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Spectral Riesz–Cesàro means: How the square root function helps us to see around the world *

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Abstract

The heat-kernel expansion for a nonanalytic function of a differential operator, and the integrated (Cesàro-smoothed) spectral densities associated with the corresponding nonanalytic function of the spectral parameter, exhibit a certain nonlocal behavior. Because of this phenomenon, care is needed in applying the pseudodifferential symbolic calculus to nonanalytic functions. We demonstrate this effect by both analytical and numerical calculations for the square root of the Laplace operator in the model of a "twisted" scalar field over the circle. This shows that the effect cannot be attributed solely to boundaries.

1 Motivation

In the late seventies many physicists working in quantum gravity or gauge theories learned about the symbolic calculus of pseudodifferential operators, primarily from the writings of Peter Gilkey [11]. In particular, let H be a positive self-adjoint second-order linear differential operator (on scalars, for simplicity). Then the small-time asymptotic expansion of the heat kernel (on diagonal) can be calculated from the high-frequency expansion of the "resolvent symbol" σ by the relation

$$e^{-tH}(x,x) \sim \int_{T_x^*} d\xi \int_C d\lambda\, e^{-t\lambda}\sigma(x,\xi,\lambda). \tag{1}$$

(C is an appropriate complex contour, and T_x^* is "momentum space" relativized to x so that the formula will make sense on a manifold. $A(x,y)$ is the integral kernel of the operator A (written $\langle x|A|y\rangle$ in Dirac notation).)

It is tempting to extrapolate this construction to such objects as

$$e^{-t\sqrt{H}}(x,x) \sim \int_{T_x^*} d\xi \int_C d\lambda\, e^{-t\sqrt{\lambda}}\sigma(x,\xi,\lambda) \tag{2a}$$

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or

$$\sqrt{H}e^{-tH}(x,x) \sim \int_{T_x^*} d\xi \int_C d\lambda\, \sqrt{\lambda} e^{-t\lambda} \sigma(x,\xi,\lambda). \tag{2b}$$

Then (2b) or the time derivative of (2a) would give a regularized expression for (at least part of) the vacuum energy density of the quantum field theory with field equation $H\phi = -\frac{\partial^2 \phi}{\partial t^2}$, from which the exact energy could be obtained by renormalization and taking the limit $t \downarrow 0$ [20, 14].

However, if this procedure were correct in general, there would be no Casimir effect! The asymptotic series $\sigma(x,\xi,\lambda)$ is completely determined by the symbol of H *at the point* x (that is, the coefficients in the differential operator H and their derivatives at the point); it is merely a *parametrix*, not an exact formula for the resolvent operator $(H-\lambda)^{-1}$. Therefore, the small-t expansions of (1) and (2) also are purely *local* expressions. Now consider the system with $H = -\nabla^2$ and $M = I \times \mathbb{R}^{m-1}$, where I is a finite interval furnished with suitable boundary conditions. Then the expansions of (1) and (2) are identical to those for $M = \mathbb{R}^m$ (namely, all terms after the first are zero). It would therefore appear that the renormalized vacuum energy is zero, contrary to the well-known direct calculations for such models ([3], [5], ...).

There are actually two kinds of "Casimir" energy in the scalar-field scenario just described, with $m = 3$ and I an interval of length L. First, there is a constant energy density throughout the space, proportional to L^{-4}. This is also present (with a different coefficient) if the boundary conditions are periodic — i.e., there actually is no boundary. It is not visible at all in the heat-kernel expansion, local or integrated. Second, there is a buildup of energy near the boundary, with an x^{-4} dependence of energy density on distance from the boundary. This contribution also does not show up in the local heat-kernel expansion in the interior of M (which is not uniform in x). However, its effect on the total energy, through the density of eigenvalues, can be calculated by expanding the exact heat kernel in distance from the boundary, integrating, and only then expanding in t (cf. [18]). Both these effects can be calculated either from the expansion of the field in normal modes or by the construction of Green functions by the method of images. (More details have been reviewed in [7, 10, 9].)

It should be noted that the local expansion of the heat kernel (1) *is correct* in Casimir-type situations, and it gives the correct counterterms for renormalizing effective actions, for instance. From the point of view of solutions by the method of images, the difference between the heat kernel and the objects (2) is that the off-diagonal heat kernel for $M = \mathbb{R}^m$ decreases exponentially as $t \downarrow 0$,

$$e^{-tH}(x,y) \le \text{const.} \times e^{-\|x-y\|^2/4t},$$

while most such Green functions do not. (We give a detailed example in the next section.) It is thus a miraculous accident that image terms *for the heat kernel* do not contribute to the small-t expansion, rather than a surprise that image terms *do* contribute to such expansions for more general functions such as (2).

