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A one dimensional Hammerstein problem *

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Abstract

Nonlinear equations of the form $L[u] = \lambda g(u)$ where L is a linear operator on a function space and g maps u to the composition function $g \circ u$ arise in the theory of spontaneous combustion. If L is invertible, such an equation can be written as a Hammerstein equation, $u = B[u]$ where $B[u] = \lambda L^{-1}[g(u)]$. To investigate the importance of the growth rate of g and the sign and magnitude of λ on the number of solutions of such problems, in this paper we consider the one-dimensional problem $L(x) = \lambda g(x)$ where $L(x) = ax$.

1 Introduction

We wish to investigate the number of solutions (and their computation) to problems of the form

$$L[u] = \lambda g(u) \tag{1.1}$$

where $L : V \to W$ is a linear operator and V and W are function spaces whose domains are the same set, say D, and whose codomains are the real numbers $\mathbb{R}$. If $u \in V$ and $x \in D$, then the value of the function $g(u)$ at x is $g(u(x))$ where $g : \mathbb{R} \to \mathbb{R}$. Thus we use the symbol g for a real valued function of a real variable as well as for the (nonlinear Nemytskii) operator from V to W that this function defines by the composition $g \circ u$. An example is

$$\begin{aligned} -\Delta u &= \lambda e^u \qquad \vec{x} = [x, y, z]^T \in \Omega \subseteq \mathbb{R}^3 \\ u(\vec{x}) &= 0 \qquad \vec{x} \in \partial\Omega \end{aligned} \tag{1.2}$$

Here L is the negative of the Laplacian operator in three spatial dimensions with homogeneous Dirichlet boundary conditions, $\vec{x}$ is a point in $\mathbb{R}^3$, Ω is an open connected region in $\mathbb{R}^3$, $\partial\Omega$ is its boundary, $V = \{u \in V_1 : u(\vec{x}) = 0 \ \forall \ \vec{x} \in \partial\Omega\}$ where $V_1 = C^2(\Omega, \mathbb{R}) \cap C(\bar{\Omega}, \mathbb{R})$, and $W = C(\bar{\Omega}, \mathbb{R})$. Hence $D = \bar{\Omega}$. Such problems arise in the theory of combustion (Deberness [1], Moseley [2, 3, 4, 5]). For this problem in $\mathbb{R}^2$, it is known that for $\lambda < 0$, there exists a unique solution. However, for $\lambda > 0$ and large, there is no solution. But for $\lambda > 0$ and small, there are at least two solutions. If $\lambda = 0$, the solution set is the null space of

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L and since L is invertible the problem has only the trivial solution $u = 0$. If $g(u) = u$, (1.1) is a spectral problem for L. Hence (1.1) is sometimes referred to as a nonlinear eigenvalue problem.

If L is invertible, then (1.1) can be written as the Hammerstein equation

$$u - \lambda L^{-1}[g(u)] = 0. \tag{1.3}$$

A solution of (1.3) is a fixed point of the combined operator $B = \lambda(L^{-1} \circ g)$. The level of difficulty of problems of type (1.1) or (1.3) varies greatly depending on the number of elements in D, the value of n, and the operator L. We list several categories, starting with the easiest.

1. One dimensional problems (i.e., D contains only one element).
2. Multidimensional problems (i.e., D is finite, but contains two or more elements).
3. Infinite dimensional problems with $D \subseteq \mathbb{R}$ $(n = 1)$ and L a first, second, or higher order differential operator.
4. Infinite dimensional problems with $D \subseteq \mathbb{R}^n$, $n = 2, 3, 4, \ldots$ and L a first, second, or higher order partial differential operator.

Since L is linear, we at most have linear coupling and often this coupling is weak. The coupling of L^{-1} may be stronger than the coupling of L and is a reason to examine (1.1) directly even when L is invertible. To investigate the fundamental importance of the growth rate of g and the sign and magnitude of λ on the number of solutions to problems of this type, in this paper we consider the one dimensional nonlinear eigenvalue problem

$$ax = \lambda g(x). \tag{1.4}$$

Here $L : \mathbb{R} \to R$ is $L(x) = ax$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function and a and λ are parameters. (If $a \neq 0$, L is invertible.) To this end, we first consider two types of behavior for a continuous function $f : \mathbb{R} \to \mathbb{R}$ (i.e., $f \in C(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is continuous } \forall x \in \mathbb{R}\}$). If L is invertible $(a \neq 0)$, we take $f(x) = x - kg(x)$ where $k = \lambda/a$. Although less restrictive conditions on f can be obtained, for convenience we assume that f has a continuous second derivative for all x in $\mathbb{R}$; that is, $f \in C^2(\mathbb{R}, R) = \{f : \mathbb{R} \to R : f''(x) \text{ exists and is continuous } \forall x \in \mathbb{R}\}$. We are interested in sufficient conditions on f that will determine the number of solutions of the equation

$$f(x) = 0. \tag{1.5}$$

The obvious advantage of considering the scalar equation (one dimensional problem) (1.4) or (1.5) over an abstract Hammerstein equation or a Hammerstein equation of the type (1.1) where D is finite or infinite dimensional is that much more (often everything) can be said for many functions $g(x)$ (and classes of functions). However, the techniques investigated here are quite different from the standard fixed point theorems (e.g., contraction mapping theorem and the Brouwer and Schauder fixed point theorems) and are expected reveal distinctive new results when extended to higher dimensions including methods for solving multidimensional problems.

