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Monotonicity and bounds on Bessel functions *

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Abstract

I survey my recent results on monotonicity with respect to order of general Bessel functions, which follow from a new identity and lead to best possible uniform bounds. Application may be made to the 'spreading of the wave packet' for a free quantum particle on a lattice and to estimates for perturbative expansions.

On my arrival as a graduate student at Berkeley in September 1964, I was amused to see a Volkswagen Beetle with Schrödinger's equation written on it drive past. (I don't recall if it was the time-dependent or time-independent equation.) As I stood in line to enroll, a table off to the side with a 'Free Speech' banner caught my eye. Soon were to begin the student demonstrations which culminated in Vietnam war protests. I managed to complete the typing of my thesis in 1969 even as tear gas wafted in through the open window. I had asked Eyvind Wichmann if he would supervise my Ph.D. studies, and after checking that Emilio Segrè had given a good report on my oral examination, he agreed to take me on. I'd like to thank Eyvind for helping to make my stay at Berkeley a successful one.

1 Motivation

The free evolution of a quantum particle is important for understanding the "spreading of the wave packet," the large time behavior of scattering states, the Dyson perturbative expansion (each term of which is expressed in terms of the free evolution), and other aspects of the evolution of the quantum particle. The free evolution of a quantum particle on a one-dimensional lattice is described by Bessel functions of integer order, as reviewed below. In higher dimensions, the free evolution is given by products of the one-dimensional evolution, and so Bessel functions again describe the evolution. A detailed study of the behavior of Bessel functions of integer order is therefore necessary if the free evolution on a lattice is to be as well understood as in the continuum.

Computer packages such as Maple yield precise plots of Bessel functions, and I carried out computer experiments which yielded a very detailed picture of

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the variation of the Bessel function with respect to order and precise bounds on the magnitude of the Bessel function. Rough bounds were no longer satisfying when I could see the precise behavior on the computer screen. (Of course, computer generated pictures can be misleading and rigorous mathematical proof is required.) Just as a physical theory should give precise agreement with experiment, so too one should prove results in precise agreement with computer experiments and hence "best possible."

Recalling that the Bessel function of integer order satisfies

$$J_{-n}(x) = J_n(-x) = (-1)^n J_n(x)$$

we need only consider $n \geq 0$ and $x \geq 0$. The dependence of $J_n(x)$ on the order n is best elucidated by replacing the discrete n with a continuous ν. Thus generalizing from Bessel functions of integer order, we are led to study the Bessel function of the first kind $J_\nu(x)$, the second kind $Y_\nu(x)$, and the general Bessel function $C_\nu(x) = aJ_\nu(x) + bY_\nu(x)$, for $\nu \geq 0$ and $x \geq 0$.

A Quantum Particle on a Lattice

A quantum particle on the one-dimensional lattice $\mathcal{L} = \{0, \pm\ell, \pm 2\ell, \ldots\}$ has a wave function $\psi(n)$, position operator Q and shift operator U, where $[Q\psi](n) = n\ell\psi(n)$ and $[U\psi](n) = \psi(n-1)$. The finite-difference Laplacian may be expressed in terms of the shift operator as

$$\nabla^2 = \frac{U + U^{-1} - 2I}{\ell^2}$$

the free Hamiltonian then being

$$H = -\frac{\hbar^2}{2m}\nabla^2 \ .$$

The position operator at time t is

$$Q(t) = e^{itH/\hbar} Q e^{-itH/\hbar}$$

and the momentum operator is

$$P = m\frac{d}{dt}\bigg|_{t=0} Q(t) = \frac{\hbar}{i}\frac{1}{2\ell}[U^{-1} - U] \ .$$

It then follows that $P(t) = P$ and

$$Q(t) = Q + \frac{t}{m}P \ . \tag{1}$$

We'll call equation (1) *Newton's Law*, which has the same form as in the continuum. A consequence of Newton's law is that when observed on a large space-time scale, the free quantum particle on a lattice follows straight line trajectories. (See [3], and for additional discussion of the large space-time limit [4].)

Bessel Functions

A comparison of the unitary evolution

$$e^{-itH/\hbar} = e^{-it\hbar/m\ell^2} e^{t\hbar/2m\ell^2\left[(iU)-(iU)^{-1}\right]}$$

with the generating function for Bessel functions

$$e^{x/2(\rho-1/\rho)} = \sum_{n=-\infty}^{\infty} \rho^n J_n(x)$$

leads to the expression

$$e^{-itH/\hbar} = e^{-it\hbar/m\ell^2} \sum_{n=-\infty}^{\infty} i^n J_n\left(t\hbar/m\ell^2\right) U^n .$$

The kernel of the free evolution on the lattice is therefore

$$K_t(n,k) = e^{-it\hbar/m\ell^2} i^{n-k} J_{n-k}(t\hbar/m\ell^2), \tag{2}$$

which may be compared with the kernel in the continuum

$$K_t(x,y) = \sqrt{\frac{m}{2\pi\hbar it}} e^{im(x-y)^2/2\hbar t} .$$

The $t^{-1/2}$ bound, uniform in x, for the continuum kernel must be replaced by a $t^{-1/3}$ bound, uniform in n, on the lattice.
Remark. It's amusing that the well-known Bessel function identity

$$nJ_n(x) = \frac{x}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

may be thought of as an expression of Newton's law, as follows by writing Newton's law as

$$e^{-itH/\hbar}Q = (Q - \frac{t}{m}P)e^{-itH/\hbar}$$

and substituting the kernel (2).