Our purpose here is to explore the true limitations of the calculation of the local energy density by (2). One approach would be to examine carefully the proof of (1) to find where it breaks down in more general contexts; in fact, the square-root function's branch point at the origin is critical here, as follows from one of the theorems of [7]. Perhaps more compelling in practice, however, would be explicit examples of nonlocality in solvable models. Among some physicists there is an impression that such effects are limited to spaces that are "cluttered" by boundaries (or perfect conductors, in electromagnetism), so we want an example without a boundary.

2 A Model

The operator considered is

$$H = -\frac{d^2}{dx^2} \tag{3}$$

($m = 1$). It acts on functions defined on the interval $(0, L)$ with "boundary" conditions

$$u(L) = e^{i\theta}u(0), \qquad u'(L) = e^{i\theta}u'(0), \tag{4}$$

where θ is a fixed parameter in $[0, 2\pi)$. There is actually no boundary: The functions u are sections of a line bundle over the circle that is completely homogeneous spatially. (In particular, for $\theta = \pi$ we have antiperiodic boundary conditions, alias the Möbius strip.)

The eigenfunctions of this operator are

$$u_n(x) = \frac{1}{\sqrt{L}} e^{ik_n x}, \tag{5}$$

where n is any integer and

$$k_n = \frac{2\pi n + \theta}{L}\,. \tag{6}$$

The corresponding eigenvalues are

$$\lambda_n = {k_n}^2 = \frac{(2\pi n + \theta)^2}{L^2}\,. \tag{7}$$

The integral kernel of the operator $e^{-t\sqrt{H}}$ has been called the *cylinder kernel* because it solves Laplace's equation on a semi-infinite cylinder with boundary data on its base, the manifold M (here, the circle). (It is also the heat kernel of the first-order *pseudodifferential* operator $\sqrt{H}$, but our main point is that it does not behave like the familiar heat kernel of a second-order *differential* operator.) This kernel is

$$\begin{aligned} T(t, x, y) &\equiv e^{-t\sqrt{H}}(x, y) \\ &= \sum_{n=-\infty}^{\infty} e^{-t\sqrt{\lambda_n}} u_n(x) u_n^*(y) \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \frac{e^{-t|k_n|}}{L}\, e^{\frac{i(2\pi n+\theta)(x-y)}{L}}. \tag{8}$$

It can be broken into a sum of geometric series and thus can be evaluated in terms of elementary functions:

$$T(t,x,y) = \frac{e^{\frac{i\theta(x-y)}{L}}\left(\sinh\frac{(2\pi-|\theta|)t}{L} + \sinh\frac{|\theta|t}{L}\cos\frac{2\pi(x-y)}{L} - i\sinh\frac{\theta t}{L}\sin\frac{2\pi(x-y)}{L}\right)}{L(\cosh\frac{2\pi t}{L} - \cos\frac{2\pi(x-y)}{L})}. \tag{9}$$

We observe the following properties and limiting cases:

- Complex conjugation and the interchange $x \leftrightarrow y$ give the same result, as should be the case for matrix elements of an Hermitian operator.
- If x is advanced by Lm, then $T(t,x,y)$ is multiplied by $e^{i\theta m}$, as expected.
- If $\theta = 0$, then (9) coincides with formula (17) of [10] for the cylinder heat kernel with periodic boundary conditions (except for a redefinition of L by a factor 2).
- If $L \to \infty$, then the cylinder kernel becomes

  $$\frac{t}{\pi(t^2+(x-y)^2)}. \tag{10}$$

  This coincides with formula (11) of [10] for the cylinder heat kernel on the real line.
- For $t \to 0$, $T(t,x,y)$ is

  $$\begin{cases} O(t) & \text{if } x \neq y + Lm, \\ \frac{e^{i\theta m}}{\pi t} & \text{if } x = y + Lm. \end{cases} \tag{11}$$

- Most significantly, for $x = y$ the cylinder kernel is real and is equal to

  $$T(t,x,x) = \frac{\sinh\frac{(2\pi-|\theta|)t}{L} + \sinh\frac{|\theta|t}{L}}{L\left(\cosh\frac{2\pi t}{L} - 1\right)}. \tag{12}$$

  As $t \downarrow 0$ this function has an expansion whose first two terms are

  $$T(t,x,x) \sim \frac{1}{\pi t} + \frac{(2\pi^2 - 6\pi|\theta| + 3|\theta|^2)t}{6\pi L^2} + \cdots. \tag{13}$$

  If $\theta = 0$, then this is

  $$\frac{1}{\pi t} + \frac{\pi t}{3L^2},$$

  which agrees with formula (18) of [10] for the cylinder kernel expansion on a circle.

