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A singular nonlinear boundary-value problem *

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Abstract

In this paper we prove an existence and uniqueness theorem for the singular nonlinear boundary-value problem

$$(|y'(t)|^p y'(t))' + \frac{\phi}{y^\lambda(t)} = 0 \text{ in } (0,1),$$
$$y(0) = 0 = y(1),$$

where $p \geq 0$, $\lambda > 0$ and ϕ is a positive function in $L^1_{\rm loc}(0,1)$. Moreover, we derive asymptotic estimates describing the behavior of the solution and its derivative at the boundary.

1 Introduction

In this paper we study the singular nonlinear boundary value problem

$$\begin{gathered}(|y'(t)|^p y'(t))' + \frac{\phi}{y^\lambda} = 0 \quad \text{in } (0,1), \\ y(0) = y(1) = 0\,, \end{gathered} \tag{1}$$

where we assume throughout that $p \geq 0$, $\lambda > 0$ and ϕ is a positive function in $L^1_{\rm loc}(0,1)$. Boundary value problems such as (1) occur in a wide variety of applications. For example, see [3], [4] and [7] for applications to fluid dynamics.

The primary motivation for our work comes from [8], in which Taliaferro studied (1) for the case where $p = 0$ and $\phi \in C(0,1)$. Taliaferro showed that (1) has a smooth positive solution iff

$$\int_0^1 t(1-t)\phi(t)\,dt < \infty,$$

and that the given solution is unique. Taliaferro also showed that $\lim_{t\to 0^+} y'(t)$ exists and is finite iff

$$\int_0^{1/2} \frac{\phi(t)}{t^\lambda}\,dt < \infty,$$

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where a similar condition applies at $t = 1$. Moreover, Taliaferro derived asymptotic formulae describing the behavior of the solution near the boundary for both the finite and infinite slope cases. Generalizations of Taliaferro's paper include Gatica, Oliker, and Waltman [5], Gatica, Hernandez, and Waltman [6], Baxley [1], and Baxley and Martin [2]. The papers [5], [6] and [1] study the case $p = 0$, and the work in [2] is related to the case $-2 \le p < 0$.

In this paper we generalize Taliaferro's work to the case $p \ge 0$. This generalization allows us to study fluids with velocity dependent diffusion, and it forces us to confront the inevitable difficulties that follow when a well understood linear differential operator is replaced by a nonlinear differential operator.

One example of the differences between the cases $p = 0$ and $p > 0$ is that in the latter case the problem has an additional singularity where $y'(t) = 0$. Moreover, this singularity cannot be avoided, because a positive solution of (1) must be concave down and must be 0 at the boundaries, and therefore must have a unique critical point where it achieves a maximum. To get a different point of view on this singularity assume that y is twice differentiable and rewrite (1) as

$$\begin{gathered} y''(t) + \frac{\phi}{(p+1)|y'|^p y^\lambda} = 0 \quad \text{in } (0,1), \\ y(0) = 0 = y(1), \end{gathered}$$

which is similar in form to the problem studied in [2]. From this representation of the problem it is clear that a solution cannot be twice differentiable at its critical point.

In Section 2 we prove that (1) has a positive solution iff $\phi \in L^1_{\mathrm{loc}}(0,1)$ such that

$$\int_0^1 \Big(\int_t^{1/2} \phi \, ds \Big)^{\frac{1}{p+1}} dt < \infty,$$

and that this solution is unique. (This is equivalent to Taliaferro's condition when $p = 0$.) The argument begins by examining initial value problems that start in the interior of the interval and then "shoot" towards the boundary. Of particular interest are the solutions whose initial slopes are 0. Thus we shoot from the singularity mentioned above, and it is important to check that suitable existence, uniqueness, comparison, and continuous dependence theorems are still available. We prove these lemmas and then show that for each $T \in (0,1)$ there exist left and right *half solutions* y_0 and y_1 defined on $[0,T]$ and $[T,1]$, respectively, such that $y_0(0) = 0 = y_1(1)$ and $y_0'(T) = 0 = y_1'(T)$. Finally, we show that the parameter T can be adjusted until $y_0(T) = y_1(T)$, thus bringing the half solutions together to create the unique solution to (1).

In Section 3 we examine the boundary behavior of the solution. We prove that the solution has finite slope at $t = 0$ iff

$$\int_0^{1/2} \frac{\phi(t)}{t^\lambda} \, dt < \infty,$$

just as in [8], where a similar condition applies at $t = 1$. For the finite slope case we quickly obtain the asymptotic formulae

$$y'(t) = \left(\frac{1}{A}\right)^{\frac{\lambda}{p+1}} \Big(\int_t^T \frac{\phi}{s^\lambda}(1 + o(1))\, ds\Big)^{\frac{1}{p+1}},$$

and

$$y(t) = \left(\frac{1}{A}\right)^{\frac{\lambda}{p+1}} \int_0^t \Big(\int_t^T \frac{\phi}{s^\lambda}(1 + o(1))\, ds\Big)^{\frac{1}{p+1}},$$

which are similar to those in [8]. For the remainder of Section 3 we concentrate on the infinite slope case. Using comparison arguments we prove that if ψ is positive and locally integrable such that $\lim_{t\to 0^+} \frac{\psi(t)}{\phi(t)} = 1$ and if $z(t)$ is any solution of

$$\begin{gathered}(|z'(t)|^p z'(t))' + \frac{\psi(t)}{z^\lambda(t)} = 0 \text{ in } (0, \delta),\\ z(0) = 0,\end{gathered} \tag{2}$$

for some $\delta > 0$, then $\lim_{t\to 0^+} \frac{z'(t)}{y'(t)} = \lim_{t\to 0^+} \frac{z(t)}{y(t)} = 1$. We apply this general result to the special case where ϕ behaves like a power of t. In this case we see that if ϕ is asymptotically comparable to ct^r, then $y(t)$ is asymptotically comparable to γt^ρ for appropriate γ and ρ . These results complement those in [8] and [2].

