_s} \\ &\quad + n[(K_R - K_J)(1 - K_Y)] P_{4_s} \ , \end{aligned} \tag{1}$$

where the symbols in this paper are defined in the Appendix. For the remainder of the lot the cost $K_{w2}(\overline{x}, \mu)$ is given as:

$$\begin{aligned} K_{w2}(\overline{x}, \mu) &= K_{c1}(N - n) P_{1_u} + K_{c2}(N - n) P_{2L} \\ &\quad + (N - n) P_{1_u} [(K_R - K_J)(1 - K_Y)] \\ &\quad + (N - n) P_{1_u} [(K_R - K_J)(1 - K_Y)] \ . \end{aligned} \tag{2}$$

Assuming $K_{c1} = K_{c2} = K_c$ and $P_{1_u} = P_{2L}$, expression (1) can be written as

$$K_{w1}(\overline{x}, \mu) = n[K_c + (K_R - K_J)(1 - K_Y)][P_{3_s} + P_{4_s}] \ . \tag{3}$$

Defining K_{R1} as

$$K_{R1} = [K_c + (K_R - K_J)(1 - K_Y)] \ , \tag{4}$$

and employing expression (4), then expressions (1) and (2) can be written respectively as:

$$K_{W1}(\overline{x}, \mu) \quad = \quad nK_{R1}\Big[\int_U^{\infty} t(x \mid \overline{x}, \mu) + \int_{-\infty}^{L} t(x \mid \overline{x}, \mu)\, dx\Big] \tag{5}$$

and

$$K_{W2}(\overline{x}, \mu) \quad = \quad (N-n)K_{R1}\Big[\int_U^{\infty} (f(x)\mu)\ dx\Big] + \Big[\int_{-\infty}^{L} f(x \mid \mu)\, dx\Big]X. \tag{6}$$

(B) Cost of Reworked Defective Items

The expected cost, $K_W(\overline{x}, \mu)$, of reworking defective items success can be obtained by adding expressions (5) and (6), thus

$$K_W(\overline{x}, \mu) = NK_{R1} - nK_{R1}Q_{1D}(\overline{x}, \mu) - (N-n)K_{R1}P_{1D}(\mu) \ . \tag{7}$$

where the two possibilities $P_{1D}(\mu)$ and $Q_{1D}(\overline{x}, \mu)$ are defined in the following form:

$$P_{1D}(\mu) = \textstyle\int_L^U f(x \mid \mu)\, dx \ , \tag{8}$$

and

$$Q_{1D}(\overline{x}, \mu) = \textstyle\int_L^U t(x \mid \overline{x}, \mu)\, dx \ . \tag{9}$$

The total expected cost can be written as:

$$\begin{aligned}
K_T \quad = \quad & \int_{-\infty}^{+\infty} \left[\int_{L_A}^{U_A} n(K_A - K_P)Q_{1D}(\overline{x}, \mu)T(\overline{x}_n \mid \mu)\, d\overline{x}\right] h(\mu)\, d\mu \\
& -K_{A^n} \int_{-\infty}^{+\infty} \left[\int_{U_L}^{U_A} P_{1D}(\mu)T(\overline{x}_n \mid \mu)\, d\overline{x}\right] h(\mu)\, d\mu \\
& + (K_A(N-n) + K_{pn}) \int_{-\infty}^{+\infty} \left[\int_{U_L}^{U_A} T(\overline{x}_n \mid \mu)\, d\overline{x}\right] h(\mu)\, d\mu \\
& + \int_{-\infty}^{+\infty} \left[\int_{L_A}^{L_{sn}} K_W(\overline{x}, \mu)T(\overline{x}_n \mid \mu)\, d\overline{x}\right] h(\mu)\, d\mu \\
& + \int_{-\infty}^{+\infty} \left[\int_{L_A}^{U_{sn}} K_W(\overline{x}, \mu)T(\overline{x}_n \mid \mu)\, d\overline{x}\right] h(\mu)\, d\mu + K_{I^n} \ .
\end{aligned} \tag{10}$$