In contrast, the heat kernel for this operator is

$$\begin{aligned} K(t,x,y) &\equiv e^{-tH}(x,y) \\ &= \sum_{n=-\infty}^{\infty} \frac{e^{-t\lambda_n}}{L} e^{\frac{i(2\pi n+\theta)(x-y)}{L}}. \end{aligned} \tag{14}$$

The alternative form

$$K(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} e^{-\left(\frac{(x-y+Ln)^2}{4t}+in\theta\right)} \tag{15}$$

is obtained from (14) by the Poisson sum formula, or directly as an image sum from the heat kernel for the real line. Unlike T, the K of this model cannot be expressed through elementary functions.

The natural counterparts of the first four properties and limits stated above hold. The contrast comes in the fifth and sixth properties:

- For $t \to 0$, $K(t,x,y)$ is

$$\begin{cases} O(t^\infty) & \text{if } x \neq y + Lm, \\ \frac{e^{i\theta m}}{\sqrt{4\pi t}} + O(t^\infty) & \text{if } x = y + Lm. \end{cases} \tag{16}$$

- For $x = y$ the heat kernel is real and is equal to

$$K(t,x,x) = \frac{\theta_3\left(\frac{\theta}{2}, e^{\frac{-L^2}{4t}}\right)}{\sqrt{4\pi t}}, \tag{17}$$

where [15], for $|q| < 1$,

$$\theta_3(u,q) \equiv \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2nu). \tag{18}$$

It is clear from (16) that the diagonal heat kernel expansion is local and does not "feel" the finiteness of space (L) or the nontrivial bundle boundary condition (θ). That expansion is the same as for ordinary periodic boundary conditions or for the whole line. On the other hand, the corresponding expansion (13) for the cylinder kernel contains this nonlocal information, because it manifestly depends on θ and L. Those parameters are not part of the pseudodifferential resolvent-symbol expansion, so the latter is inherently incapable of providing (13).

Note also that (13) is uniform (in fact, constant) in x, so it can be integrated rigorously to give eigenvalue asymptotics that depend on θ. The latter will be investigated directly in Sec. 4.

Here we have concentrated on the cylinder kernel, (2a). Similar calculations can be conducted for the kernel (2b), with similar conclusions; details will be published elsewhere.

3 General Theory

We need to put the previous discussion into a general framework of Riesz–Cesáro means for eigenvalue distributions and other spectral functions. We will be brief, referring to earlier papers for the details [9, 6, 10, 7, 8]. Some aspects of the subject were known to Hörmander by 1968 [17], and others to Riesz and Hardy by 1916 [16]. One of the most recent related papers is by Gilkey and Grubb [13]; it is primarily concerned with the logarithmic terms that, in the most general circumstances, accompany those power terms that we call "nonlocal", but its Theorem 1.7 proves this nonlocality property in generality. (We should emphasize, incidentally, that even when the coefficient of such a logarithmic term turns out to be zero, the accompanying pure power term is generally present and is not locally determined by the symbol of the [pseudo]differential operator involved.)

Consider a function defined by a Stieltjes integral with respect to some measure:

$$f(\lambda) \equiv \int_0^\lambda a(\sigma)\, d\mu(\sigma). \tag{19}$$

Its iterated indefinite integrals can be expressed as single integrals,

$$\begin{aligned} \partial_\lambda^{-\alpha} f(\lambda) &\equiv \int_0^\lambda d\sigma_1 \cdots \int_0^{\sigma_{\alpha-1}} d\sigma_\alpha\, f(\sigma_\alpha) \\ &= \frac{1}{\alpha!} \int_0^\lambda (\lambda - \sigma)^\alpha\, df(\sigma), \end{aligned} \tag{20}$$

and then its *Riesz means* are defined by dividing by the volume of the simplex of size λ:

$$R_\lambda^\alpha f(\lambda) \equiv \alpha!\, \lambda^{-\alpha} \partial_\lambda^{-\alpha} f(\lambda) = \int_0^\lambda \Bigl(1 - \frac{\sigma}{\lambda}\Bigr)^\alpha\, df(\sigma). \tag{21}$$