2 Linear and quadratic properties

In addition to $f \in C^2(\mathbb{R}, \mathbb{R})$ we also assume some of the following:

H1 $\lim_{x\to\infty} f(x) = +\infty$

H2 $\lim_{x\to\infty} f(x) = -\infty$

H3 $\lim_{x\to-\infty} f(x) = -\infty$

H4 $\lim_{x\to-\infty}(x) = +\infty$

H5 $f'(x) > 0$

H6 $f'(x) < 0$

H7 $f''(x) > 0$

H8 $f''(x) < 0$

We consider properties that $f : \mathbb{R} \to R$ may have that are also possessed by the linear function $F(x) = ax + b$ $(a \neq 0)$:

Definition If $f(x)$ satisfies H1 and H3, we say that it is **mainly increasing**. If it satisfies H1, H3 and H5, we say that it is **consistently increasing**. On the other hand, if $f(x)$ satisfies H2 and H4, we say it is **mainly decreasing**. If it satisfies H2, H4, and H6, we say it is **consistently decreasing**. If $f(x)$ is mainly increasing or decreasing, then we say $f(x)$ is **mainly monotonic**. If $f(x)$ is consistently increasing or decreasing, then we say $f(x)$ is **consistently monotonic**.

For completeness we review a theorem that establishes the existence and uniqueness of solutions to (1.5) if $f(x)$ is mainly or consistently monotonic. Like the Jordan Curve Theorem, it is geometrically obvious and an analytic proof can be given [8].

Theorem 2.1 *If a function f is mainly monotonic, then for any c in $\mathbb{R}$, there exists at least one x in $\mathbb{R}$ such that $f(x) = c$. If the function is consistently monotonic, then for any c in $\mathbb{R}$, there exists exactly one x in $\mathbb{R}$ such that $f(x) = c$.*

If $f(x)$ is consistently monotonic it is similar to the linear function $F(x) = ax + b$ $(a \neq 0)$ in that (1.5), like $F(x) = 0$, has exactly one solution. We say that $f(x)$ has the **linear property**. We also consider some properties that $f(x)$ may have that are also possessed by the quadratic function $G(x) = ax^2 + bx + c$ $(a \neq 0)$.

Definition If $f(x)$ satisfies H1,H4 (H2,H3), we say $f(x)$ **opens upwards (downwards)**. If $f(x)$ satisfies H1, H4, H7 (H2, H3, H8), we say $f(x)$ is **completely concave up (down)**.

Again for completeness, we review two theorems that establish when (1.5) has exactly zero, one, or two solutions. Again they are geometrically obvious and can be proved analytically [8].

Theorem 2.2 *If f opens upwards (downwards), then there exists at least one $x_0 \in R$ such that $f(x_0) = min f(x) \quad (f(x_0) = \max f(x))$.*

Theorem 2.3 *If f opens upwards (downwards) and $m = f(x_0) = \min\ f(x)$ $(m = f(x_0) = \max f(x))$, then*

1. *$c < m$ $(c > m)$ implies that there is no x such that $f(x) = c$,*
2. *$c = m$ implies that there is at least one x such that $f(x) = c$, namely $x = x_0$,*
3. *$c > m$ $(c < m)$ implies that there are at least two (distinct) values of x for which $f(x) = c$.*

If $f(x)$ is completely concave up or down, we see that $f(x)$ has similar properties to the quadratic function $G(x) = ax^2 + bx + c$ $(a \neq 0)$ in that (1.5), like $G(x) = 0$, has exactly 0, 1, or 2 solutions. We say $f(x)$ has the **quadratic property**.

3 Zero and nonzero values for the parameters

Returning to (1.4) we consider whether the values for the parameters a and λ are zero or not. Recall that if $\lambda = 0$ the solution set of (1.1) is just the null space of L. If L is the zero operator and $\lambda \neq 0$, then solutions are those functions u which map all values of x into the zeros of g. If L is the zero operator and $\lambda = 0$, then all u in V are solutions. For completeness, we stand these results explicitly for (1.4).

Theorem 3.1 *If $a = \lambda = 0$, then $\forall x \in \mathbb{R}$, x is a solution. If $a \neq 0$, $\lambda = 0$, we have $x = 0$ is the unique solution. If $a = 0$, $\lambda \neq 0$, then x is a solution of (1.4) if and only if x is a solution of $g(x) = 0$.*

From now on we assume $a \neq 0$ and $\lambda \neq 0$. We may then divide by a and have one nonzero parameter.

