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A perspective on the contributions of Alan C. Lazer to critical point theory *

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Abstract

Over the last thirty five years Professor Alan C. Lazer has been a leading figure in the development of min-max methods and critical point theory for applications to partial differential equations. The author, his former student, summarizes from his own perspective Professor Lazer's contributions to the subject.

Critical point theory has proven to be one of the most important tools in the study of nonlinear equations. Among the various critical point techniques, minmax principles leading to the existence of *saddle points* have played a central role. Professor Lazer has been one of the main architects of these developments. Let us begin by stating the following basic principle proven by Professor Lazer jointly with Professor E. M. Landesman and D. R. Meyers [16].

Theorem 1 *Let H be a real Hilbert space and X, Y closed subspaces with $\dim(X) < \infty$ and $H = X \oplus Y$. Let $f : H \to R$ be a functional of class C^2. Let ∇f and $D^2 f$ denote the gradient and the Hessian of f respectively. If there exits a positive constant m such that*

$$\langle D^2 f(u)x, x\rangle \le -m\|x\|^2 \quad \text{for all } x \in X,\ u \in H \tag{1}$$

and

$$\langle D^2 f(u)y, y\rangle \ge m\|y\|^2 \quad \text{for all } y \in Y,\ u \in H, \tag{2}$$

then there exists a unique $u_0 \in H$ such that $\nabla f(u_0) = 0$. Moreover

$$f(u_0) = \max_{x\in X} \min_{y\in Y} f(x + y). \tag{3}$$

The proof of Theorem 1 is based on the observation that, by (2), the function f is convex on linear manifolds of the form $\{x+y; y \in Y\} \equiv Y_x$ and $f(x+y)$ tends to $+\infty$ as $\|y\| \to \infty$. Hence for each $x \in X$ there exists a unique $y = \phi(x) \in Y$ such that

$$f(x + \phi(x)) = \min_{y\in Y} f(x + y). \tag{4}$$

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Through a clever use of the implicit function theorem Professor Lazer and his coauthors show that the function ϕ is actually of class C^1. Since $f(x) \to -\infty$ as $\|x\| \to \infty$ (see (1)) and $\tilde{f}(x) \equiv f(x + \phi(x)) \leq f(x)$, $\tilde{f}$ attains its maximum value at some point x_0. This implies that $u_0 = x_0 + \phi(x_0)$ satisfies the claims of Theorem 1.

Noting that the latter argument only uses the convexity of f on the manifold Y_x and that $f(x) \to -\infty$ as $\|x\| \to \infty$, Professor Lazer with the help of one of his students, this author, proved in [10] the following result.

Theorem 2 *Let H be a real Hilbert space and X, Y closed subspaces with* $\dim(X) < \infty$ *and $H = X \oplus Y$. Let $f : H \to R$ be a functional of class C^2. Let ∇F and $D^2 f$ denote the gradient and the Hessian of f respectively. If there exits a positive constant m such that*

$$\langle D^2 f(u)y, y\rangle \geq m\|y\|^2 \quad \textit{for all } y \in Y,\ u \in H \tag{5}$$

and

$$f(x) \to -\infty \quad \textit{as } \|x\| \to \infty, \tag{6}$$

then there exists $u_0 \in H$ such that $\nabla f(u_0) = 0$. Moreover

$$f(u_0) = \max_{x \in X} \min_{y \in Y} f(x + y). \tag{7}$$

In [5], condition (5) is further relaxed by using the variational characterization in (4) to prove that ϕ is continuous and $\tilde{f}$ of class C^1 when f is of class C^1. These observations bypass the usage of the implicit function theorem.

The latter results were motivated by the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) + g(x, u(x)) = 0 \quad & \text{for } x \in \Omega \\ u(x) = 0 \quad & \text{for } x \in \partial\Omega, \end{aligned} \tag{8}$$

where Ω is a smooth bounded region in R^n, and $g : \Omega \times R \to R$ is a sufficiently regular function satisfying an adequate *growth condition.* Indeed if we let H denote the Sobolev space of square integrable functions in Ω whose first order partial derivatives are also square integrable in Ω and which vanish on $\partial\Omega$ (see [1]), $G(x, t) = \int_0^t g(x, s)ds$, and $J : H \to R$ is the functional given by

$$J(u) = \int_\Omega \{\|\nabla(u(x))\|^2/2 - G(x, u(x))\}\, dx, \tag{9}$$

then the critical points of J are the solutions to (8). Moreover, J obeys the assumptions of Theorem 1 when there exits an integer N and real numbers a, b such that

$$\lambda_N < a \leq \frac{\partial g}{\partial t}(x, t) \leq b < \lambda_{N+1} \quad \text{for all} \quad (x, t) \in \Omega \times R, \tag{10}$$

where λ_N, λ_{N+1} are consecutive eigenvalues of

$$\begin{aligned} \Delta u(x) + \lambda_k u(x) = 0 \quad \text{for } x \in \Omega \\ u(x) = 0 \quad \text{for } x \in \partial\Omega\,. \end{aligned} \tag{11}$$