2 Existence and Uniqueness

In this section we prove

Theorem 1 *The boundary-value problem (1) has a positive solution iff*

$$\int_0^1 \Big(\int_t^{1/2} \phi\, ds\Big)^{\frac{1}{p+1}} dt < \infty\,.$$

Moreover, this solution is unique.

Let $T \in (0, 1)$ and consider the initial value problem

$$\begin{gathered}(|y'(t)|^p y'(t))' + \frac{\phi}{y^\lambda} = 0,\\ y(T) = h,\ y'(T) = k,\end{gathered} \tag{3}$$

where $h > 0$ and $k \geq 0$. We begin this section by investigating the positive solutions of (3) on intervals whose right endpoint is T.

Before proceeding it is helpful to rewrite (3) as an equivalent integral equation. Assuming that y is a smooth positive solution of (3) on the interval $(a, T]$, we integrate once to get

$$|y'(t)|^p y'(t) = k^{p+1} + \int_t^T \frac{\phi(s)}{y^\lambda(s)}\, ds,\ t \in (a, T].$$

Clearly, y' is nonnegative, so

$$y'(t) = \Big(k^{p+1} + \int_t^T \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}},\ t \in (a,T].$$

A second integration yields the equation

$$y(t) = h - \int_t^T \Big(k^{p+1} + \int_\tau^T \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}} d\tau,\ t \in (a,T]. \tag{4}$$

Our proofs will be based upon this representation of the problem. Several straightforward comments regarding (4) are collected in the following lemma.

Lemma 1 *Suppose that $y \in C(a,T]$ is a positive function satisfying (4). Then y is increasing and concave down, $y \in C[a,T] \bigcap C^1(a,T]$, $|y'|^p y' = (y')^{p+1}$ is differentiable a.e., and y satisfies (3) .*

Thus for any positive $y \in C(a,T]$ we define

$$Fy(t) = h - \int_t^T \Big(k^{p+1} + \int_\tau^T \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}} d\tau,$$

and it is clear that positive solutions of (3) correspond to fixed points of F.

Theorem 2 *Suppose that $T > 0$, $h > 0$, and $k \geq 0$. Then there exists an $a \in [0,T)$ and a unique positive $y \in C[a,T] \bigcup C^1(a,T]$ such that y is a solution of (3) on $[a,T]$. Moreover we may assume that the interval $[a,T]$ is maximal in the sense that either $a = 0$ or $y(a) = 0$.*

Proof: We begin by proving existence and uniqueness over some interval $[a_0,T]$. Our primary tool will be the Contraction Mapping Theorem. The estimates below will be useful in later arguments, so they are proved in slightly greater generality than is needed for this theorem. Suppose that $y \in C[a_0,T]$ such that $0 < m \leq y(t)$, where a_0 is to be determined. Then, for $t \in (a_0,T)$, we have

$$0 \leq (Fy)' - k = \Big(k^{p+1} + \int_t^T \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}} - \big(k^{p+1}\big)^{\frac{1}{p+1}} \leq \Big(\int_t^T \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}},$$

since $|b^{\frac{1}{p+1}} - a^{\frac{1}{p+1}}| \leq |b-a|^{\frac{1}{p+1}}$ for any $a, b \geq 0$. Therefore

$$k \leq (Fy)' \leq k + \Big(\frac{1}{m}\Big)^{\frac{\lambda}{p+1}} \Big(\int_t^T \phi\, ds\Big)^{\frac{1}{p+1}}.$$

Integrating from t to T yields

$$k(T-t) \leq h - Fy \leq k(T-t) + \Big(\frac{1}{m}\Big)^{\frac{\lambda}{p+1}} \int_t^T \Big(\int_\tau^T \phi\, ds\Big)^{\frac{1}{p+1}} d\tau. \tag{5}$$

Recall that $\phi \in L^1_{\text{loc}}(0,1)$. Thus, given any $\epsilon \in (0,h)$, we can choose a_0 close enough to T so that

$$k(T-t) \le h - Fy \le k(T-t) + \frac{\epsilon}{2}.$$

Further restricting a_0 so that $|a_0 - T| < \frac{\epsilon}{2k}$ we have $|Fy(t) - h| \le \epsilon$ for all $t \in [a_0, T]$. In particular, for $m = h - \epsilon$, and an appropriate choice of a_0, we get $F : \overline{B}_\epsilon(h) \to \overline{B}_\epsilon(h)$, where $\overline{B}_\epsilon(h) := \{y \in C[a_0,T] : |y(t) - h| \le \epsilon \; \forall t \in [a_0,T]\}$.

Now we show that, given a further restriction on a_0, F is a contraction on $\overline{B}_\epsilon(h)$. Assume that $y_1, y_2 \in C[a_0,T]$ such that $0 < m \le y_1(t), y_2(t) \le M < \infty$ for all $t \in [a_0,T]$. An application of the Mean Value Theorem implies

$$\begin{aligned} \big|(Fy_1)' - (Fy_2)'\big| &= \Big|\Big(k^{p+1} + \int_t^T \frac{\phi}{y_1^\lambda}\,ds\Big)^{\frac{1}{p+1}} - \Big(k^{p+1} + \int_t^T \frac{\phi}{y_2^\lambda}\,ds\Big)^{\frac{1}{p+1}}\Big|. \\ &\le \Big(\frac{1}{p+1}\Big)(c(t))^{\frac{1}{p+1}-1}\Big(\int_t^T \phi\Big|\frac{1}{y_1^\lambda} - \frac{1}{y_2^\lambda}\Big|\,ds\Big), \end{aligned}$$

where

$$k^{p+1} + \int_t^T \frac{\phi}{y_i^\lambda}\,ds \le c(t) \le k^{p+1} + \int_t^T \frac{\phi}{y_j^\lambda}\,ds,$$

and either $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$. In either case we see that

$$c(t) \ge \Big(\frac{1}{M}\Big)^\lambda \int_t^T \phi\,ds.$$

Substituting the minimal value for $c(t)$ leads to

$$\begin{aligned} &|(Fy_1)' - (Fy_2)'| \\ &\le \Big(\frac{1}{p+1}\Big)\Big(\frac{1}{M}\Big)^{\frac{\lambda}{p+1}-\lambda}\Big(\int_t^T \phi\,ds\Big)^{\frac{1}{p+1}-1}\Big(\int_t^T \phi\Big|\frac{1}{y_1^\lambda} - \frac{1}{y_2^\lambda}\Big|\,ds\Big), \end{aligned}$$