The decision points L_A, U_A, L_{sn}, U_{sn}, relative to a lot acceptance and screening, respectively, are defined in the Appendix. For estimating the decision points, the total cost must be optimized relative to U_A. Taking the partial derivative of K_T relative to U_A yields:

$$\begin{aligned}
\frac{\partial K_T}{\partial U_A} &= n(K_{An} - K_P)\int_{-\infty}^{+\infty} Q_{1D}\left((U_A,\mu)T(U_A) \mid \mu\right) h(\mu)\, d\mu \\
&\quad -K_A N \int_{-\infty}^{+\infty} P_{1D}(\mu)T(U_A \mid \mu)h(\mu)\, d\mu \\
&\quad -NK_{R1}\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu)\, d\mu \\
&\quad +nK_{R1}\int_{-\infty}^{+\infty} Q_{1D}(U_A,\mu)T(U_A \mid \mu)h(\mu)\, d\mu \\
&\quad +(N-n)K_{R1}\int_{-\infty}^{+\infty} P_{1D}(\mu)T(U_A \mid \mu)h(\mu)\, d\mu\ . \qquad (11)
\end{aligned}$$

Arranging the terms in expression (11) yields:

$$\begin{aligned}
\frac{\partial K_T}{\partial U_A} &= n\left[(K_A - K_P) + K_{R1}\right]\int_{-\infty}^{+\infty} Q_{1D}(U_A,\mu)T(U_A,\mu)T(U_A \mid \mu)h(\mu\, d\mu) \\
&\quad + \left[(N-n)K_{R1} - K_A n\right]\int_{-\infty}^{+\infty} P_{1D}(\mu)T(U_A \mid \mu)h(\mu\, d\mu) \\
&\quad + \left[K_A(N-n) + K_{pn} - nK_{R1} - NK_{R1} + nK_{R1}\right] \\
&\quad \times \int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu)\, d\mu\ . \qquad (12)
\end{aligned}$$

Setting $\frac{\partial K_T}{\partial U_A} = 0$, and dividing each term of expression (12) by $\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu)\, d\mu$, the resulting expression is

$$\begin{aligned}
&\frac{\int_{-\infty}^{+\infty} P_{1D}(\mu)T(U_A \mid \mu)h(\mu\, d\mu)}{\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu\, d\mu)} \\
&+\frac{n\left[(K_A - K_P) + K_{R1}\right]}{(N-n)K_{R1} - K_A n}\,\frac{\int_{-\infty}^{+\infty} Q_{1D}(U_A,\mu)T(U_A \mid \mu)h(\mu)\, d\mu}{\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu)\, d\mu} \\
&= \frac{NK_{R1} - \left[K_A(N-n) + K_p n\right]}{(N-n)K_{R1} - K_A N} \qquad (13)
\end{aligned}$$

Define the following qualities $Q_1(U_A,n)$ *and* $Q_2(U_A,n)$ as

$$Q_1(U_A,n) = \frac{\int_{-\infty}^{+\infty} Q_{1D}(U_A,\mu)T(U_A \mid \mu)h(\mu)\, d\mu}{\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu)\, d\mu} \qquad (14)$$

and

$$Q_2(U_A,n) = \frac{\int_{-\infty}^{+\infty} P_{1D}(\mu)T(U_A \mid \mu)h(\mu\, d\mu)}{\int_{-\infty}^{+\infty} T(U_A \mid \mu)h(\mu\, d\mu)}\ . \qquad (15)$$