$R_\lambda^\alpha f$ is to be thought of as an averaged or smoothed version of f itself. In our applications, $\mu(\lambda)$ is $N(\lambda)$ (the number of eigenvalues less than or equal to λ) or $E_\lambda(x,y)$ (the kernel of the spectral projection) for some self-adjoint operator H. Then one can study the spectral quantities themselves by taking $a = 1$, $f = \mu$; these Riesz means have asymptotic expansions that give rigorous meaning to the formal high-energy expansions of μ obtained from the rigorous asymptotics of the corresponding heat kernel by formally inverting the Laplace transform. Alternatively, taking $a(\sigma) = e^{-i\sigma t}$ (for instance) and letting $\lambda \to \infty$ with $\alpha > 0$ fixed yields a generalization (to possibly continuous spectrum) of the classic Cesàro summation of the eigenfunction expansion for the Green function of the time-dependent Schrödinger equation (for instance).

Of principal importance to us is the matter of *Riesz means with respect to different variables*. Let

$$\lambda \equiv \omega^2 \tag{22}$$

and consider the new set of means, $R_\lambda^\alpha f$, defined by (20)–(21) with integration over ω instead of λ. When one works out the relationship between the $R_\lambda^\alpha f$

and the $R^\beta_\omega f$ [16, 17, 10], one finds that the asymptotic behavior as $\lambda \to \infty$ of the R_λ quantities is entirely determined by that of the R_ω quantities, but the asymptotics of the R_ω depends on integrals of the R_λ over all (arbitrarily small) λ. Put another way, in passing from λ-means to ω-means one encounters constants of integration from the lower limit ($\lambda = 0$) that *cannot, in principle, be determined from the large-λ behavior*; in going the other way, there is a purely algebraic (nonintegral) relationship, and furthermore, the extra constants in the asymptotic formulas for the R_ω get multiplied by zero, so that they disappear from the λ asymptotics.

For eigenvalue distributions on compact manifolds, and for local spectral densities $E_\lambda(x, x)$ in general (for second-order linear operators), the typical behaviors are [10]

$$R^\alpha_\lambda \mu(\lambda) = \sum_{s=0}^{\alpha} a_{\alpha s} \lambda^{(m-s)/2} + O(\lambda^{(m-\alpha-1)/2}), \tag{23}$$

$$R^\alpha_\omega \mu(\omega) = \sum_{s=0}^{\alpha} c_{\alpha s} \omega^{m-s} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\alpha} d_{\alpha s} \omega^{m-s} \ln \omega + O(\omega^{m-\alpha-1} \ln \omega), \tag{24}$$

where m is the dimension of the manifold and

- $c_{\alpha s} =$ constant $\times\, a_{\alpha s}$ if $s \le m$ or $s - m$ is even;
- $c_{\alpha s}$ is undetermined by $a_{\alpha s}$ if $s > m$ and $s - m$ is odd;
- $d_{\alpha s} =$ constant $\times\, a_{\alpha s}$ if $s > m$ and $s - m$ is odd.

(The error terms are those that arise in passing from heat-kernel expansions (which extend to arbitrarily high order) to Riesz-mean asymptotics by generalized Tauberian theorems (e.g., [17]).)

The major conclusion, therefore, is that *the coefficients of the Riesz–Cesàro ω expansion of μ contain more information that the coefficients of its λ expansion*. In the principal application to differential operators and quantum field theory, these new data are "global" and the old ones are "local".

The λ and ω expansion coefficients are in direct correspondence with those in the small-t expansions of the Laplace transforms of μ with respect to λ and ω, respectively; the latter generalize the heat and cylinder kernels. If

$$K(t) \equiv \int_0^\infty e^{-\lambda t}\, d\mu(\lambda) \sim \sum_{s=0}^{\infty} b_s t^{(-m+s)/2} \tag{25}$$

and

$$T(t) \equiv \int_0^\infty e^{-\omega t}\, d\mu \sim \sum_{s=0}^{\infty} e_s t^{-m+s} + \sum_{\substack{s=m+1 \\ s-m \text{ odd}}}^{\infty} f_s t^{-m+s} \ln t, \tag{26}$$

