15th Annual Conference on Applied Mathematics, Univ. of Central Oklahoma,
Electronic Journal of Differential Equations, Conference 02, 1999, pp. 133–136.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Traveling-wave solutions of a modified Hodgkin-Huxley type neural model via Novel analytical results for nonlinear transmission lines with arbitrary $I(V)$ characteristics *

Valentino Anthony Simpao

Abstract

Herein an enhanced Hodgkin-Huxley (H-H) type model of neuron dynamics is solved analytically via formal methods. Our model is a variant of an earlier one by M.A. Mahrous and H.Y. Alkahby [1]. Their modified model is realized by a hyperbolic quasi-linear diffusion operator with time-delay parameters; this compared to the original H-H model with standard parabolic quasi-linear diffusion operator and no time-delay parameters. Besides these features, the present model also incorporates terms describing signal dissipation into the background substrate (e.g., conductance to ground), making it more experimentally amenable. The solutions which results via the present scheme are of traveling-wave profile, which agree qualitatively with those observed in actual electro-physiological measurements made on the neural systems originally studied by H-H These results confirm the physiological soundness of the enhanced model and of the preliminary assumptions which motivated the present solution strategy; the comparison of the present results with actual electro-physiological data displays shall appear in later publications.

1 Introduction

Consider the nonlinear transmission-line model equation (viz. (3.8) in Mahrous and Alkahby in [3])

$$(\partial_x^2 - \frac{1}{\theta^2}\partial_t^2)V = \frac{2RC}{a}\partial_t V + \frac{2R}{a}I_i + \frac{2L}{a}\partial_t I_i\,. \tag{1}$$

Where all the parameters are as defined in [3], except for the J-terms in their section 5. In the present analysis, only the time-asymptotically stable expressions are being considered as $t \to \infty$ for the various J-terms, particularly

*1991 Mathematics Subject Classifications: 35L70, 92C20, 35K57.
Key words and phrases: Hodgkin-Huxley, hyperbolic quasilinear diffusion operator, non-linear transmission line, analytical solution.

Published January 21, 2000.

$J = \frac{\alpha_J(V)}{\alpha_J(V)+\beta_J(V)}$. As a consequence of [3] and the present J-term considerations, the ionic current I_i is here clearly a function of V and the constant parameters of the system; the variation of I_i with respect to (x,t) is implicit, being here determined exclusively by $I_i(V(x,t))$. Since traveling waves are physiologically useful constructs [ubiquitous in natural phenomena], the present work is dedicated to obtaining traveling-wave solutions to (1). Specifically sought are solutions of form $V(x,t) = V(\mu_\pm)$ with $\mu_\pm = (x \pm \theta t)$. Concerning the empirically determined forms of the J-terms in [2], along with the aforementioned stipulation about the asymptotically stable terms, the particular form of the ionic current $I_i(V(x,t))$ is

$$\begin{aligned} I_i(v) & \qquad\qquad (2)\\ = \; & \overline{G}_k\left(\frac{(0.1+0.01V)^4(V-V_k)}{(e^{1+0.1V}-1)^4\big(0.125e^{v/80}+(0.1+0.01V)(e^{1+0.1V}-1)^{-1}\big)^4}\right)\\ & +\overline{G}_{Na}\left(\frac{0.07e^{V/20}(2.5+0.1V)^3(V-V_{Na})}{(e^{2.5+0.1V}-1)^3\big(0.07e^{V/20}+\frac{1}{e^{3+0.1V}+1}\big)\big(4e^{V/18}+\frac{2.5+0.1v}{e^{2.5+0.1V}-1}\big)^3}\right)\\ & +\overline{G}_L(V-V_L)\,. \end{aligned}$$

To solve (1), we consider an analytical result for a general class of Non Linear Transmission Line equations.

2 Main result

Consider the class of Non Linear Transmission Line (NLTL) equations, which arise in the context of transmission line models for systems with 1-configuration space variable x degree of freedom (the longitudinal axis of the cable), a single time variable t, and a specified but otherwise arbitrary dependence of the line current $I(x,t)$ upon the line voltage $V(x,t)$, i.e., $I(V(x,t))$. Then

$$\begin{aligned} \partial_x V(x,t) &= -RI(x,t) - L\partial_t I(x,t) \\ \partial_x I(x,t) &= -GV(x,t) - C\partial_t V(x,t) \end{aligned} \qquad (3)$$

where R, G, L, C are the constant resistance per unit of length, constant conductance (leakage loss to ground) per unit of length, constant inductance per unit of length and constant capacitance per unit of length. Re-arranging (3), we obtain

$$\begin{aligned} & \big[(\partial_x V(x,t))^2 - LC(\partial_t V(x,t))^2\big] D^2_{V(x,t)} I(V(x,t)) \qquad (4)\\ & +\big[\partial_x^2 V(x,t) - LC\partial_t^2 V(x,t) - (GL+RC)\partial_t V(x,t)\big] D_{V(x,t)} I(V(x,t)) \\ & \quad = \quad GRI(V(x,t))\,, \end{aligned}$$

where x and t are real variables and G, L, R, C are constants.

