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DECAY TIMES IN QUANTUM MECHANICS

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ABSTRACT. We consider the problem of determining the probability distribution for the time of decay for a one-dimensional Schrödinger operator.

An important feature of quantum physics is that times of occurrences like decay of a radioactive nucleus are random; typically the decay time is observed to have an exponential probability distribution. But quantum mechanics does not directly prescribe a probability distribution for such times. Recall how probability distributions arise: Any "observable" $\mathcal{A}$ is represented by a self-adjoint operator A on a Hilbert space $\mathcal{H}$; if at time t the system is in a state represented by the vector φ in $\mathcal{H}$, and $f(\mathcal{A})$ is measured (for any reasonable function f) then the expected value is

$$E(f(\mathcal{A})) = \langle\varphi|f(A)\varphi\rangle \ .$$

By the spectral theory of self-adjoint operators this determines a probability distribution function F such that

$$E(f(\mathcal{A})) = \int f(\lambda)dF(\lambda)$$

so that the probability of finding the value of $\mathcal{A}$ in $(\lambda_1, \lambda_2]$ is $F(\lambda_2) - F(\lambda_1)$.

An "observable" is something that can, in principle, be measured at any given instant. But the most straightforward way to measure the time of decay would be to place detectors that respond to the products of the decay whenever that happens; this is not a measurement made at a specific time. Still, the probability distributions for such experiments must be determined in principle by quantum mechanics, since the detector could be regarded as a part of the quantum system, and its result read off ("measured") at some later time. But the quantum mechanical description of such an apparatus seems hopelessly complicated, and its details should be irrelevant.

For a simple model, the Schrödinger operator $H = -d^2/dr^2 + V(r)$ on the half-line $[0,\infty)$, with $V \geq 0$, we will propose a self-adjoint operator, on the Hilbert space $L^2([0,\infty))$ of the model, whose probability distribution should be that of the decay time.

A potential $V(r)$ with a barrier that tends to trap a particle inside $[0, R]$ is often used to model decay. It has been shown [1] that in this case there is a state φ located inside $[0, R]$ such that $|\langle\varphi|e^{-iHt}\varphi\rangle|^2$, the probability of the particle being

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found in the initial state at time t, is approximately given by an exponentially decaying function. (This gives the probability of a result if a certain measurement is made at time t, not the probability distribution for when decay will happen.)

First note that certain operators have been considered as measuring trapping time:

$$T_{[0,R]} = \int_{-\infty}^{\infty} e^{iHt}\chi_{[0,R]}e^{-iHt}dt \tag{1}$$

represents the time spent in $[0,R]$, where $\chi_{[0,R]}$ is (multiplication by) the characteristic function of $[0,R]$. This operator commutes with H.

The time delay T_D also commutes with H; it satisfies [2], [3]

$$HT_D = \int_{-\infty}^{\infty} e^{iHt}(V + \frac{r}{2}V')e^{-iHt}dt\ . \tag{2}$$

The time spent in state φ is

$$T_\varphi = \int_{-\infty}^{\infty} e^{iHt}|\varphi\rangle\ \langle\varphi|e^{-iHt}dt\ , \tag{3}$$

which also commutes with H.

None of these operators has probability distribution close to exponential; in fact they are all bounded operators, at least when restricted to a spectral subspace for any closed energy interval contained in $(0,\infty)$. The probability of finding a time longer than $\|T\|$ is zero while the exponential distribution allows arbitrarily large times.

So it appears that particles can not be trapped for more than a given finite time! This does not necessarily contradict the arbitrarily long times observed until decay happens, because the particle is not unequivocally inside $[0,R]$ until the time of decay.

To find an operator to represent a physical quantity, we can try to quantize a classical function of p and q, or lay down enough requirements to determine the operator. Following the latter approach, we might require

$$e^{iHt}Te^{-iHt} = T - t \tag{4}$$

if T gives the time until decay happens. This implies

$$i[H,T] = -I \tag{5}$$

which is notoriously impossible for H bounded below. But it is possible to find operators A such that $e^{iHt}Ae^{-iHt}$ decreases at a constant rate; in fact if $H_0 = -d^2/dr^2$ $(V=0)$ and

$$A_0 = \frac{i}{4}\left(r\frac{d}{dr} + \frac{d}{dr}r\right)\ , \tag{6}$$

we have

$$i[H_0, A_0] = -H_0\ . \tag{7}$$

The most general operator with this commutation relation is $A_0 + f(H_0)$, so the time until decay, multiplied by energy must be of this form. A metastable state is typically highly concentrated in energy, so that (7) is close to (5), up to a scalar multiple.