then

$$b_s = \frac{\Gamma((m+s)/2+1)}{\Gamma(s+1)}\, a_{ss}\,, \tag{27}$$

and

$$e_s = \frac{\Gamma(m+1)}{\Gamma(s+1)}\, c_{ss} \quad \text{if } s-m \text{ is even or negative,} \tag{28}$$

but

$$e_s = \frac{\Gamma(m+1)}{\Gamma(s+1)}[c_{ss} + \psi(m+1)d_{ss}], \qquad f_s = -\frac{\Gamma(m+1)}{\Gamma(s+1)}\, d_{ss} \tag{29}$$

if $s-m$ is odd and positive. (Coefficients with $\alpha \neq s$ contain no additional information.) Therefore (by comparison with the discussion of (24)), the e_s in case (29) are "new" objects containing information not present in the heat coefficients b_s .

4 Numerical Verification

Modern computer algebra systems make it possible to investigate directly the sums and Riesz means of eigenvalues, spectral densities, energy densities, etc., providing "experimental" verification of the conclusions obtained from rigorous analysis of Green functions. We have used *Maple* to study the model in Sec. 2 (with $L = \pi$) and some other simple problems; details appear on a Web page [19]. A crucial tool is *Maple*'s `floor` (greatest integer) function.

In (21) let f be $N(\lambda)$, the number of the eigenvalues (7) less than λ. Then $R^\alpha_\omega N$ is a finite sum, and we begin by defining it in *Maple*:

```
exact := (alpha,omega) ->
     Sum((1 - (2*n+(theta/Pi))/omega)^alpha,
         n=0..floor((omega-(theta/Pi))/2))
   + Sum((1 - (2*n-(theta/Pi))/omega)^alpha,
         n=1..floor((omega+(theta/Pi))/2));
```

$$exact := (\alpha,\, \omega) \to \left(\sum_{n=0}^{\mathrm{floor}(1/2\,\omega - 1/2\,\frac{\theta}{\pi})} \left(1 - \frac{2\,n + \frac{\theta}{\pi}}{\omega}\right)^{\alpha} \right) + \left(\sum_{n=1}^{\mathrm{floor}(1/2\,\omega + 1/2\,\frac{\theta}{\pi})} \left(1 - \frac{2\,n - \frac{\theta}{\pi}}{\omega}\right)^{\alpha} \right).$$

There are two separate series of eigenfunctions, since n in (6) can be either positive or negative.

We define the identity function

```
id := p -> p
```

and tell the computer to replace `floor` by `id` in $R^0_\omega N$; the result is $\omega + 1$, showing that $c_{00} = 1$ in (24), in agreement with Weyl's eigenvalue distribution

law and with (13), (26), (28). (The π in the denominators of (13) is removed by the integration over the manifold.)

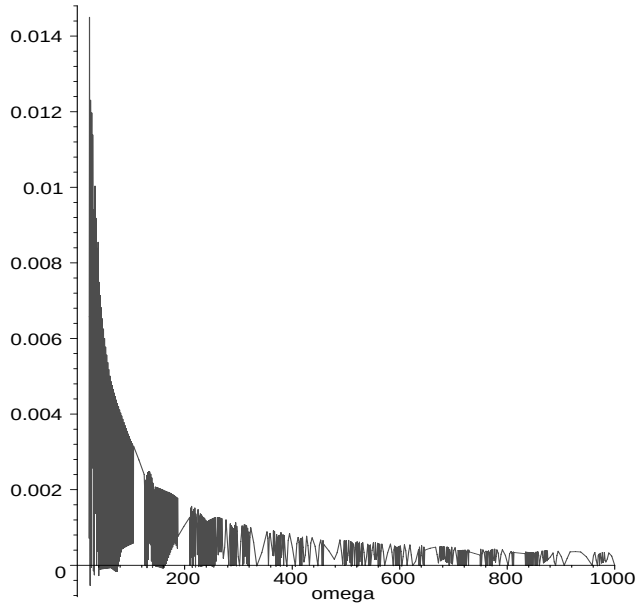
The leading term, ω, in $R^0_\omega N$ corresponds via (21) to a leading term $\frac{1}{2}\omega$ in $R^1_\omega N$. Subtracting this from the exact sum $R^1_\omega N$ yields