Apply the Mean Value Theorem a second time to get

$$\Big|\frac{1}{y_1^\lambda(t)} - \frac{1}{y_2^\lambda(t)}\Big| \le \lambda(c(t))^{-\lambda-1}|y_1(t) - y_2(t)|,$$

where $y_i(t) \le c(t) \le y_j(t)$ and either $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$. In either case we have $c(t) \ge m$. Substituting the minimal value for $c(t)$ leads to

$$|(Fy_1)' - (Fy_2)'| \le \Big(\frac{\lambda}{p+1}\Big)\Big(\frac{1}{M}\Big)^{\frac{\lambda}{p+1}-\lambda}\Big(\frac{1}{m}\Big)^{\lambda+1}\Big(\int_t^T \phi\,ds\Big)^{\frac{1}{p+1}}\|y_1 - y_2\|,$$

where $\|\cdot\|$ represents the sup-norm on $C[a_0,T]$. Integration leads to

$$|Fy_1 - Fy_2| \le \Big(\frac{\lambda}{p+1}\Big)\Big(\frac{1}{M}\Big)^{\frac{\lambda}{p+1}-\lambda}\Big(\frac{1}{m}\Big)^{\lambda+1}\int_t^T\Big(\int_\tau^T \phi\,ds\Big)^{\frac{1}{p+1}}d\tau\|y_1 - y_2\|. \tag{6}$$

For the purposes of this theorem we choose $m = h - \epsilon$ and $M = h + \epsilon$, and then refine our previous choice of a_0 to get

$$\|Fy_1 - Fy_2\| \leq \frac{1}{2}\|y_1 - y_2\|.$$

Hence F is a contraction on $\overline{B}_\epsilon(h)$, and we have established the existence of a unique local solution.

We now extend the solution to a maximal subinterval of $[0,T]$ by a standard form of argument. Suppose that $[a,T]$ is the maximal subinterval of $[0,T]$ such that (3) has a unique positive solution $y \in C[a,T] \bigcap C^1(a,T]$. If either $a = 0$ or $y(a) = 0$, we are done, so suppose that $a > 0$ and $y(a) > 0$. We know that $y'(t)$ is decreasing and nonnegative. Also

$$0 \leq y'(t) = \Big(k^{p+1} + \int_t^T \frac{\phi}{y^\lambda}\,ds\Big)^{\frac{1}{p+1}} \leq \Big(k^{p+1} + \big(\frac{1}{y(a)}\big)^\lambda \int_a^T \phi\,ds\Big)^{\frac{1}{p+1}} < \infty,$$

because $\phi \in L^1_{\rm loc}(0,1)$. Therefore $\lim_{t\to a^+} y'(t)$ exists and $y \in C^1[a,T]$. Using $h = y(a) > 0$ and $k = y'(a) \geq 0$, we apply the local existence result above to extend y to a unique positive solution of (3) over some interval $[a-\delta, T]$, contradicting the assumption that $[a,T]$ is maximal. The theorem is proved.

Observe that for the case $k > 0$ Theorem 1 follows from the standard existence and uniqueness theory for ordinary differential equations, because we can work on an interval where potential solutions are bounded away from the singularities. It was our preference to treat the case $k \geq 0$ as a whole. For the remainder of this section we focus primarily on the case $k = 0$, and invite the interested reader to generalize.

Before continuing we introduce some useful notation. We refer to the initial value problem (3) with the additional restriction $k = 0$ as $(3)_0$. Given $T \in (0,1)$, let y_h denote the positive solution of $(3)_0$ satisfying $y(T) = h$. We define

$$\begin{aligned}\mathcal{H}_T^- &:= \{h > 0 : y_h \text{ has maximal interval } [0,T]\},\\ h_T^- &:= \inf \mathcal{H}_T^-\end{aligned}$$

The $-$ sign is a reminder that we are currently working to the left of T.

Lemma 2 *If $\int_0^T \big(\int_s^T \phi\,ds\big)^{\frac{1}{p+1}}\,dt = \infty$ then $(3)_0$ has no positive solution whose maximal interval is $[0,T]$, i.e. $\mathcal{H}_T^-$ is empty.*

Proof: If $y \in C[0,T]$ is such a solution, then

$$\begin{aligned} y &= h - \int_t^T \Big(\int_\tau^T \frac{\phi}{y^\lambda}\,ds\Big)^{\frac{1}{p+1}} d\tau \\ &\leq h - \Big(\frac{1}{h}\Big)^{\frac{\lambda}{p+1}} \int_t^T \Big(\int_s^T \phi\,ds\Big)^{\frac{1}{p+1}} d\tau, \end{aligned} \tag{7}$$

because $y \leq h$ on $(0,T]$. It follows that $\lim_{t\to 0^+} y(t) = -\infty$, a contradiction.

An immediate consequence of this lemma is that $\int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$ is a necessary condition for the existence of a solution to (1). Similarly, $\int_T^1 \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$ is necessary. Hence

$$\int_0^1 \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$$

is necessary. Since ϕ is locally integrable is suffices to write this condition using $T = 1/2$.