Thus under this hypothesis the problem (8) has a unique solution. On the other hand if (10) is replaced by the weaker condition

$$\frac{\partial g}{\partial t}(x,t) \le b, \quad \text{and} \quad G(x,t) \ge \frac{at^2}{2} + C, \tag{12}$$

where a, b are as in (10) and C is an arbitrary real number then J satisfies the assumptions of Theorem 2. Hence (8) has a solution which need not be unique (see [10], [12], and Theorem 6 below).

Conditions (10) and (12), also known as *non-resonance conditions*, open the issue of what happens when the range of $\frac{\partial g}{\partial t}$ includes an eigenvalue λ_k. Professor Lazer, in collaboration with Professor E. M. Landesman, provided in [14] a breakthrough in this problem by considering the case in which

$$\begin{gathered} g(x,u) = h(u) + \lambda_k u - p(x), \\ h(-\infty) \equiv \lim_{t\to-\infty} h(t) < h(x) < h(+\infty) \equiv \lim_{t\to\infty} h(t) \quad \text{for all } x \in \mathbb{R}, \\ p \in L^2(\Omega), \text{ and} \\ \lambda_k \text{ is a simple eigenvalue.} \end{gathered} \tag{13}$$

They proved that if w is an eigenfunction corresponding to the eigenvalue λ_k then (8) has a solution if and only if

$$\begin{aligned} & h(-\infty)\int_{\Omega^+} |w| dx - h(\infty)\int_{\Omega^-} |w|\, dx \\ < \quad & \int_\Omega p \cdot w dx \\ < \quad & h(\infty)\int_{\Omega^+} |w| dx - h(-\infty)\int_{\Omega^-} |w|\, dx, \end{aligned} \tag{14}$$

here $\Omega^+ = \{x; w(x) \ge 0\}$, and $\Omega^- = \{x; w(x) \le 0\}$.

This surprising result is, without doubt, a corner stone in the study of nonlinear boundary value problems and, hence, one of the most cited papers in this area. Actually, as pointed out by Professor Landesman to the author in a personal communication, previously Professor Lazer had considered a somewhat more complicated case with his student D. E. Leach in [15].

By 1974, when this author had the privilege of meeting Professor Lazer, he was interested in providing a variational proof of his result in [14]. He concluded this successfully in [2], where he and his coauthors established in a *semivariational way* the following result.

Theorem 3 *Let Y denote a subspace solutions to the problem $\Delta u + \lambda_k u = 0$ in Ω with $u = 0$ on $\partial\Omega$. Assume $g(x,t) - \lambda_k t$ is a continuous bounded function. If for $w \in Y$*

$$\int_\Omega (G(x, w(x)) - \frac{\lambda_k w^2(x)}{2})dx \to \pm\infty \quad as \quad \|w\| \to \infty, \tag{15}$$

then (8) has a solution.

Motivated by this result, Professor P. H. Rabinowitz (see [19]) provided a variational proof of Theorem 3. Rabinowitz's variational proof includes the following general *saddle point principle.*

Theorem 4 *Let H be a real Banach space and X, Y closed subspaces with* $\dim(X) < \infty$ *and $H = X \oplus Y$. Let $f : H \to \mathbb{R}$ be a functional of class C^1 that satisfies de Palais-Smale condition (i.e., every sequence $\{u_n\}$ for which $\{f(u_n)\}$ is bounded and $\{f'(u_n)\}$ converges to zero, has a convergent subsequence.) Suppose that*

$$\inf_{y \in Y} f(y) = d > -\infty \tag{16}$$

and

$$f(x) \to -\infty \quad as \quad \|x\| \to \infty. \tag{17}$$

Let $D = \{x \in X; \|x\| \leq r\}$ with r big enough so that $\|x\| = r$ implies $F(x) > d$. If Γ denotes the set of continuous mappings $p : D \to H$ such that $G(x) = x$ if $\|x\| = r$ then

$$c \equiv \inf_{p \in \Gamma} \max_{x \in D} f(p(x)) > -\infty \tag{18}$$

and there exists $u_0 \in H$ such that $f(u_0) = c$ and $f'(u_0) = 0$.