Theorem 3.2 *If $a \neq 0$, $\lambda \neq 0$, the solution set of $ax = \lambda g(x)$ is the same as the solution set of*

$$x = kg(x) \tag{3.1}$$

where $k = \lambda/a$.

4 Conditions for infinite limits as $x \to \pm\infty$

We let

$$f(x) = x - kg(x)(k \neq 0) \tag{4.1}$$

which we may also write as

$$f(x) = x - kg(x) = x\left(1 - \frac{kg(x)}{x}\right) = g(x)\left(\frac{x}{g(x)} - k\right), \tag{4.2}$$

provided $x \neq 0$ and $g(x) \neq 0$. From this it is easy to establish that $\lim_{x\to\infty} f(x) = \infty$ for the following conditions. Although less restrictive conditions can be stated, for convenience we consider only limit conditions on $g(x)$, $g(x)/x$, and $x/g(x)$ that are finite or $\pm\infty$.

PP1 $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to\infty} x/g(x) = M$, $M \in \mathbb{R}$ and $M < k$ (e.g. $g(x) = -1/3x + 2, k = -1$)

PP2 $\lim_{x\to\infty} g(x) = -\infty$ and $k > 0$. (e.g. $g(x) = -e^x$ or $g(x) = -x^2$ and $k = 2$)

PP3 $\lim_{x\to\infty} g(x) = M$, (e.g. $g(x) = \text{Arctan } x$)

PP4 $\lim_{x\to\infty} g(x) = \infty$ and $k < 0$. (e.g. $g(x) = e^x$ or $g(x) = x^2$ and $k = -1$)

PP5 $\lim_{x\to\infty} g(x) = \infty$ and $\lim_{x\to\infty} x/g(x) = \infty$. (e.g. $g(x) = x^{1/3}$)

PP6 $\lim_{x\to\infty} g(x)/x = M$, and $kM < 1$. (e.g. $g(x) = 3x + 2$ and $k = -1$)

PP7 $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to\infty} x/g(x) = M$, and $M > k$. (e.g. $g(x) = 3x + 2$ and $k = -1$)

To obtain conditions such that $\lim_{x\to\infty} f(x) = -\infty$ we must completely counteract the effect of the linear term.

PN1 $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to\infty} x/g(x) = M$, and $M > k$. (e.g. $g(x) = -3x + 2$ and $k = -2$)

PN2 $\lim_{x\to\infty} g(x)/x = M$, and $kM > 1$. (e.g. $g(x) = 2x+$ Arctan x and $k = 1$)

PN3 $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to\infty} x/g(x) = M$ and $M < k$. (e.g. $g(x) = 3x^2 + 4$ and $k = 1$)

Similar to the conditions above, we can obtain properties as x approaches $-\infty$. It is easy to establish that $\lim_{x\to-\infty} f(x) = -\infty$ for the following conditions.

NN1 $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to-\infty} x/g(x) = M$ and $M < k$. (e.g. $g(x) = -1/3x$, $k = -1$ or $g(x) = e^{-x}$, $k = 1$)

NN2 $\lim_{x\to-\infty} g(x) = \infty$ and $k > 0$. (e.g. $g(x) = e^{-x}$ or $g(x) = -x^2$ and $k = 1$)

NN3 $\lim_{x\to-\infty} g(x) = M$ (e.g. $g(x) = e^x$ or $g(x) = \text{Arctan } x$)

NN4 $\lim_{x\to-\infty} g(x) = -\infty$ and $k < 0$. (e.g. $g(x) = -e^{-x}$ or $g(x) = -x^2$ and $k = -1$)

NN5 $\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{x\to-\infty} x/g(x) = \infty$. (e.g. $g(x) = x^{1/3}$)

NN6 $\lim_{x\to-\infty} g(x)/x = M$ and $kM < 1$. (e.g. $g(x) = 3x + 2$ and $k = -2$)

NN7 $\lim_{x\to-\infty} g(x) = -\infty$, $\lim_{x\to-\infty} x/g(x) = M$, and $M > k$. (e.g. $g(x) = 3x + 2$ and $k = -1$)

Finally, we can establish that $\lim_{x\to-\infty} f(x) = \infty$ for the following conditions.