Lemma 3 *Let* $K := \int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt$. *If* $K < \infty$ *then* $\mathcal{H}_T^-$ *is nonempty, and is bounded below by* $K^{\frac{p+1}{p+1+\lambda}}$

Proof: Let y be a solution of $(3)_0$. Apply inequality (5) with $m = \frac{h}{2}$, $k = 0$, and $Fy = y$ to get

$$0 \le h - y \le \left(\frac{2}{h}\right)^{\frac{\lambda}{p+1}} K.$$

For large h we have $0 \le h - y \le \frac{h}{4}$, so $y \ge \frac{3h}{4} > 0$. Thus if $y \ge \frac{h}{2}$ on an interval, then $y \ge \frac{3h}{4}$ on that interval. Hence, for large h, the solution remains above $\frac{3h}{4}$ no matter how far it is extended. Therefore the solution must have maximal interval $[0, T]$.

Substitute $t = 0$ into inequality (7) to get

$$0 \le y(0) \le h - \left(\frac{1}{h}\right)^{\frac{\lambda}{p+1}} K,$$

for $h \in \mathcal{H}_T^-$. This simplifies to

$$h \ge K^{\frac{p+1}{p+1+\lambda}}.$$

Before characterizing $\mathcal{H}_T^-$ and h_T^- further we need a comparison lemma.

Lemma 4 *Let* $T \in (0,1)$ *and any let* $y_1, y_2 \in C[a_0, T]$ *be solutions of (3) with* $0 < y_1(T) \le y_2(T)$ *and* $0 \le y_2'(T) \le y_1'(T)$. *Moreover, assume that at least one of these inequalities is strict. Then* $y_2 - y_1$ *is strictly decreasing in* $[a_0, T]$.

Proof: Since at least one of the inequalities is strict, we know that $y_1(t) < y_2(t)$ in some interval $(T - \delta, T]$. Suppose that there exists a $t_0 \in (a_0, T)$ such that $y_1(t_0) = y_2(t_0)$ and $y_1(t) < y_2(t)$ for $t \in (t_0, T)$. Therefore, for $t \in (t_0, T)$,

$$\begin{aligned} y_2'(t) &= \left((y_2'(T))^{p+1} + \int_t^T \frac{\phi}{y_2^\lambda}\, ds\right)^{\frac{1}{p+1}} \\ &< \left((y_1'(T))^{p+1} + \int_t^T \frac{\phi}{y_1^\lambda}\, ds\right)^{\frac{1}{p+1}} = y_1'(t). \end{aligned}$$

Thus $y_2 - y_1$ is strictly decreasing in $[t_0, T]$ with $y_2(T) - y_1(T) \le 0$, which contradicts $y_1(t_0) = y_2(t_0)$. Hence $y_1(t) < y_2(t)$ in $[a_0, T)$, and it follows, by the same inequality, that $y_2'(t) < y_1'(t)$ in $[a_0, T)$.

Corollary 1 *Let $T \in (0,1)$ and assume that $\int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$. If $h > h_T^-$ then $h \in \mathcal{H}_T^-$, and if $h < h_T^-$ then $h \notin \mathcal{H}_T^-$.*

Proof: We have already shown that $\mathcal{H}_T^-$ is nonempty and bounded below by $K^{\frac{p+1}{p+1+\lambda}}$, so $h_T^- > 0$. Suppose that $h_1 \in \mathcal{H}_T^-$ and $h_2 > h_1$. Let y_{h_1} and y_{h_2} represent the corresponding solutions of $(3)_0$. Then Lemma 4 shows that $y_{h_2} > y_{h_1}$ on any common interval of definition, and therefore $h_2 \in \mathcal{H}_T^-$. The lemma follows.

Given Lemma 4 and its consequences, it is reasonable to guess that the positive solution of $(3)_0$ satisfying $y(T) = h_T^-$ will have maximal interval $[0, T]$ and will satisfy $y(0) = 0$. This is indeed the case as the following lemmas will show.

Lemma 5 *Let $T \in (0,1)$ and assume that $\int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$. Let $y_1, y_2 \in C[a_0, T]$ be solutions of $(3)_0$ and $0 < m \le y_1(t) \le y_2(t) \le M$ for all $t \in [a_0, T]$. Then given any $\epsilon > 0$ there is a $\delta > 0$ such that if $|y_1(T) - y_2(T)| < \delta$, then $|y_1(t) - y_2(t)| < \epsilon$ for all $t \in [a_0, T]$.*

Proof: The proof will follow from an extension of estimate (6) to this special case. Notice that we can substitute y' for $(Fy)'$. Also, Lemma 4 shows that $y_2(t) - y_1(t)$ is positive and decreasing, so $\sup_{[t,T]} |y_2 - y_1| = (y_2(t) - y_1(t))$ on any $[t, T] \subset [a_0, T]$. Thus

$$0 \le y_1'(t) - y_2'(t) \le \Big(\frac{\lambda}{p+1}\Big)\Big(\frac{1}{M}\Big)^{\frac{\lambda}{p+1} - \lambda}\Big(\frac{1}{m}\Big)^{\lambda+1}\Big(\int_t^T \phi\, ds\Big)^{\frac{1}{p+1}}(y_2(t) - y_1(t)).$$

Let $C := \left(\frac{\lambda}{p+1}\right)\left(\frac{1}{M}\right)^{\frac{\lambda}{p+1} - \lambda}\left(\frac{1}{m}\right)^{\lambda+1}$, $P(t) := C\left(\int_t^T \phi\, ds\right)^{\frac{1}{p+1}}$ and $w := y_1 - y_2$. We restate the inequality above as

$$w'(t) + P(t)w(t) \le 0.$$

Thus

$$\left(e^{P(t)}w(t)\right)' \le 0,$$

and so

$$e^{P(T)}w(T) - e^{P(t)}w(t) \le 0.$$

Notice that $P(T) = 0$ and $w(T) = y_1(T) - y_2(T)$, so

$$e^{-P(t)}(y_1(T) - y_2(T)) \le y_1(t) - y_2(t) \le 0.$$

The assumption $\int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$ guarantees that $P(t)$ is bounded even in the case $a_0 = 0$. Therefore the lemma follows.