Also, expressions (14) and (15) can be written as

$$Q_1(U_A, n) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+\sigma_n^2}} \int_L^U e^{-\frac{1}{2}\frac{(x-m_n)^2}{\sigma^2+\sigma_n^2}}\, dx \tag{16}$$

where

$$\delta_n^2 = \frac{\sigma^2}{n} \quad , \quad m_n^2 = \frac{m\delta_n^2 + \sigma_\mu^2 U_A}{\delta_n^2 + \sigma_\mu^2} \quad , \quad \sigma_n^2 = \frac{\sigma_n^2 \cdot \delta_n^2}{\delta_n^2 + \sigma_n^2} \ , \tag{17}$$

and U_A can be obtained by employing expression (17) and can be written as

$$U_A = \frac{m_n^2(\delta_n^2 + \sigma_n^2) - m\delta_n^2}{\sigma_\mu^2} \tag{18}$$

and in the same way

$$Q_2(U_A, n) = \frac{\sigma_u \cdot \delta_n}{\sqrt{2\pi}\sqrt{\sigma_\mu^2 + \delta_n^2}} \int_L^U e^{-\frac{1}{2}\frac{(x-U_A)^2}{\sigma^2\frac{(n-1)}{n}}}\, dx \ . \tag{19}$$

Employing expressions (16), (17) and (20), expression (13) can be

$$\begin{aligned}
&\Phi\left(\frac{U-m_n}{\sqrt{\sigma^2+\sigma_n^2}}\right) - \Phi\left(\frac{L-m_n}{\sqrt{\sigma^2+\sigma_n^2}}\right) \\
&+\frac{n[(K_A-K_P)+k_{r1}}{(n-N)k_{R1}-K_A N}\frac{\sigma_\mu \cdot \sigma^2\sqrt{n-1}}{n\sqrt{\sigma_\mu^2+\delta_n^2}}\left[\Phi\left(\frac{U-U_A}{\sigma\sqrt{\frac{n-1}{n}}}\right) - \Phi\left(\frac{L-U_A}{\sigma\sqrt{\frac{n-1}{n}}}\right)\right] \\
&= \quad \frac{NK_{R1} - [K_A(N-n) + K_{pn}]}{(N-n)K_{R1} - K_A N} \ .
\end{aligned} \tag{20}$$

Optimizing the total cost relative to the screening limit of X yields the upper and lower limits for lot screening. Thus, taking the partial derivative of K_T relative to U_{sn} yields

$$\frac{\partial K_T}{\partial U_{sn}} = \int_{-\infty}^{+\infty} K_W(U_{sn}, \mu) T(U_{sn} \mid \mu) h(\mu)\, d\mu \ . \tag{21}$$

The partial derivative of K_T relative to U_{sn} is

$$\begin{aligned}
\frac{\partial K_T}{\partial U_{sn}} \quad &= \quad NK_{R1}\frac{\partial K_T}{\partial U_{sn}} - (N-n)K_{R1}\int_{-\infty}^{+\infty} P_{1D}(\mu) T(U_{sn} \mid \mu) h(\mu)\, d\mu \\
&\quad -nK_{R1}\int_{-\infty}^{+\infty} Q_{1D}(U_{sn}, \mu) T(U_{sn} \mid \mu) h(\mu)\, d\mu \ .
\end{aligned} \tag{22}$$

Setting $\frac{\partial K_T}{\partial U_{sn}} = 0$, and dividing the above expression by $\int_{-\infty}^{+\infty} T(U_{sn} \mid \mu) h(\mu)\, d\mu$ and simplifying yields

$$\begin{aligned}
&\frac{\int_{-\infty}^{+\infty} P_{1D}(\mu) T(U_{sn} \mid \mu) h(\mu)\, d\mu}{\int_{-\infty}^{+\infty} T(U_{sn} \mid \mu) h(\mu)\, d\mu} + \frac{n}{N-n}\frac{\int_{-\infty}^{+\infty} Q_{1D}(U_{sn}, \mu) T(U_{sn} \mid \mu) h(\mu)\, d\mu}{\int_{-\infty}^{+\infty} T(U_{sn} \mid \mu) h(\mu)\, d\mu} \\
&= \quad \frac{N}{N-n} \ .
\end{aligned} \tag{23}$$