NP1 $\lim_{x\to-\infty} g(x) = -\infty$, $\lim_{x\to-\infty} x/g(x) = M$, and $M < k$. (e.g. $g(x) = 3x + 2$ and $k = 2$)

NP2 $\lim_{x\to-\infty} g(x)/x = M$ and $kM > 1$. (e.g. $g(x) = 2x+\text{Arctan}(x)$ and $k = 1$)

NP3 $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to-\infty} x/g(x) = M$ and $M > k$. (e.g. $g(x) = 3x^2 + 4$ and $k = -1$)

5 Linear property

For $f(x)$ given by (4.1), we consider conditions such that$f(x)$ is mainly increasing (and mainly decreasing). We then add conditions so that $f(x)$ is consistently increasing (and consistently decreasing) so that it has the linear property. We begin by looking at conditions so that $f(x)$ is mainly increasing when $k > 0$. Of those conditions such that $\lim_{x\to-\infty} f(x) = \infty$, only PP4 explicitly requires $k < 0$ and must be eliminated. For PP1 we note

Theorem 5.1 *If* $\lim_{x\to\infty} g(x) = -\infty$ *and* $\lim_{x\to\infty} x/g(x) = M$*, then* $M \leq 0$.

Proof. $\lim_{x\to\infty} g(x) = -\infty$, then there exists $M_0 > 0$ such that $\forall x > M_0$, $g(x) < 0$. Hence $\forall x > M_0$, $x/g(x) < 0$. Hence $\lim_{x\to\infty} x/g(x) = M \leq 0$.

Since we are assuming $k > 0$, the requirement that $k > M$ in PP1 is superfluous. Also assuming $k > 0$ allows us to combine PP2 and PP3 into one statement. For emphasis, we modify the new statement (PPKP2) as well as PP1 and PP7 (PPKP1 and PPKP5) to include the constraint $k > 0$. The sign of k has no impact on PP5 (PPKP3). The impact of $k > 0$ on PP6 (PPKP4) is examined later. We do a similar reorganization for those cases for which $\lim_{x\to-\infty} f(x) = -\infty$. Hence we have the following sufficient conditions for the function $f(x) = x - kg(x)$ to be mainly increasing.

Theorem 5.2 *Suppose* $k > 0$ *and one of the following hypothesis holds:*

PPKP1 $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to-\infty} x/g(x) = M$*, and* $k > 0$ $(\geq M)$. *(PP1)*

PPKP2 $\lim_{x\to\infty} g(x) = -\infty$ *or* $M > 0$ *and* $k > 0$. *(PP2 or PP3)*

PPKP3 $\lim_{x\to\infty} g(x) = \infty$ *and* $\lim_{x\to\infty} x/g(x) = \infty$. *(PP5)*

PPKP4 $\lim_{x\to\infty} x/g(x) = M$ *and* $kM > 1$. *(PP6)*

PPKP5 $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to\infty} x/g(x) = M$ *and* $M > k > 0$. *(PP7).*

Suppose further that one of the following hypothesis holds

NNKP1 $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to-\infty} x/g(x) = M$, *and* $k > 0$ $(k \geq M)$. *(NN1)*

NNKP2 $\lim_{x\to-\infty} g(x) = \infty$ *or* $M > 0$ *and* $k > 0$. *(NN2 or NN3)*

NNKP3 $\lim_{x\to-\infty} g(x) = -\infty$ *and* $\lim_{x\to-\infty} x/g(x) = \infty$. *(NN5)*

NNKP4 $\lim_{x\to-\infty} x/g(x) = M$ *and* $kM < 1$. *(NN6)*

NNKP5 $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to-\infty} x/g(x) = M$, *and* $M > k > 0$. *(NN7)*

Then f *given by (4.1) is mainly increasing.*

Note that by requiring $k > 0$, PPKP1 is now contained in PPKP2 and can be eliminated. Similarly for NNKP1. To consider PPKP4 and PPKP5, we note that these depend on the growth rate of $g(x)$ as compared with x. If $\lim_{x\to\infty} g(x)/x = M$ and $M \neq 0$, then the growth rate of $g(x)$ is essentially linear. In this case the linear part of $g(x)$ can be moved to the other side of the equation and combined with x. Hence we really wish to consider only the case when $M = 0$. But if $M = 0$ and $k > 0$, neither PPKP4 or PPKP5 is possible. We consider the details. Suppose $\lim_{x\to\infty} g(x)/x = M$, $M \neq 0$ and $\lim_{x\to-\infty} g(x)/x = N$, $N \neq 0$. We then consider

$$g_1(x) = \begin{cases} g(x) - Mx & x > 0 \\ g(x) - Nx & x < 0 \end{cases} \tag{5.1}$$

Theorem 5.3 *Suppose* $\lim_{x\to\infty} g(x)/x = M$, $M \neq 0$ *and* $\lim_{x\to-\infty} g(x)/x = N$, $N \neq 0$. *Let* $g_1(x)$ *be given by (5.1).*
Then $\lim_{x\to\infty} g_1(x)/x = 0$, $\lim_{x\to-\infty} g_1(x)/x = 0$,
$\lim_{x\to\infty} g_1(x)/g(x) = 0$, *and* $\lim_{x\to-\infty} g_1(x)/g(x) = 0$.