It is important to observe that the choice of δ in the previous proof depends upon ϵ, m, M and K, but does not depend upon the interval $[a_0, T]$.

Lemma 6 *Let $T \in (0,1)$ and assume that $\int_0^T \left(\int_s^T \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$. Then $h_T^- \in \mathcal{H}_T^-$, and the solution, y, of $(3)_0$ with $y(T) = h_T^-$ has maximal interval $[0,T]$ and satisfies $y(0) = 0$.*

Proof: Suppose that y has a maximal interval $[a, T]$ with $a > 0$. Recall that $y(a) = 0$ in this case. Let h_a^- be defined as before, and choose $a_1 \in (a, T)$ such that $0 < y(a_1) < h_a^-$. Let $m = y(a_1), M = h_T^- + 1$ and $\epsilon < h_a^- - y(a_1)$, and choose δ as in Lemma 5. Let y_δ represent the solution of $(3)_0$ satisfying $y_\delta(T) = h_T^- + \delta$. Since $y_\delta(T) > h_T^-$ we know that y_δ has maximal interval $[0,T]$, and by Lemma 5, we know that $y(a_1) < y_\delta(a_1) < h_a^-$. Since y_δ is increasing it follows that $y_\delta(a) < h_a^-$. Now compare y_δ to the solution, $\overline{y}$, of $(3)_0$ satisfying $\overline{y}(a) = y_\delta(a)$. It is clear that $y_\delta'(a) > 0$, so, by Lemma 4, we know that $y_\delta < \overline{y}$ on any common interval of definition. However, since $\overline{y}(a) < h_a^-$ we know that $\overline{y}$ has maximal interval $[\overline{a}, a]$ for some $\overline{a} \in (0, a)$ and we know $\overline{y}(\overline{a}) = 0$. Thus $y_\delta(\overline{a}) < 0$, a contradiction. Therefore $a = 0$, and $y \in C[0,T] \bigcap C^1(0,T]$.

Suppose that $y(0) = \alpha > 0$. Let $m = \frac{\alpha}{2}, M = h_T^-$ and $\epsilon = \frac{\alpha}{4}$, and choose δ as in Lemma 5. Let y_δ represent the solution of $(3)_0$ with $y_\delta(T) = h_T^- - \delta$. By Lemma 5 we know that if $\frac{\alpha}{2} \le y_\delta \le y \le h_T^-$ on some interval $[a_0, T]$, then $|y_\delta - y| \le \frac{\alpha}{4}$ and thus $\frac{3\alpha}{4} \le y_\delta \le y \le h_T^-$. It follows that y_δ is bounded below by $\frac{3\alpha}{4}$ on any subinterval of $[0,T]$, and so its maximal interval is $[0,T]$. However, this contradicts the fact that $y_\delta(T) = h_T^- - \delta \notin \mathcal{H}_T^-$. The theorem is proved.

It is easy to show, by Lemma 4, that h_T^- is the unique $h \in \mathcal{H}_T^-$ such that the associated solution satisfies $y(0) = 0$. We refer to the solution, y, of $(3)_0$ such that $y(T) = h_T^-$ as the *left half solution* on $[0,T]$. By identical arguments, which we omit, we introduce the quantity h_T^+ and the *right half solution* on the interval $[T, 1]$.

The remaining lemmas in this section will prove that $h_T^- = h_T^+$ for exactly one T. This will allow us to join the two half solutions to create the unique positive solution of (1).

Lemma 7 *h_T^- is a monotone increasing function in T.*

Proof: Let $T_1, T_2 \in (0,1)$ such that $T_2 < T_1$. Let y_1, y_2 represent the left half solutions on $[0,T_1]$ and $[0,T_2]$, respectively. Suppose that $y_2(T_2) = h_{T_2}^- \ge h_{T_1}^- = y_1(T_1)$. Since y_1 is increasing and y_1' is decreasing, it is clear that $y_1(T_2) < y_2(T_2)$ and $y_1'(T_2) > 0 = y_2'(T_2)$. By Lemma 4 we know that $y_2 - y_1$ is strictly decreasing, which implies that $y_1(0) < y_2(0) = 0$, a contradiction. Hence $h_{T_2}^- < h_{T_1}^-$.

Lemma 8 *h_T^- is a continuous function of T.*

Proof: Let $\{T_n\} \subset (0,1)$ be an increasing sequence that converges to $T \in (0,1)$, and let $\{y_n\}$ and y represent the corresponding left half solutions. From the previous lemma we know that $\{h_{T_n}^-\}$ is monotone increasing and bounded above by h_T^-, and thus converges to some $h \le h_T^-$.

Observe that for $m < n$ we can compare y_m and y_n on the interval $[0, T_m]$. We know that $y_n'(T_m) < 0$. If $y_n(T_m) \le h_{T_n}^-$, then Lemma 4 implies that $y_n < y_m$ in $[0, T_m)$. Thus $y_n(0) < 0$, a contradiction. Hence $y_n(T_m) > h_{T_m}^-$, A similar contradiction arises if $y_n(t) = y_m(t)$ for any $t \in (0, T_m]$, so $y_m(t) < y_n(t)$ for $t \in (0, T_m]$.

In order to have a common interval for comparison we define

$$\overline{y}_n(t) := \begin{cases} y_n(t), 0 \le T_n \\ h_{T_n}, T_n < t \le T \end{cases}$$

It is clear that $\overline{y}_n \in C[0,T] \bigcap C^1(0,T]$ and that for $m < n$ we have $\overline{y}_m(t) < \overline{y}_n(t) \le y(t)$ in $(0,T]$. Thus $\{\overline{y}_n(t)\}$ is bounded and increasing for any $t \in (0,T]$ and we can define $f(t) = \lim_{n\to\infty} \overline{y}_n(t)$. Moreover, if $t < T$, then $t < T_n$ for all n large enough and we have $f(t) = \lim_{n\to\infty} y_n(t)$.