$$-\frac{1}{2\pi\omega}\left(\omega^2\,\pi - 2\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi-\theta}{\pi}\right)\,\omega\,\pi - 2\,\omega\,\pi + 2\,\theta\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi-\theta}{\pi}\right) + 2\,\theta \right.$$
$$+ 2\,\pi\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi-\theta}{\pi}\right)^2 + 2\,\pi\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi-\theta}{\pi}\right) - 2\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi+\theta}{\pi}\right)\,\omega\,\pi$$
$$\left. - 2\,\theta\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi+\theta}{\pi}\right) + 2\,\pi\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi+\theta}{\pi}\right)^2 + 2\,\pi\,\mathrm{floor}\left(\frac{1}{2}\,\frac{\omega\,\pi+\theta}{\pi}\right)\right).$$

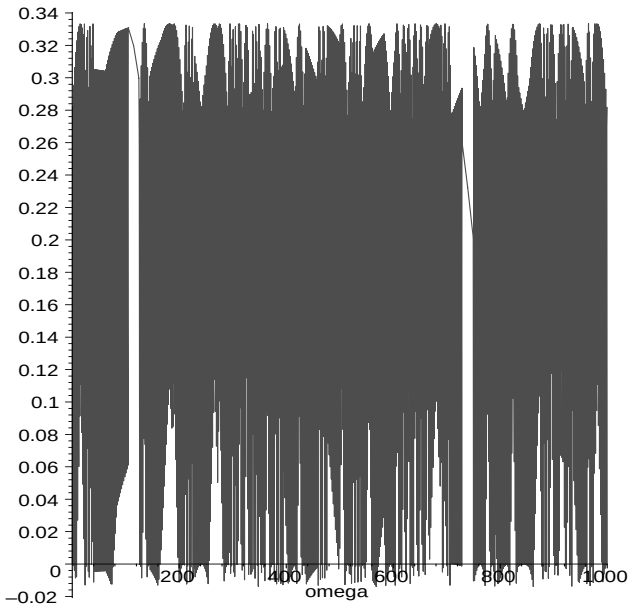
(Fortunately, *Maple* is able to evaluate in closed form all the sums that arise in this simple model.) Again substituting the identity function for the floor function, we get

$$\frac{1}{2}\,\frac{\theta\,(\theta-2\,\pi)}{\pi^2\,\omega}.$$

The relevant term at this order is the one of order ω^0, which is zero. Thus $c_{11} = 0$, consistently with (13), (26), (28), which show that $c_{ss} = 0$ for all odd s. Conceptually subtracting this term also, we obtain the exact remainder in the first Riesz mean (the O term in (24) — where the $\ln\omega$ factor can actually be ignored, because in fact there are no logarithmic terms in this case). Setting $\theta = \frac{\pi}{6}$ for definiteness, we plot this remainder:



Perhaps more revealingly, we plot the remainder times ω:



This amplified error is neatly confined between horizontal lines, demonstrating that the error is precisely of order $O(\omega^{-1})$. The gaps and striations in these graphs are artifacts purely of the sampling in the plotting routine; plotting over a smaller interval always reveals beautifully periodic oscillations, as we shall exemplify below.

Continuing in this way, we can empirically determine each c_{ss} in turn, and we can plot the remainder times each power of ω until the noise has been amplified to the level $O(1)$. In particular, at the second step we get

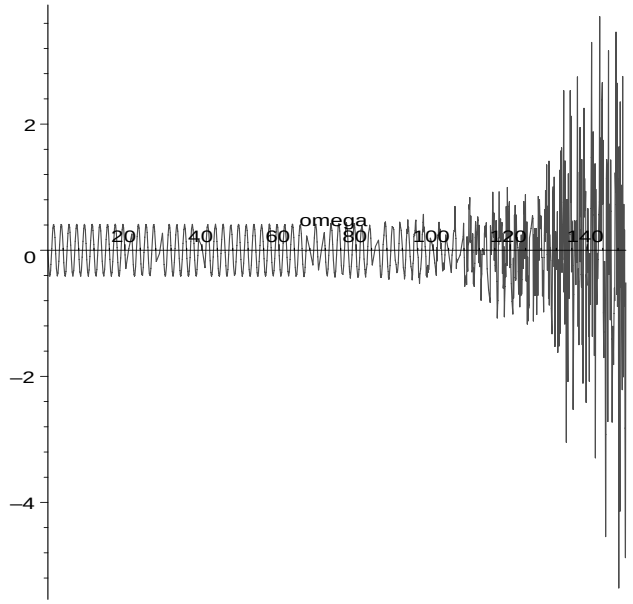
$$c_{22}\omega^{-1} = \frac{1}{3}\,\frac{2\,\pi^2 - 6\,\theta\,\pi + 3\,\theta^2}{\pi^2\,\omega},$$