Proof. $\lim_{x\to\infty} g_1(x)/x = \lim_{x\to\infty}(g(x) - Mx)/x = \lim_{x\to\infty}(g(x)/x - M) = M - M = 0$. Similarly for $\lim_{x\to-\infty} g_1(x)/x = \lim_{x\to\infty} g_1(x)/g(x) = 0$,
$\lim_{x\to\infty}(g(x)Mx)/g(x) = \lim_{x\to\infty}(1 - M/(g(x)/x)) = \lim_{x\to\infty}(1 - M/M) = 0$.
Similarly for $\lim_{x\to-\infty} g_1(x)/g(x)$. ♢

Thus when $\lim_{x\to\infty} g(x)/x = M$, $M \neq 0$ and/or $\lim_{x\to-\infty} g(x)/x = N$, $N \neq 0$, we rewrite (3.1) as

$$x + k(g_1(x) - g(x)) = kg_1(x).$$

Hence f can be written as $f(x) = x + k(g_1(x) - g(x)) - kg_1(x) = x(1 + k(g_1(x) - g(x))/xkg_1(x)/x) = g_1(x)(x/g_1(x) + k))$. Conditions can now be obtained so that $f(x)$ approaches $\pm\infty$ as x approaches $\pm\infty$ using $g_1(x)$ instead of $g(x)$. However, care must be taken since $x + k(g_1(x) - g(x))$ need not be linear, but is only guaranteed to be "piecewise" linear. Also $g_1(x)$ is continuous at $x = 0$, but not differentiable unless $M = N$. This is only a minor inconvenience and well worth the elimination of a linear component in g by moving it to the other side of the equation. This is a small price to pay to reap the benefits of the function on the right hand side having a truly nonlinear growth rate. We leave such issues to future work and from now on only allow $\lim_{x\to\infty} g(x)/x = M$ where $M = 0$ or $M = \pm\infty$ and $\lim_{x\to-\infty} g(x)/x = N$ where $N = 0$ or $N = \pm\infty$. We now modify Theorem 7 to consider only these possibilities.

Theorem 5.4 *Suppose $k > 0$ and one of the following hypothesis holds:*

PPKPN1 $\lim_{x\to\infty} g(x) = -\infty$ *or* 0, *(e.g.,* $g(x) = -e^x$ *or* $g(x) = -x^3$*),*

PPKPN2 $\lim_{x\to\infty} g(x) = \infty$ *and* $\lim_{x\to\infty} x/g(x) = \infty$. *(e.g.,* $g(x) = x^{1/3}$*).*

Suppose further that one of the following hypothesis holds:

NNKPN1 $\lim_{x\to-\infty} g(x) = \infty$ *or* 0, *(e.g.,* $g(x) = -e^x$ *or* $g(x) = -x^3$*),*

NNKPN2 $\lim_{x\to-\infty} g(x) = -\infty$ *and* $\lim_{x\to-\infty} x/g(x) = \infty$, *(e.g.,* $g(x) = x^{1/3}$*).*

Then the function f given by (4.1) is mainly increasing.

Similarly, we consider mainly increasing functions $f(x)$ when $k < 0$.

Theorem 5.5 *Suppose $k < 0$ and one of the following hypothesis holds:*

PPKNN1 $\lim_{x\to\infty} g(x) = \infty$ *or* 0, *(e.g.,* $g(x) = e^x$ *or* $g(x) = x^3$*),*

PPKNN2 $\lim_{x\to\infty} g(x) = -\infty$ *and* $x/g(x) = -\infty$. *(e.g.,* $g(x) = -x^{1/3}$*).*

Suppose further that one of the following hypothesis holds:

NNKNN1 $\lim_{x\to-\infty} g(x) = -\infty$ *or* $\lim_{x\to-\infty} g(x) = 0$, *(e.g.,* $g(x) = e^x$ *or* $g(x) = x^3$*).*

NNKNN2 $\lim_{x\to-\infty} g(x) = \infty$ *and* $\lim_{x\to-\infty} x/g(x) = -\infty$ *(e.g.,* $g(x) = -x^{1/3}$*).*

Then the function f given by (4.1) is mainly increasing.

We now consider functions $f(x)$ that are mainly decreasing. Following the previous procedure, we retain only the cases where $\lim_{x\to\infty} g(x)/x = M$ where $M = 0$ or $\pm\infty$ and $\lim_{x\to-\infty} g(x)/x = N$ where $N = 0$ or $\pm\infty$. We first consider the case of $k > 0$. This eliminates PN1 and PN2 as well as NP2 and NP3.

Theorem 5.6 *Suppose* $k > 0$, *if*

PNKPN1 $\lim_{x\to\infty} g(x) = \infty$, $\lim_{x\to\infty} x/g(x) = 0$ *(e.g.,* $g(x) = \sinh x$*) and*

NPKPN1 $\lim_{x\to-\infty} g(x) = -\infty$, $\lim_{x\to-\infty} x/g(x) = 0$, *(e.g.,* $g(x) = \sinh x$*)*

then $f(x)$ *is mainly decreasing.*

Now, we consider the case $k < 0$.