Next we argue that the convergence is better than pointwise. For all $t \ge t_0 > 0$ we have

$$0 \le \overline{y}_n'(t) \le y_n'(t_0) = \Big(\int_{t_0}^{T} \frac{\phi}{y_n^\lambda}\, ds \Big)^{\frac{1}{p+1}} \le \left(\frac{1}{y_1(t_0)} \right)^{\frac{\lambda}{p+1}} \Big(\int_{t_0}^{T} \phi\, ds \Big)^{\frac{1}{p+1}} < \infty$$

Thus, by the Arzela-Ascoli theorem, me may assume, without loss of generality, that $\{\overline{y}_n\}$ converges uniformly to f on compact subsets of $(0,T]$.

Thus $f \in C(0,T]$ and for $t \in (0,T)$

$$\begin{aligned} f(t) &= \lim_{n\to\infty} y_n(t) \\ &= \lim_{n\to\infty} \Big(h_{T_n}^- - \int_t^{T_n} \Big(\int_\tau^{T_n} \frac{\phi}{y_n^\lambda}\, ds \Big)^{\frac{1}{p+1}} d\tau \Big) \\ &= h - \int_t^T \Big(\int_\tau^T \frac{\phi}{f^\lambda}\, ds \Big)^{\frac{1}{p+1}} d\tau. \end{aligned}$$

Hence f is a solution of $(3)_0$ with maximal interval $[0,T]$. Thus $h \in \mathcal{H}_T^-$ and $h \ge h_T^-$. But we already proved that $h \le h_T^-$ so $h = h_T^-$, and thus $\lim_{n\to\infty} h_{T_n}^- = h_T^-$. Moreover, $f(t) \equiv y(t)$.

If $\{T_n\}$ is a sequence converging to T from the right, then an analagous argument holds. In fact the argument is simpler, because there is no need to extend the functions to a common interval. We omit this part of the proof. The lemma is proved.

Lemma 9 $\lim_{T\to 0^+} h_T^- = 0$.

Proof: Let y_1 represent the left half solution of $(3)_0$ over $[0, \frac{1}{2}]$, and let y_2 represent the left half solution of $(3)_0$ over $[0, T]$ where $T \in (0, \frac{1}{2})$. Clearly, $y_1'(T) > 0 = y_2'(T)$. If $y_1(T) \leq h_T^- = y_2(T)$, then Lemma 4 implies that $y_1(0) < y_2(0) = 0$, a contradiction. Thus $0 \leq h_T^- < y_1(T)$. Hence $\lim_{T\to 0^+} h_T^- = 0$.

Identical arguments show that h_T^+ is continuous and decreasing in $[0, 1)$ with $\lim_{T\to 1^-} h_T^+ = 0$. Therefore there is a unique $T \in (0, 1)$ such that $h_T^- = h_T^+$. For this T let y_0 represent the left half solution over $[0, T]$ and let y_1 represent the right half solution over $[T, 1]$, and define

$$y(t) := \begin{cases} y_0(t), & t \in [0, T] \\ y_1(t), & t \in (T, 1] \end{cases}$$

This y is the unique solution of (1). Thus Theorem 1 has been proved.

3 Boundary Behavior

In this section we assume throughout that $\int_0^{1/2} \left(\int_s^{1/2} \phi\, ds\right)^{\frac{1}{p+1}} dt < \infty$ is satisfied, and we investigate the boundary behavior of the unique positive solution of (1). More specifically, we concentrate on behavior near 0 for left half solutions of $(3)_0$. Similar results apply for the behavior of right half solutions near 1.

We begin with the question of whether the slope at the boundary is finite or infinite. Recall that the solution, y, is concave down, so the quantity $y'(0) = \lim_{t\to 0^+} y'(t) \in (0, \infty]$ is well-defined.

Theorem 3 *Let y be the left half solution of $(3)_0$ on $[0, T]$. Then $y'(0)$ is finite if and only if $\int_0^{1/2} \frac{\phi}{t^\lambda}\, dt < \infty$.*

Proof: Assume that $y'(0) = A < \infty$. We know that

$$(y'(t))^{p+1} = \int_t^T \frac{\phi}{y^\lambda}\, ds,$$

and that $0 \leq y'(t) \leq A$ and thus $0 \leq y(t) \leq At$ in $[0, T]$. Therefore

$$(y'(t))^{p+1} \geq \left(\frac{1}{A^\lambda}\right) \int_t^T \frac{\phi}{s^\lambda}\, ds \geq 0,$$

and it follows that

$$A^{p+1+\lambda} \geq \int_0^T \frac{\phi}{t^\lambda}\, ds.$$

Hence $\int_0^{1/2} \frac{\phi}{y^\lambda}\, dt < \infty$.

Assume $y'(0) = \infty$. Let $A > 0$ such that $y(t) \geq At$ on $[0, T]$. Then

$$(y'(t))^{p+1} \leq \frac{1}{A^\lambda} \int_t^T \frac{\phi}{s^\lambda}\, ds.$$

Thus

$$\lim_{t\to 0^+}\int_t^T \frac{\phi}{s^\lambda}\,ds \geq A^\lambda \lim_{t\to 0^+}(y'(t))^{p+1} = \infty.$$

For the finite slope case we derive asymptotic formulae for y that are similar to those in [8].