in precise agreement with (13). Similarly, we find

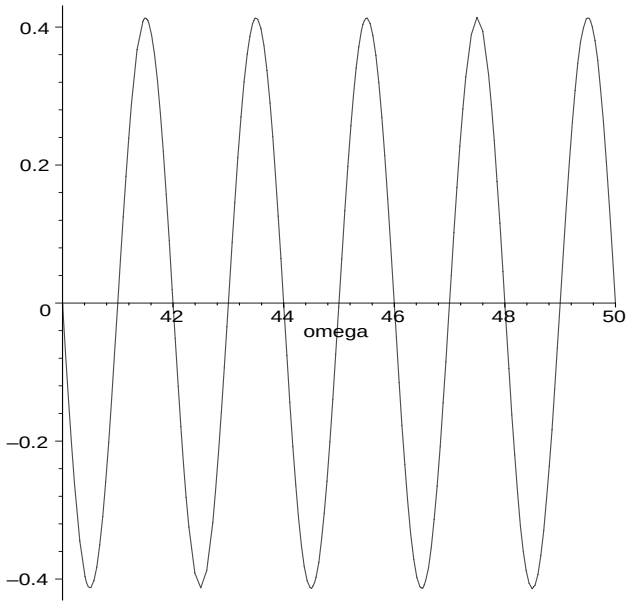
$$c_{44}\omega^{-3} = \frac{1}{15}\,\frac{-60\,\theta^3\,\pi - 8\,\pi^4 + 60\,\theta^2\,\pi^2 + 15\,\theta^4}{\omega^3\,\pi^4}, \tag{30}$$

$$c_{66}\omega^{-5} = \frac{1}{21}\,\frac{-126\,\theta^5\,\pi + 21\,\theta^6 + 210\,\theta^4\,\pi^2 - 168\,\theta^2\,\pi^4 + 32\,\pi^6}{\omega^5\,\pi^6}. \tag{31}$$

The plot of the remainder in $R_\omega^6 N$ times ω^6 is



At large ω there is roundoff error in the numerical evaluation of the function (now a very long sum of `floor` terms of both signs, which cancel to a high order). This is a good excuse for replotting on a smaller interval, which shows the periodicity advertised earlier:



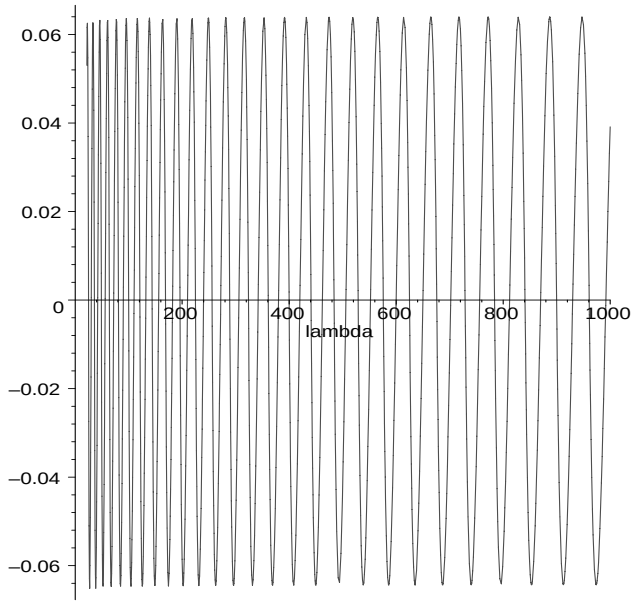
We can do the same thing for $R^\alpha_\lambda N$, with, of course, a more trivial result. For variety let's look at the Dirichlet case, where the eigenvalues are

$$E_n = n^2$$

with n positive only. The exact Riesz mean is

$$exact := (\alpha,\, \lambda) \to \sum_{n=1}^{\mathrm{floor}(\sqrt{\lambda})} \left(1 - \frac{n^2}{\lambda}\right)^\alpha .$$