Theorem 5.7 *Suppose* $k < 0$, *if*

PNKNN1 $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to\infty} x/g(x) = 0$ *(e.g.,* $g(x) = \sinh x$*) and*

NPKNN1 $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to-\infty} x/g(x) = 0$, *(e.g.,* $g(x) = -\sinh x$*)*

then $f(x)$ *is mainly decreasing.*

Corollary 5.8 *If* $a \neq 0$, $\lambda \neq 0$ *and*
$k > 0$, *PPKPN1 or 2, and NNKPN1 or 2 (mainly increasing)*
or
$k < 0$, *PPKNN1 or 2, and NNKNN1 or 2 (mainly increasing)*
or
$k > 0$, *PNKPN1, and NPKPN1 (mainly decreasing)*
or
$k > 0$, *PNKNN1, and NPKNN1 (mainly decreasing)*
so that $f(x)$ *given by (4.1) is mainly monotonic, then (3.1) (and (1.3)) has at least one solution.*

We now investigate sufficient conditions for $f(x)$ given by (4.1) to be consistently monotonic. Since we require $f'(x) > 0$, we obtain conditions on k and $g(x)$.

Theorem 5.9 *If* $a \neq 0$, $\lambda \neq 0$, $k = \lambda/a > 0$, *and*
PPKPN1 or 2, NNKPN1 or 2, and $g'(x) > 1/k$*(consistently increasing)*
or
PNKPN1, NPKPN1, and $g'(x) < 1/k$ *(consistently decreasing)*
then the function $f(x)$ *given by (4.1) is consistently monotonic.*

Theorem 5.10 *If* $a \neq 0$, $\lambda \neq 0$, $k = \lambda/a < 0$, *and*
PPKNN1 or 2 , NNKNN1 or 2, and $g'(x) > 1/k$ *(consistently increasing)*
or
PNKNN1 or 2, NPKNN1 and $g'(x) < 1/k$ *(consistently decreasing)*
then the function $f(x)$ *given by (4.1) is consistently monotonic.*

Corollary 5.11 *If* $a \neq 0$, $\lambda \neq 0$ *and*

$k > 0$, *PPKPN1 or 2, NNKPN1 or 2, and* $g'(x) > 1/k$ *(consistently increasing)*
or

$k < 0$. PPKNN1 or 2, NNKNN1 or 2, and $g'(x) > 1/k$ (consistently increasing)
or
$k > 0$, PNKPN1, NPKPN1, and $g'(x) < 1/k$ (consistently decreasing)
or
$k < 0$, PNKNN1, NPKNN1, and $g'(x) < 1/k$ (consistently decreasing)
so that the function $f(x)$ given by (4.1) is consistently monotonic, then (3.1) (and (1.3)) has exactly one solution.

6 Quadratic property

We now wish to consider cases when f opens upwards or opens downwards. As with the linear cases, we consider the sign of k separately and only cases where the growth rate of $g(x)$ is not linear. We begin with the case where f opens upwards and $k > 0$.

Theorem 6.1 *Suppose $k > 0$ and one of the following hypothesis holds:*

PPKPN1 $\lim_{x\to\infty} g(x) = -\infty$ *or* 0, *(e.g., $g(x) = -e^{-x}$).*

PPKPN2 $\lim_{x\to\infty} g(x) = \infty$ *and* $\lim_{x\to\infty} x/g(x) = \infty$. *(e.g., $g(x) = e^{-x} + x^{1/3}$).*

Suppose further that the following hypothesis holds:

NPKPN1 $\lim_{x\to-\infty} g(x) = -\infty$ *and* $\lim_{x\to-\infty} x/g(x) = 0$. *(e.g., $g(x) = -e^{-x}$ or $g(x) = -e^{-x} + x^{1/3}$.*

Then the function $f(x)$ given by (4.1) opens upwards.

Now we consider the case where f opens upwards and $k < 0$.

Theorem 6.2 *Suppose $k < 0$ and one of the following hypothesis holds:*

PPKNN1 $\lim_{x\to\infty} g(x) = \infty$ *or* 0, *(e.g., $g(x) = e^{-x}$)*
or

PPKNN2 $\lim_{x\to\infty} g(x) = -\infty$ *and* $\lim_{x\to\infty} x/g(x) = -\infty$. *(e.g., $g(x) = e^{-x} - x^{1/3}$*

Suppose further that the following hypothesis holds:

NPKNN1 $\lim_{x\to-\infty} g(x) = \infty$ *and* $\lim_{x\to-\infty} x/g(x) = 0$. *(e.g., $g(x) = e^{-x}$ or $g(x) = e^{-x} - x^{1/3}$).*

Then the function $f(x)$ given by (4.1) opens upwards.