Theorem 4 *Let y be a left half solution of (3)$_0$ on $[0,T]$ with $y'(0)=A<\infty$. Then*

$$y'(t) = \left(\frac{1}{A}\right)^{\frac{\lambda}{p+1}} \Big(\int_t^T \frac{\phi}{s^\lambda}(1+o(1))\,ds\Big)^{\frac{1}{p+1}},$$

and

$$y(t) = \left(\frac{1}{A}\right)^{\frac{\lambda}{p+1}} \int_0^t \Big(\int_\tau^T \frac{\phi}{s^\lambda}(1+o(1))\,ds\Big)^{\frac{1}{p+1}}\,d\tau\,.$$

Proof: Substitute $\frac{1}{y^\lambda(s)} = \frac{1}{(As)^\lambda}(1+o(1))$.

For the remainder of this section we concentrate on the infinite slope case and assume $\int_0^{1/2}\frac{\phi}{t^\lambda}\,dt=\infty$. The following theorem provides a general tool for comparing the boundary behavior of solutions.

Theorem 5 *Let $\psi\in L^1_{\rm loc}(0,\delta)$ be a positive function satisfying $\int_0^\delta \frac{\psi}{t^\lambda}\,dt=\infty$, $\int_0^\delta\Big(\int_t^\delta \psi\,ds\Big)^{\frac{1}{p+1}}\,dt<\infty$, and $\lim_{t\to 0^+}\frac{\psi}{\phi}=1$. Let y be the left half solution of (3)$_0$ on $[0,T]$, and let z be a particular solution of*

$$\begin{aligned}&(|z'|^p z')' + \frac{\psi}{z^\lambda} = 0 \ \textit{in}\ (0,\delta),\\ &z(0)=0.\end{aligned}$$

Then $\displaystyle\lim_{t\to 0^+}\frac{z'}{y'} = \lim_{t\to 0^+}\frac{z}{y} = 1.$

Before proving Theorem 5 we provide an interesting application.

Corollary 2 *Assume that $\lim_{t\to 0^+}\frac{ct^r}{\phi}=1$ where $c>0$ and $-p-2<r\leq\lambda-1$. Let y be the left half solution of (3)$_0$ on $[0,T]$, and let $z(t)=\gamma t^\rho$ such that $\rho=\frac{r+p+2}{\lambda+p+1}$ and $\gamma=\left(\frac{c}{\rho^{p+1}(1-\rho)}\right)^{\frac{1}{\lambda+p+1}}$. Then $\lim_{t\to 0^+}\frac{z'}{y'}=\lim_{t\to 0^+}\frac{z}{y}=1$.*

Proof: The restrictions on r guarantee that $\psi=ct^r$ satisfies

$$\int_0^1\Big(\int_t^{1/2}\psi\,ds\Big)^{\frac{1}{p+1}}\,dt<\infty$$

and $\int_0^{1/2} \frac{\psi}{\tau^\lambda} = \infty$. It is straight forward to check that $z(t)$ is a solution of

$$
\begin{gathered}
(|z'|^p z')' + \frac{\psi}{z^\lambda} = 0 \quad \text{in } (0,\infty), \\
z(0) = 0\,.
\end{gathered}
$$

Hence the result follows from Theorem 5.

The proof of Theorem 5 depends upon several lemmas.

Lemma 10 *Let $\psi \in L^1_{\rm loc}(0,\delta)$ be a positive function satisfying $\int_0^\delta \frac{\psi}{t^\lambda}\,dt = \infty$, and $\int_0^\delta \left(\int_t^\delta \psi\,ds\right)^{\frac{1}{p+1}} dt < \infty$. Let let z_1 and z_2 be solutions of*

$$
\begin{gathered}
(|z'|^p z')' + \frac{\psi}{z^\lambda} = 0 \quad \textit{in } (0,\delta), \\
z(0) = 0.
\end{gathered}
$$

Then $\displaystyle\lim_{t\to 0^+} \frac{z_1'}{z_2'} = \lim_{t\to 0^+} \frac{z_1}{z_2} = 1.$

Proof: Assume that z_1 and z_2 are distinct solutions with $z_1(t) < z_2(t)$ at some point. If $z_1'(t) \geq z_2'(t)$ at this same point, then, by Lemma 4, $z_2 - z_1$ must be decreasing on $(0,t)$. But this leads to a contradiction of $z_1(0) = z_2(0) = 0$. Thus $z_1'(t) < z_2'(t)$. It follows that $0 \leq z_1'(t) < z_2'(t)$ and $0 < z_1(t) < z_2(t)$ in $[0,\delta]$. Moreover,

$$
\begin{aligned}
z_2'(t) &= \Big((z_2'(\delta))^{p+1} + \int_t^\delta \frac{\psi}{z_2^\lambda}\,ds\Big)^{\frac{1}{p+1}} \\
&< \Big((z_2'(\delta))^{p+1} + \int_t^\delta \frac{\psi}{z_1^\lambda}\,ds\Big)^{\frac{1}{p+1}} \\
&= \Big((z_2'(\delta))^{p+1} + (z_1'(t))^{p+1} - (z_1'(\delta))^{p+1}\,ds\Big)^{\frac{1}{p+1}}.
\end{aligned}
$$

Thus

$$
1 < \frac{z_2'(t)}{z_1'(t)} < \Big(\big(\frac{z_2'(\delta)}{z_1'(t)}\big)^{p+1} + 1 - \big(\frac{z_1'(\delta)}{z_1'(t)}\big)^{p+1}\Big)^{\frac{1}{p+1}}.
$$

Since $\int_0^\delta \frac{\psi}{\tau^\lambda}\,d\tau = \infty$, we know that $\lim_{t\to 0^+} z_1'(t) = \infty$. Hence $\lim_{t\to 0^+} \frac{z_2'(t)}{z_1'(t)} = 1$. By L'Hospital's rule we have $\lim_{t\to 0^+} \frac{z_2(t)}{z_1(t)} = 1$. The proof is done.