2 Method and Results

Our approach is based on a new Bessel function identity which leads to monotonicity properties and in turn to best possible uniform bounds. The main ingredients in the derivation of the new identity are the Wronskian [8, p.76(1)]

$$J_\nu(x)Y_\nu'(x) - Y_\nu(x)J_\nu'(x) = \frac{2}{\pi x} \tag{3}$$

where $'$ denotes derivative with respect to the argument x, and the Nicholson integral (this one actually proved by Watson) [8, p.444(2)] relating derivatives with respect to the order ν:

$$J_\nu(x)\frac{\partial Y_\nu(x)}{\partial \nu}(x) - Y_\nu(x)\frac{\partial J_\nu(x)}{\partial \nu}(x) = -\frac{4}{\pi}A_\nu(x) \tag{4}$$

where

$$A_\nu(x) = \int_0^\infty K_0(2x \sinh t)e^{-2\nu t}\, dt$$

and K_0 is the modified Bessel function of the second kind of order 0, where in general:

$$K_\nu(x) = \int_0^\infty e^{-x\cosh u}\cosh \nu u\, du\ .$$

The identity concerns the derivative with respect to order of the function

$$f_\nu(x) = F(x)C_\nu(x)$$

where $C_\nu(x) = aJ_\nu(x) + bY_\nu(x)$ and a and b are real constants (independent of ν and x), and $F(x)$ is a differentiable function of x. The analysis [5] proceeds by introducing also the function

$$g_\nu(x) = F(x)D_\nu(x)$$

where

$$D_\nu(x) = cJ_\nu(x) + dY_\nu(x)$$

and $\gamma \doteq ad - bc \neq 0$. A straightforward computation using (3) and (4) yields

$$\left[\frac{g_\nu(x)}{f_\nu(x)}\right]' = \frac{2\gamma F^2}{\pi x f_\nu^2} \tag{5}$$

$$\frac{\partial}{\partial \nu}\left[\frac{g_\nu(x)}{f_\nu(x)}\right] = -\frac{4\gamma F^2 A_\nu}{\pi f_\nu^2} \tag{6}$$

Now setting the derivative of (5) with respect to ν equal to the derivative of (6) with respect to x gives [5]

$$\boxed{\frac{\partial f_\nu}{\partial \nu} = x\left[\frac{(F^2A_\nu)'}{F^2}f_\nu - 2A_\nu f_\nu'\right]\ .} \tag{7}$$

This is the new identity which leads to monotonicity and then to best possible uniform bounds. Notice that it expresses a derivative with respect to order ν (which is in general difficult to analyze) in terms of derivatives with respect to argument x (which are easier to deal with). The main advantage of (7) becomes apparent at a stationary point of f_ν, where $f_\nu'(x) = 0$, and hence

$$\frac{\partial f_\nu}{\partial \nu} = x\frac{\left[F^2A_\nu\right]'}{F^2}f_\nu\ .$$

Multiplying through by f_ν then yields

$$\frac{\partial f_\nu^2}{\partial \nu} = 2x \frac{f_\nu^2}{F^2} \left[F^2 A_\nu\right]' \ . \tag{8}$$

Notice that the sign of the right-hand-side of (8) is the same as the sign of $[F^2 A_\nu]'$. (Recall that we are taking $x \geq 0$ and $\nu \geq 0$.) Thus *the magnitude of f_ν at a stationary point is increasing or decreasing in ν depending on whether $F^2 A_\nu$ is increasing or decreasing in x at the stationary point.*

Case 1: $F(x) = 1$

Here $f_\nu(x) = C_\nu(x)$, the general Bessel function. According to equation (8) we need to consider A'_ν. But as is easily seen, $K_0(x)$ decreases monotonically in x and hence $A_\nu(x)$ decreases monotonically in x. Indeed, $A'_\nu(x) < 0$ for all positive x. We conclude that *the magnitude of $C_\nu(x)$ is decreasing in ν at all its positive stationary points.* In the case of $J_\nu(x)$, its value at the first stationary point is equal to $\sup_x |J_\nu(x)|$, which therefore decreases monotonically in ν.