Moreover, expression (23) can be written as

$$\begin{aligned}
&\Phi\left(\frac{U-m_n}{\sqrt{\sigma^2+\sigma_n^2}}\right)-\Phi\left(\frac{L-m_n}{\sqrt{\sigma^2+\sigma_n^2}}\right)\\
&+\frac{n}{N-n}\left[\frac{\sigma_\mu\cdot\sigma^2\sqrt{n-1}}{n\sqrt{\sigma_\mu^2+\delta_n^2 n}}\right]\left[\Phi\left(\frac{U-U_{sn}}{\sigma\sqrt{\frac{n-1}{n}}}\right)-\Phi\left(\frac{LU-U_{sn}}{\sigma\sqrt{\frac{n-1}{n}}}\right)\right]\\
&=\frac{N}{N-n}
\end{aligned} \tag{24}$$

where

$$\delta_n^2=\frac{\sigma^2}{n}\quad,\quad m_n^2=\frac{m\delta_n^2+\sigma_\mu^2 U_{sn}}{\delta_n^2+\sigma_\mu^2}\quad,\quad \sigma_n^2=\frac{\sigma_\mu^2\cdot\delta_n^2}{\delta_n^2+\sigma_\mu^2}\;. \tag{25}$$

3 Example

A manufacturer of an electronic device used as a temperature probe in a space satellite, use fuses of high quality for the device. The mission time of each of the fuses is intended to be six to seven thousand hours. The quality control engineers constructed a control chart in terms of the decision points relative to upper and lower limits X based on the statistical and economical cost parameters to test the fuses. The chart is designed to keep the quality of the items produced under control by accepting, rejecting or reworking the items before installing them to meet their standard. The specifications and the outcome of the test procedure are listed below.

Input:

MODEL SPECIFICATIONS

Upper limit of the Q.C. X:	7.50000
Lower limit of the Q.C. X:	6.50000
Variance of X:	0.06250
Variance of the mean of X:	0.00420
Unit cost of screening:	0.30000
Unit cost of acceptance:	5.00000
Cost of scrapping or replacing a defective unit found during sampling or screening inspection:	0.60000
Unit cost of scrapping:	0.60000
Lot size:	1000

Output 1: Sample size, roots of the cost function, posterior and sampling costs per unit.

Column	Description
1	Sample size
2	$(a*b)/(b+n*a)$, where a is the variance of the mean of X, b is the variance of X, and n is the sample size
3	Lower disposition limit for lot screening
4	Upper disposition limit for lot screening
5	Upper disposition limit of the sample mean
6	Lower disposition limit of the sample mean
7	Screening cost per lot
8	P2 is the fraction defective at which the costs of screening and scrapping are equal

1	2	3	4	5	6	7	8
52	0.00093	6.47421	7.52579	7.67626	6.32374	997.41631	0.33031
53	0.00092	6.47597	7.52403	7.67116	6.32884	997.36340	0.33025
54	0.00091	6.47765	7.52235	7.66629	6.33371	997.31037	0.33019
55	0.00089	6.47926	7.52074	7.66163	6.33837	997.25720	0.33013
56	0.00088	6.48081	7.51919	7.65716	6.34284	997.20392	0.33007
57	0.00087	6.48228	7.51772	7.65288	6.34712	997.15050	0.33001

Output 2:

Column	Description
1	Sample size
2	Second derivative of the cost relative to the variables involved
3	Scrapping cost per lot
4	Value of the cumulative probability of X given the mean of the lot between the limits $(-\infty \cdot LS)$ and $(US \cdot \infty)$
5	Expected value of the cost obtained by summing over all sample means and lot means

1	2	3	4	5
52	0.0000 E 00	0.9974 E 03	0.6697E 00	0.1352 E 04
53	0.0000 E 00	0.9974 E 03	0.6697 E 00	0.1350 E 04
54	0.0000 E 00	0.9973 E 03	0.6698 E 00	0.1352 E 04
55	0.0000 E 00	0.9973 E 03	0.6699 E 00	0.1357 E 04
56	0.0000 E 00	0.9972 E 03	0.6699 E 00	0.1366 E 04

The density product factor have the following values values: 0.6777283865338088, 0.6766359442639973, 0.6777177776626595, 0.6805999893591675, 0.6849525069706924, 0.6904866061698254.