Now define the characteristic variable as $\mu_\pm = x \pm t/\sqrt{LC}$. Substituting the characteristic variable as the particular $V(x,t) = V(\mu_\pm)$, the functional dependence in (2) yields

$$\begin{aligned}&\big[(D_{\mu_\pm}V(\mu_\pm))^2 - \frac{LC}{LC}(D_{\mu_\pm}V(\mu_\pm))^2\big]D^2_{V(\mu_\pm)}I(V(\mu_\pm)) \\ &+\big[D^2_{\mu_\pm}V(\mu_\pm) - \frac{LC}{LC}D^2_{\mu_\pm}V(\mu_\pm) - \frac{GL+RC}{\pm\sqrt{LC}}D_{\mu_\pm}V(\mu_\pm)\big]D_{V(\mu_\pm)}I(V(\mu_\pm)) \\ &= \; GRI(V(\mu_\pm))\,.\end{aligned} \tag{5}$$

Simplifying this equation, we obtain

$$-\frac{GL+RC}{\pm\sqrt{LC}}D_{V(\mu_\pm)}I(V(\mu_\pm))D_{V(\mu_\pm)}V(\mu_\pm) = GRI(V(\mu_\pm))\,. \tag{6}$$

Since $I(V(x,t))$ is specified but otherwise arbitrary, (6) has an analytical implicit solution given by

$$\int \frac{D_{V(\mu_\pm)}I(V(\mu_\pm))}{I(V(\mu_\pm))}\,dV(\mu_\pm) = \int -\frac{\pm GR\sqrt{LC}}{GL+RC}D_{V(\mu_\pm)}\mu_\pm(V(\mu_\pm))\,dV(\mu_\pm)$$

Therefore, $\ln(I(V)) = -\pm GR\sqrt{LC}(\mu_\pm + \mu_{\text{const.}})/(GL+RC)$ and

$$I(V) = \exp\big(-\frac{\pm GR\sqrt{LC}}{GL+RC}(\mu_\pm + \mu_{\text{const.}})\big)\,. \tag{7}$$

By the inversion theorem on power series [1], the explicit analytical form of V ascends

$$V(\mu_\pm) = \sum_{n=1}^{\infty}\frac{1}{n!}D_V^{n-1}\big(\frac{V}{I(V)}\big)^n\Big|_{V=0}\exp\big(-\frac{\pm GR\sqrt{LC}}{GL+RC}(\mu_\pm + \mu_{\text{const.}})\big)\,. \tag{8}$$

Regarding the arbitrary constant $\mu_{\text{const.}}$, it may be used to designate advances or delays in the time and/or space domains of the solution.

With these results in place, consider the fundamental system of coupled partial differential equations (3) defining the transmission line equation (1),

$$\begin{aligned}\partial_x v(x,t) &= -ri_a(x,t) - l\partial_t i_a(x,t) \\ \partial_x i_a(x,t) &= -i_i(x,t) - c_a\partial_t v(x,t)\,.\end{aligned}$$

Identifying $v = V$, $i_a = I$, $l = L$, $c_a = C$, $r = R$, and $i_i(x,t) = 2\pi a(C_m\partial_t V(x,t) + I_i(V(x,t)) = -GV(x,t)$, terms in (3) with terms in [3] (with the (x,t) dependence suppressed for notational simplicity) yields

$$2\pi a(C_m\partial_t V(x,t) + I_i(V(x,t)) = -GV(x,t)$$

$$\begin{aligned}&I_i(v) \\ &= \; \overline{G}_k\Big(\frac{(0.1+0.01V)^4(V-V_k)}{(e^{1+0.1V}-1)^4\big(0.125e^{v/80} + (0.1+0.01V)(e^{1+0.1V}-1)^{-1}\big)^4}\Big) \\ &\quad +\overline{G}_{Na}\Big(\frac{0.07e^{V/20}(2.5+0.1V)^3(V-V_{Na})}{(e^{2.5+0.1V}-1)^3\big(0.07e^{V/20} + \frac{1}{e^{3+0.1V}+1}\big)\big(4e^{V/18} + \frac{2.5+0.1v}{e^{2.5+0.1V}-1}\big)^3}\Big) \\ &\quad +\overline{G}_L(V-V_L)\,.\end{aligned}$$

So the line current, defined in terms of the ionic current $I_i(V)$, and $V(x,t)$ are given by

$$I(V) = \exp\big(-\frac{\pm GR\sqrt{LC}}{GL+RC}(x \pm \frac{t}{\sqrt{LC}} + \mu_{\text{const.}})\big)$$

$$V(\mu_\pm) = \sum_{n=1}^{\infty} \frac{1}{n!} D_V^{n-1} \big(\frac{V}{I(V)}\big)^n \Big|_{V=0} \exp\big(-\frac{\pm GR\sqrt{LC}}{GL+RC}(x \pm \frac{t}{\sqrt{LC}} + \mu_{\text{const.}})\big)\,.$$

Explicit calculation of the above formula with numerical values for the system parameters indicates that the functional form, $V(x \pm t/\sqrt{LC})$, theoretically-predicted traveling-wave potential solution matches the experimentally observed action potential of the neuron; these results shall appear in later reports.

References

[1] G. Arfken, *Mathematical Methods for Physicists*, Academic Press, 1970, 2nd Ed, pp. 366-367.

[2] A. L. Hodgkin and A. F. Huxley, *Quantitative Description of Membrane Current and its Application to Conduction and Excitation in Nerve*, J. Physiol. London, 117 (1952), pp. 500-544.

[3] M. A. Mahrous and H.Y. Alkahby, *Mathematical Model for Signal Transmission on Nerve Axon with Time Delay*, Proceedings of the 14th Annual Conference on Applied Mathematics, University of Central Oklahoma, 1998.

Valentino Anthony Simpao
Mathematical Consultant Services
108 Hopkinsville St.
Greenville, Kentucky 42345 USA
e-mail: mcs007@muhlon.com Tel.: 502-338-5543