Evaluating for $\alpha = 0$ and replacing `floor` by `id` yields the Weyl term, $\sqrt{\lambda}$. Subtracting the corresponding term, $\frac{2}{3}\sqrt{\lambda}$, from the first Riesz mean, one sees that there is still an error of order $O(\lambda^0)$, whose leading term is $-\frac{1}{2}$. This is precisely what is expected from two Dirichlet endpoints in dimension 1 [12]. Subtracting this and replacing `floor` by `id`, one finds that the rest of the second Riesz mean, $R^2_\lambda N$, behaves like $O(\lambda^{-3/2})$. That is, the term $a_{22}\lambda^{-1/2}$ in (23) is zero. The error in this approximation to $R^2_\lambda N$ is of order $O(\lambda^{-1})$, as we check by plotting it times λ:



Continuing in this way, one sees that the correction term is *always* zero from this point on. This reflects the triviality of the local heat kernel expansion (16) (with (25), (27)), and also the fact that there are no higher-order contributions from the endpoints in this case [12].

5 Future Questions

Presumably the reader is now convinced that the nonlocality of the cylinder kernel has nothing to do with boundaries. However, it may be objected that

the effect exhibited in our model is "topological". Perhaps there is nothing wrong with (2a) or (2b) when applied to a system whose configuration space is $\mathbb{R}^m$? Consider a Hamiltonian

$$H = -\frac{d^2}{dx^2} + V(x)$$

for which $V(x) = 0$ in a neighborhood of $x = 0$ but $V(x)$ has a huge hump (or well) elsewhere. We believe that it is clear from the general properties of Green functions that the existence of the distant potential hill will affect the energy density (and the local spectral density, etc.) at 0. After all, there is little physical difference between a high potential barrier and a perfectly reflecting boundary. Unfortunately, we have not yet found a model of this sort in which calculations can be done in an elementary way.

A related question is whether the energy density at, say, the origin of the harmonic oscillator potential $V(x) = x^2$ reflects the quadratic growth of the potential at infinity, or is determined completely by the power series of V at the point concerned. This question may be algebraically tractable, given the solvability of the Schrödinger equations with potentials x^2 and $1 - \cosh^{-2} x$; we hope to return to it in later work. The issue is more subtle than it may appear at first glance, because for analytic potentials there is no distinction between local and distant behavior, while for potentials that are not C^∞ the situation will be clouded by boundary-like effects associated with the singular points.

Here is another question of the same sort, whose answer can actually be extracted from the physics literature of a decade or two ago [2, 1]. In quantum field theory on a curved manifold, one expects a contribution to the renormalized energy density from the local curvature of space. Thus, for example, in a static universe whose spatial sections are m-spheres ($m \geq 2$) there should be a vacuum energy that is a (decreasing) function of the radius of the sphere. On the other hand, we have seen that when $m = 1$ there is already such an energy ($L \equiv 2\pi \times$ radius), *even though the 1-sphere has no intrinsic curvature*. In higher dimensions, can these two effects be disentangled? How much of the vacuum energy of the m-sphere is due to its curvature, and how much simply to its finite volume? In view of Gauss–Bonnet-type theorems, does the question even make any sense? The answer that emerges from the work of Bunch and others [2, 1] is this: For a massless, conformally coupled scalar field in a static, infinite universe of constant *negative* curvature, the vacuum energy is zero. By analytic continuation or algebraic analogy, the vacuum energy produced by local *positive* curvature is also zero for such a field, and the nonzero vacuum energy of a sphere arises entirely from the replacement of an integration over normal modes by a discrete sum — that is, it is just Casimir energy related to the finite volume of the space. However, if the field's mass is positive, or it has nonconformal coupling to the curvature scalar, or the metric is time-dependent, then in general there is also a genuine curvature energy.

A more general question is this: Through the work of Balian and Bloch, Gutzwiller, Duistermaat and Guillemin, and others, it is now generally understood (e.g., [4]) that the unsmoothed structure of an eigenvalue distribution has

periodicities (in energy) reciprocal to the periods of the closed trajectories of the associated classical mechanical system. What is the relation between this fact and the nonlocal structure of the asymptotics of the cylinder kernel and the square-root Riesz means? In the simple one-dimensional models we have studied, the periodic orbits are dictated by the length L, which also plays the major role in the asymptotics of T and $R^\alpha_\omega N$. If there is a more general connection, if all or part of the information about periodic orbits is encoded in $R^\alpha_\omega N$ (or in $R^\alpha_{\lambda^{1/p}} N$ for all p), the implications for calculational tractability could be significant.

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