Taking the approach via classical mechanics, the time a free particle, moving on the half-line, takes to reach a curve $r = f(p^2)$ (on its way out) is $2|p|^{-1}f(p^2) -$

$(2p)^{-1}r$. Multiplication by the energy p^2 gives $\frac{1}{2}[|p|f(p^2) - pr]$. In particular, for the time to reach $r = R$ we get $\frac{1}{2}(|p|R - pr)$; quantization gives the operator $\frac{1}{2}H_0^{1/2}R + A_0$.

If V is short range, e.g. $|V(r)| \leq (1+r)^{-\alpha}$ with $\alpha > 1$, then the wave operators $\Omega_\pm$ exist (and are unitary since $V \geq 0$) and the operators

$$A_\pm = \Omega_+ A_0 \Omega_\pm^*$$

satisfy

$$i[H, A_\pm] = -H \ . \tag{8}$$

The probability distribution for A_0 in a state φ can be found by obtaining a spectral representation for A_0, i.e., a unitary W diagonalizing A_0 so that WAW^* is a multiplication operator. In the standard spectral representation of H_0 as multiplication by λ on $L^2([0,\infty))$, A_0 is given by

$$-\frac{i}{8}\left[\lambda\frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right] \ . \tag{9}$$

Then the operator W given by

$$\begin{aligned} W\varphi(\tau) &= (2\pi)^{-1/2}\int_0^\infty \lambda^{-4i\tau-\frac{1}{2}}\varphi(\lambda)d\lambda \\ &= (2\pi)^{-1/2}\int_{-\infty}^\infty e^{-is\tau}e^{s/8}\varphi(e^{s/4})ds \end{aligned} \tag{10}$$

is unitary from $L^2([0,\infty))$ to $L^2(\mathbb{R})$, since it is the composition of the Fourier transform and a unitary change of variables. It gives a spectral representation of A_0, since

$$(WA_0\varphi)(\tau) = \tau(W\varphi)(\tau) \ . \tag{11}$$

More generally,

$$(W_F[A_0 + F(H_0)]\varphi)(\tau) = \tau(W_F\varphi)(\tau) \tag{12}$$

where

$$W_F\varphi(\tau) = (2\pi)^{-1/2}\int \lambda^{-4i\tau-1/2}e^{if(\lambda)}\varphi(\lambda)d\lambda \tag{13}$$

and $\lambda f'(\lambda) = F(\lambda)$.

For example, if $\varphi(\lambda) = c(\lambda - z)^{-1}$ with $\operatorname{Im} z < 0$, a contour integral gives

$$(W\varphi)(\tau) = \frac{icz^{-4i\tau-\frac{1}{2}}}{1+e^{-8\pi\tau}} \ . \tag{14}$$

If $z = |z|e^{-i\theta}$, the probability distribution is

$$|W\varphi(\tau)|^2 = \frac{|c|^2 e^{-8\theta\tau}|z|^{-1}}{(1+e^{-8\pi\tau})^2} \ . \tag{15}$$

For small θ, this is very close to 0 for $\tau < 0$, and to a decaying exponential for $\tau > 0$.

By the second expression for $W_\varphi(\tau)$ in (10), if $e^{s/8}\varphi(e^{s/4})$ is close in $L^2(\mathbb{R})$ to

$$c(\lambda - \lambda_0 + i\epsilon)^{-1}e^{i[a+b(s-s_0)]}, \tag{16}$$

where $\lambda_0 = e^{s_0}$, then $(W\varphi(\tau))^2$ is close in $L^1(\mathbb{R})$ to (13), translated by b.

For the case of short range $V \neq 0$, the operator (7) in a spectral representation for H can be diagonalized in exactly the same way. Spectral representations of H are given by expansions in eigenfunctions $\psi(r,\lambda)$ corresponding to $\Omega_\pm$. Let

$$\tilde{\varphi}(\lambda) = \int_0^\infty \psi(r,\lambda)\varphi(r)dr \; . \tag{17}$$

In [1] a value z was found so that if $\varphi(r) = X_{[0,R]}(r)\psi(z,r)$, then $|\langle\varphi|e^{-iHt}\varphi\rangle|$ decays exponentially. For such φ, integration by parts gives

$$\tilde{\varphi}(\lambda) = \frac{\varphi'(R)\psi(\lambda,R) - \varphi(R)\psi'(\lambda,R)}{\lambda - z} \; . \tag{18}$$

Thus exponential decay of $|W\tilde{\varphi}|^2$ reduces to estimates of the difference between (16) and (18).

References

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