Now we consider the case where f opens downwards and $k > 0$.

Theorem 6.3 *Suppose $k > 0$ and the following hypothesis holds:*

PNKPN1 $\lim_{x\to\infty} g(x) = \infty$ *and* $\lim_{x\to\infty} x/g(x) = 0$. *(e.g.,* $g(x) = x^2$ *or* $g(x) = e^x + x^{1/3}$*).*

Suppose further that one of the following hypothesis holds:

NNKPN1 $\lim_{x\to-\infty} g(x) = \infty$ *or* 0, *(e.g.,* $g(x) = x^2$*)*

NNKPN2 $\lim_{x\to-\infty} g(x) = -\infty$ *and* $\lim_{x\to-\infty} x/g(x) = \infty$. *(e.g.,* $g(x) = e^x + x^{1/3}$*).*

Then the function $f(x)$ given by (4.1) opens downwards.

Now we consider the case that $k < 0$.

Theorem 6.4 *Suppose $k < 0$ and the following hypothesis holds:*

PNKNN1 $\lim_{x\to\infty} g(x) = -\infty$ *and* $\lim_{x\to\infty} x/g(x) = 0$. *(e.g.,* $g(x) = -x^2$ *or* $g(x) = -e^x - x^{1/3}$.

Suppose further that one of the following hypothesis holds:

NNKNN1 $\lim_{x\to-\infty} g(x) = -\infty$ *or* 0, *(e.g.,* $g(x) = -x^2$)

NNKNN2 $\lim_{x\to-\infty} g(x) = \infty$ *and* $\lim x \to -\infty x/g(x) = -\infty$. *(e.g.,* $g(x) = -e^{-x} - x^{1/3}$*).*

Then the function $f(x)$ given by (4.1) opens downwards.

We next state theorems when there exists no solution, at least one solution, and at least two solutions.

Theorem 6.5 *Suppose $m = \min_{x\in\mathbb{R}} f(x) = f(x_0)$ where $f'(x_0) = 0$ and*

$k > 0$, PPKPN1 or 2, and NPKPN1, (e.g., $g(x) = -e^{-x}$)
or

$k < 0$, PPKNN1 or 2, and NPKNN1. (e.g., $g(x) = e^{-x}$).

Then f given by (4.1) opens upward. Hence

(C1) $m > 0$ implies (3.1) (and (1.3)) has no solution,

(C2) $m = 0$ implies (3.1) (and (1.3)) has at least one solution, namely $x = x_0$.

(C3) $m < 0$ implies (3.1) (and (1.3)) has at least two solutions, say x_1 and x_2. Furthermore, we may assume $x_1 < x_0 < x_2$.

Similarly,

Theorem 6.6 *Suppose $m = \max x \in \mathbb{R} f(x) = f(x_0)$ where $f'(x_0) = 0$ and*

$k > 0$, PNKPN1, and NNKPN1 or 2, (e.g., $g(x) = x^2$)
or

$k < 0$, PNKNN1, and NNKNN1 or 2. (e.g., $g(x) = -x^2$).

Then f given by (4.1) opens downward. Hence

(C1) $m < 0$ implies (3.1) (and (1.3)) has no solution,

(C2) $m = 0$ implies (3.1) (and (1.3)) has at least one solution, namely $x = x_0$.

(C3) $m > 0$ implies (3.1) (and (1.3)) has at least two solutions, say x_1 and x_2. Furthermore, we may assume $x_1 < x_0 < x_2$

Note that if f is given by (4.1), then $f''(x) = kg''(x)$. Hence we can consider cases where f is completely concave up or down. We can then state conditions where there exist no solutions, exactly one solution, or exactly two solutions.

Theorem 6.7 *Suppose $m = \min_{x\in\mathbb{R}} f(x) = f(x_0)$ where $f'(x_0) = 0$ and*
$k > 0$, PPKPN1 or 2, NPKPN1, and $\forall x \in \mathbb{R}$, $g''(x) > 0$
or
$k < 0$, PPKNN1 or 2, NPKNN1 and $\forall x \in \mathbb{R}$, $g''(x) < 0$.
Then f given by (4.1) is completely concave up. Hence

(C1) $m > 0$ implies (3.1) (and (1.3)) has no solution,

(C2) $m = 0$ implies (3.1) (and (1.3)) has exactly one solution, namely $x = x_0$.

(C3) $m < 0$ implies (3.1) (and (1.3)) has exactly two solutions, say x_1 and x_2. Furthermore, we may assume $x_1 < x_0 < x_2$.

Similarly,

Theorem 6.8 *Suppose $m = \max_{x\in\mathbb{R}} f(x) = f(x_0)$ where $f'(x_0) = 0$ and*
$k > 0$, PNKPN1, NNKPN1 or 2, and $\forall x \in \mathbb{R}$, $g''(x) < 0$
or
$k < 0$, PNKNN1, NNKNN1 or 2, and $\forall x \in \mathbb{R}$, $g''(x) > 0$.
Then f given by (4.1) is completely concave down. Hence

(C1) $m < 0$ implies (3.1) (and (1.3)) has no solution,

(C2) $m = 0$ implies (3.1) (and (1.3)) has exactly one solution, namely $x = x_0$.