Output of program prog5aa

Column	Description
1	Sample size
2	Second derivative relative of the variables involved (sample size, upper and lower limits of the sample mean for lot acceptance, the upper and lower for lot screening)
3	Total expected cost

1	2	3
52	0.4142 E 05	0.3451 E 04
53	0.4197 E 05	0.3443 E 04
54	0.4252 E 05	0.3438 E 04
55	0.4304 E 05	0.3437 E 04
56	0.4355 E 05	0.3439 E 04

4 Conclusions

The data shows that the cost is optimum if the sample size if 52. Estimation of the upper and lower limits of $\overline{x}$ are $L_{sn} = 6.47597$. $U_{sn} = 7.52597$, $L_A = 6.85999$, $U_A = 7.14001$. Select a sample size of $n = 52$ from a lot of size $N = 1000$ out of a production line. Estimate the sample mean $\overline{x}$. If $6.85999 < \overline{x} < 7.14001$, the entire lot will be accepted. If $\overline{x} > 7.14001$ or $\overline{x} < 6.85999$, the lot requires screening. If $7.14001 < \overline{x} < 7.5257$ or $6.47597 < \overline{x} < 6.8599$, the lot should be screened. The items in a rejected lot can be either scrapped or reworked with success. The fraction of items reworked is 13% of the total items rejected which is 26%. The total expected cost per item is 1.799 units. The cost per scrapped item is 0.330625 units, and that per item scrapped is 0.0341. From the data generated it is obvious that the cost of reworking defectives add to the total cost. In this case the cost of acceptance is reduced. The cost of screening is the same while the cost of scrapping is reduced. The quality of the items manufactured in the while process is highly critical for both the consumers and producers. Finally, the costs per item are represented graphically by Figures 1 and 2.

5 Appendix: Notation

$\overline{x}$	Sample mean.
L	Lower specification limit of the quality characteristic.
U	Upper specification limit of the quality characteristic.
μ	Mean of the quality characteristic.
σ	Standard deviation of the quality characteristic.
σ_μ	Standard deviation of the mean μ.
$h(\mu)$	Distribution of the lot mean μ.
L_A	Lower disposition limit of $\overline{x}$ for accepting the lot.
U_A	Upper disposition limit of $\overline{x}$ for accepting the lot.
L_{sn}	Lower disposition limit for $\overline{x}$ for screening inspection.
U_{sn}	Upper disposition limit for $\overline{x}$ for inspection.
K_J	Junk value of the scrapped item.
K_P	Production cost of an item.
K_R	Sale price of an item.
K_y	Rework yield rate.
K_c	Cost of an item reworked with success.
K_{c1}	Cost per unit of repairing an item above the specification limit of the lot acceptance.
K_{c2}	Cost per unit of repairing an item below the specification limit of the lot acceptance.
K_4	Cost of an item reworked without success.
P_{4s}	Probability that an individual measurement in a sample drawn from a lot is below the lower specification limit in a single variable acceptance sampling plan.
P_{3s}	Probability that an individual measurement in a sample drawn from a lot is above the upper specification limit in a single variable acceptance sampling plan.
P_{1u}	Probability that an individual measurement in a lot of mean μ is above the upper specification limit when a single variable is involved.
P_{2L}	Probability that an individual measurement in a lot of mean μ is below the lower specification limit when a single variable is involved.

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