Case 2: $F(x) = x^{1/2}$

Here $f_\nu(x) = x^{1/2} C_\nu(x)$. According to equation (8) we need to consider $[xA_\nu(x)]'$. Now by a change in the variable of integration we may express $xA_\nu(x)$ as

$$xA_\nu(x) = \int_0^\infty K_0(2x \sinh(y/x)) e^{-2\nu y/x} dy \ . \tag{9}$$

Since $2\nu y/x$ and $x \sinh(y/x)$ decrease with x, it follows that the integrand (9) increases and thus $xA_\nu(x)$ is increasing in x. Indeed $[xA_\nu(x)]' > 0$ for all positive x. We conclude that *the magnitude of $x^{1/2} C_\nu(x)$ in increasing in ν at all its positive stationary points.*

Case 3: $F(x) = x^\alpha$, $0 < \alpha < 1/2$

Here $f_\nu(x) = x^\alpha C_\nu(x)$. According to equation (8) we need to consider $[x^{2\alpha} A_\nu(x)]'$. An analysis [5] of $x^{2\alpha} A_\nu(x)$ shows that it tends to 0 as $x \to 0$ and ∞, and has a unique stationary point at $x = x_\nu$, which is the location of its maximum. The point x_ν increases with ν [5]. The magnitude of $x^\alpha C_\nu(x)$ at a stationary point $x = X_\nu$ is increasing in ν if $X_\nu < x_\nu$ and decreasing in ν if $X_\nu > x_\nu$.

The most important case is $\alpha = 1/3$ and $C_\nu = J_\nu$, so $f_\nu(x) = x^{1/3} J_\nu(x)$. We first locate the maximum of $x^{1/3} J_\nu(x)$ at its first stationary point $x = X_\nu$, using a Sturm comparison argument. If we could show that $X_\nu > x_\nu$ for all ν, we could conclude that $\sup_x |x^{1/3} J_\nu(x)$ decreases in ν and hence is bounded by its value at $\nu = 0$, which is $c = 0.7857\cdots$ (which would therefore be the best possible constant in such a bound):

$$|J_\nu(x)| \le c|x|^{-1/3} . \tag{10}$$

However, we do not show $X_\nu > x_\nu$ for all ν, but nevertheless we are able to prove (10) by a combination of monotonicity for $\nu \le 3$ and a bound for $\nu \ge 3$. Thus we prove (10) in two steps:

Step 1. For $0 \le \nu \le 3$, we prove (10) by showing $X_\nu > x_\nu$ for $0 \le \nu \le 3$. This is shown by computing the values (given in table 1) of X_ν and x_ν for $\nu = 0, 0.5, 1, 2$ and 3, and using the fact that both x_ν and X_ν are increasing in ν. (For the increase in X_ν see [7],[2], or [1]. Note that X_ν is a root of the equation $\frac{1}{3}J_\nu(x) + xJ'_\nu(x) = 0$. The general qualitative behavior with respect to order ν, including monotonicity and multiplicity, of all the positive roots of $\alpha J_\nu(x) + xJ'_\nu(x) = 0$ for all real α and ν, is derived in [6].)

Then for $0 \le \nu \le 0.5$,

$$x_\nu \le x_{0.5} < X_0 \le X_\nu .$$

A similar argument works for the other intervals, finally giving $x_\nu < X_\nu$ for all ν in the interval $[0, 3]$.

ν	x_ν	X_ν
0	0.1726	0.7837
0.5	0.5918	1.4569
1	1.0595	2.0694
2	2.0336	3.2315
3	3.0231	4.3540

Table 1: Values of x_ν (the location of the maximum of $A_\nu(x)$) and X_ν (the location of the maximum of $|x^{1/3}J_\nu(x)|$).

Step 2. For $\nu \ge 3$, we prove (10) by the bound:

$$\sup_x |x^{1/3}J_\nu(x)| = X_\nu^{1/3}J_\nu(X_\nu) = \left[\frac{X_\nu}{\nu}\right]^{1/3} \nu^{1/3}J_\nu(X_\nu) \le \left[\frac{X_3}{3}\right]^{1/3} b \tag{11}$$

This bound uses two facts:

- the decrease in ν of X_ν/ν ([2] and [1], see also [6])
- the bound
$$\nu^{1/3}\sup_x |J_\nu(x)| < b \tag{12}$$
where $b = 0.6748\cdots$ is the best possible such constant. This bound is proved in [5] using a Sturm comparison argument, which shows that $\nu^{1/3}\sup_x |J_\nu(x)|$ strictly increases to b.

Substituting values into the right-hand-side of (11) gives $0.7641\cdots$, which is less than c. Hence (10) is proved for $\nu \geq 3$.

3 Summary

1. The magnitude of the general Bessel function $C_\nu(x)$ of order ν is decreasing in ν at all its positive stationary points. It follows that $\sup_x |J_\nu(x)|$ is decreasing in ν.

2. The magnitude of $x^{1/2}C_\nu(x)$ is increasing in ν at all its positive stationary points.

3. $\nu^{1/3}\sup_x |J_\nu(x)|$ is increasing in ν to the value b, yielding the bound, uniform in the argument x,

$$|J_\nu(x)| < \frac{b}{\nu^{1/3}}$$

   which is best possible in the exponent 1/3 and constant $b = 0.6748\cdots$.

4. The bound, uniform in the order ν,

$$|J_\nu(x)| \leq \frac{c}{x^{1/3}}$$

   is best possible in the exponent 1/3 and constant $c = 0.7857\cdots$.

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