(C3) $m > 0$ implies (3.1) (and (1.3)) has exactly two solutions, say x_1 and x_2. Furthermore, we may assume $x_1 < x_0 < x_2$.

7 Future work

In the future, we will consider multidimensional systems beginning with the scalar equations

$$ax + by = \lambda g(x) \tag{7.1}$$
$$cx + dy = \lambda g(y). \tag{7.2}$$

These can be written as the vector or matrix equation

$$A\vec{x} = \lambda \vec{g}(\vec{x}) \tag{7.3}$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \text{ and } \vec{g}(\vec{x}) = \begin{bmatrix} g(x) \\ g(y) \end{bmatrix}.$$

Extensions to n dimensions should follow. The problem (7.3) has five parameters in addition to the function g. If $b \neq 0$, (7.1) may be solved for y and substituted into (7.2) to obtain a single equation. The steps in the above process can then be followed to obtain conditions where the linear property of the quadratic property obtains for (7.3). This will include direct applications of the theorems in this paper. Hence we will obtain linear conditions where (7.3) has exactly one solution as well as quadratic conditions where there are no solutions or exactly one or two solutions. Tridiagonal matrices should also be amenable to this process.

8 Summary

We give sufficient conditions to determine exactly the number of solutions for the one dimensional nonlinear eigenvalue problem $ax = \lambda g(x)$. We first considered the cases when a or λ (or both) are zero. We then assumed $a \neq 0$, $\lambda \neq 0$, let $k = \lambda/a$ and focused on the solution set of $f(x) = 0$. For any function $f(x)$ we then defined the linear and quadratic properties depending on the limits of f as x approaches $\pm\infty$, and on its monotonicity and concavity properties. We then considered sufficient conditions on k and $g(x)$ so that $f(x) = x - kg(x)$ approaches $\pm\infty$ as x approaches $\pm\infty$ and hence obtained sufficient conditions for $f(x)$ to be mainly monotonic and consistently monotonic. This allowed us to establish sufficient conditions so that $f(x)$ has the linear property and hence conditions so that $ax = \lambda g(x)$ has at least one and exactly one solution. We also gave sufficient conditions for $f(x)$ to open upwards and downwards and to be completely concave up and down so that f has the quadratic property. This leads to sufficient conditions so that $ax = \lambda g(x)$ has no solution, at least one solution, and at least two solutions as well as sufficient conditions where exactly one solution, and exactly two solutions exist. Examples are given for both the linear and quadratic cases. Future work on multidimensional systems includes two dimensional problems and tridiagonal systems.

References

[1] Deberness, Jerrold, "Some Mathematical Problems From Combustion", Invited Address, Conference of Applied Mathematics (CAM), University of Central Oklahoma, March 27-28, 1992.

[2] Moseley, James L., "Some Extensions of Known Asymptotic Results for a Partial Differential Equation with an Exponential Nonlinearity", Ph.D. Thesis, Purdue University, May 1979.

[3] Moseley, James L.,"A Nonlinear Eigen Value Problem with an Exponential Nonlinearity", American Mathematical Society series on Contemporary Mathematics, Vol. 4, 1981, p. 11-24.

[4] Moseley, James L., "Asymptotic Solutions for a Dirichlet Problem with an Exponential Nonlinearity", SIAM Journal on Mathematical Analysis, 14, 714-735, July, 1983.

[5] Moseley, James L., "A Two Dimensional Dirichlet Problem with an Exponential Nonlinearity", SIAM Journal on Mathematical Analysis, 14, 934-946, September, 1983.

[6] Moseley, James L., "A Set Theoretic Framework for the Formulation of Some Problems in Mathematics", Proceedings of the Tenth Annual Conference on Applied Mathematics, University of Central Oklahoma, Edmond, Oklahoma, February, 1994, pp. 148-153.

[7] Moseley, James L., "A Set Theoretic Framework for the Formulation and Solution of Some Problems in Mathematics", Proceedings of the Eleventh Annual Conference of Applied Mathematics, University of Central Oklahoma, Edmond, Oklahoma, February, 1995, pp. 153-164

[8] Moseley, James L., "Some Problems where the Number of Solutions may be Zero, One, or Two", Proceedings of the Twelfth Annual Conference on Applied Mathematics, University of Central Oklahoma, Edmond, Oklahoma, February 9-10, 1996, pp. 141-153.

[9] Moseley, James L. and Jun Hua, "A One Dimensional Nonlinear Eigenvalue Problem", Applied Math Report #7, AMR#7, May, 1997, West Virginia University, Morgantown, West Virginia.

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