Lemma 11 *Let $\psi \in L^1_{\rm loc}(0,\delta)$ be a positive function satisfying $\int_0^\delta \frac{\psi}{t^\lambda}\,dt = \infty$, $\int_0^\delta \Big(\int_s^\delta \psi\,ds\Big)^{\frac{1}{p+1}} dt < \infty$, and $\psi \geq \phi$ ($\psi \leq \phi$). Let y be the left half solution of (3)$_0$ on $[0,T]$, and let z be a particular solution of*

$$
\begin{gathered}
(|z'|^p z')' + \frac{\psi}{z^\lambda} = 0 \quad \textit{in } (0,\delta), \\
z(0) = 0.
\end{gathered}
$$

Then $\liminf_{t\to 0^+} \frac{z}{y} \geq 1$ $(\limsup_{t\to 0^+} \dfrac{z}{y} \leq 1)$.

Proof: By the previous lemma we know that all solutions of

$$\begin{gathered}(|z'|^p z')' + \frac{\psi}{z^\lambda} = 0 \quad \text{ in } (0,\delta),\\ z(0) = 0,\end{gathered}$$

have asymptotically identical boundary behavior. Thus we may compare y to the particular solution where $z'(\delta) = y'(\delta)$.

Suppose that $y(t) > z(t)$ at some point in $(0, \delta)$. Then $y(t) > z(t)$ in some interval $(a_0, b_0) \subset (0, \delta)$ such that $z'(b_0) \geq y'(b_0)$. For $t \in (a_0, b_0)$ we have

$$\begin{aligned} lly'(t) &= \Big((y'(b_0))^{p+1} + \int_t^{b_0} \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}} \\ &< \Big((z'(t_0))^{p+1} + \int_t^{b_0} \frac{\psi}{z^\lambda}\, ds\Big)^{\frac{1}{p+1}} \\ &= z'(t)\,. \end{aligned}$$

Thus $y - z$ is decreasing in (a_0, b_0). Thus the maximal interval where $y > z$ must be $(0, b_0)$, and, by the same argument, $y - z$ is decreasing on $(0, b_0)$. But this implies $0 = y(0) - z(0) < y(b_0) - z(b_0) \leq 0$, a contradiction. Hence $z(t) \geq y(t)$ in $[0, \delta]$. The case $\psi \leq \phi$ can be argued similarly so the lemma is proved.

Proof of Theorem 5: Let $\epsilon > 0$ be given. Observe that for $c > 0$ we have that $w = cy$ is a left half solution on $[0, T]$ of the problem

$$\begin{gathered}(|w'|^p w')' + \frac{c^{\lambda+p+1}\phi_1}{w^\lambda} = 0,\\ w(T) = h,\ w'(T) = 0,\end{gathered} \tag{8}$$

Also, without loss of generality, we may assume that $(1-\epsilon)^{\lambda+p+1}\phi \leq \psi \leq (1+\epsilon)^{\lambda+p+1}\phi$ in $(0, \delta)$. Therefore, the previous lemma and the observation about (8) show that $\liminf_{t\to 0^+} \frac{z}{(1-\epsilon)y} \geq 1$ and $\limsup_{t\to 0^+} \frac{z}{(1+\epsilon)y} \leq 1$. Hence $\lim_{t\to 0^+} \frac{z}{y} = 1$.

By further restricting the size of the interval $(0, \delta)$ we may now assume that $(1-\epsilon) \leq \frac{\phi z^\lambda}{\psi y^\lambda} \leq (1+\epsilon)$. We have

$$\begin{aligned} y'(t) &= \Big((y'(\delta))^{p+1} + \int_t^{\delta} \frac{\phi}{y^\lambda}\, ds\Big)^{\frac{1}{p+1}} \\ &= \Big((y'(\delta))^{p+1} + \int_t^{\delta} \frac{\psi}{z^\lambda}\frac{\phi z^\lambda}{\psi y^\lambda}\, ds\Big)^{\frac{1}{p+1}}. \end{aligned}$$

Thus

$$\begin{aligned}\Big((y'(\delta))^{p+1}+\frac{1}{(1+\epsilon)}\int_t^\delta \frac{\psi}{z^\lambda}\,ds\Big)^{\frac{1}{p+1}} &\leq y'(t)\\ &\leq \Big((y'(\delta))^{p+1}+\frac{1}{(1-\epsilon)}\int_t^\delta \frac{\psi}{z^\lambda}\,ds\Big)^{\frac{1}{p+1}}.\end{aligned}$$

Divide this inequality through by $z'(t) = \big((z'(\delta))^{p+1} + \int_t^\delta \frac{\psi}{z^\lambda}\,ds\big)^{\frac{1}{p+1}}$ and let $t \to 0^+$ to get

$$\left(\frac{1}{1+\epsilon}\right)^{\frac{1}{p+1}} \leq \liminf_{t\to 0^+}\frac{y'(t)}{z'(t)} \leq \limsup_{t\to 0^+}\frac{y'(t)}{z'(t)} \leq \left(\frac{1}{1-\epsilon}\right)^{\frac{1}{p+1}}.$$

Hence $\lim_{t\to 0^+}\frac{z'(t)}{y'(t)} = 1$, and Theorem 5 is proved.

In [8] Taliaferro applies a result similar to Theorem 5 to obtain a precise description of boundary behavior for a more general collection of functions ϕ than those described in Corollary 2. Taliaferro assumes that $\lim_{t\to 0^+}\frac{tf''(t)}{f'(t)} = R$, where $f(t) := \int_t^T \psi/s^\lambda\,ds$ and where ψ is a smooth function such that $\lim_{t\to 0^+}\psi/\phi = 1$. We note that if $tf''(t)/f'(t) \equiv R$ then one can show that ψ is of the form ct^r, so Corollary 2 is applicable. Taliaferro's more general condition implies that for any $\epsilon > 0$ ψ is bounded between some $ct^{r-\epsilon}$ and $ct^{r+\epsilon}$ in some neighborhood of 0. Thus ψ still behaves much like ct^r. Our methods can be used to find corresponding estimates on the boundary behavior of the solution, but we have not generalized Taliaferro's argument and results to this case.

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