For further details on the proof and applicability of the latter result the reader is referred to [20].

If one defines the Morse index of a critical point u as the number of negative eigenvalues of $D^2 f(u)$ from the hypotheses of Theorem 1 one sees that the Morse index of u_0 is $\dim X$. Given the similarities between the assumptions of Theorem 1 and those of Theorem 4 one would be tempted to conjecture that the Morse index of the critical point arising in Theorem 4 is $\dim X$. This, in general, is not true (see [18]). Professor Lazer, in collaboration with Professor S. Solimini, in [18] provides a detailed account of this problem. They prove the following results in [18].

Theorem 5 *If f satisfies the hypotheses of Theorem 4 and has only a finite number of critical points, all of which are nondegenerate, then there exists a critical point of f with Morse index equal to* $\dim X$*.*

The reader is invited to consult [13] for extensive ellaborations on the results of Professors Lazer and Solimini in [18].

Professor Lazer has masterfully utilized the variational characterizations provided in (7) and (18) to establish the existence of multiple solutions for problems like (8). For example, in a paper with this author (see Theorem A of [11]) he proved the following multiplicity result.

Theorem 6 *If a, b are as in (12) $g(x,0) = 0$, and $\frac{\partial g}{\partial t}(x,0) < \lambda_N$ then the problem (8) has at least two solutions. Moreover, if $\frac{\partial g}{\partial t}(x,0) \neq \lambda_k$ for any k then (8) has at least three solutions.*

The key ingredient in the proof of Theorem 6 is the fact that the solution obtained via the characterization (7), if isolated, gives a critical point of Morse index N while zero is a critical point of Morse index less than N. The third solution in Theorem 6 comes from the combining the fact that the degree of ∇J in a large ball is N. Actually the argument extends to the case when the first part of the hypothesis in (12) is replaced by

$$G(x,t) \leq \frac{bt^2}{2} + C\,. \tag{19}$$

In this case one may use theorems 4 and 5 to show that either c or a *dual* value $\bar{c}$ is a critical value containing a critical point whose Morse index is not that of 0. For other results where multiple solutions for (8) are studied using the above devices the reader is referred to [7], [8], [9], and references therein.

Another line of research motivated by Professor Lazer's work in [16] is the continuous dependence of the saddle point arising in (3) and its applications. Extensive developments in this direction are due to Professor H. Amann and his coworkers (see [3], [4]). The following generalization of Theorem 1 can be found in [3].

Theorem 7 *Let H be a real Hilbert space and X, Y, Z closed subspaces with $H = X \oplus Y \oplus Z$. Let $f : H \to \mathbb{R}$ be a functional of class C^2. Let ∇f and $D^2 f$ denote the gradient f. If there exits a positive constant m such that*

$$\begin{gathered}\langle f(x+y+z) - f(x_1+y+z), x - x_1\rangle \leq -m\|x - x_1\|^2 \\ \textit{for all} \quad x, x_1 \in X, \quad y \in Y, \quad z \in Z\,,\end{gathered} \tag{20}$$

and

$$\begin{gathered}\langle f(x+y+z) - f(x+y_1+z), y - y_1\rangle \geq m\|y - y_1\|^2 \\ \textit{for all } x \in X, \quad y, y_1 \in Y, \quad z \in Z,\end{gathered} \tag{21}$$

then there exists a continuous function $\phi : Z \to X \oplus Y$ such that $< \nabla f(z + \phi(z)), x + y >= 0$ for all $(x,y,z) \in X \times Y \times Z$. Moreover

$$\hat{f}(z) \equiv f(z + \phi(z)) = \max_{x \in X} \min_{y \in Y} f(x+y+z) \tag{22}$$

is of Class C^1, and z_0 is a critical point of $\hat{f}$ if and only if $z_0 + \phi(z_0)$ is a critical point of f

This latter theorem is particularly useful for the study of equations of the form $L(u) + N(u) = 0$ where L is a self-adjoint linear operator having infinitely many eigenvalues both positive and negative. This is the case where L comes from a wave operator (see [6]) or a Hammerstein integral operator (see [5]).

The techniques involved in the construction of the function ϕ in (4) have been extended in many directions and Professor Lazer has utilized them exquisitely. For example, in [17], he and Professor P. J. McKenna use it to study the existence of multiple solutions for (8) in the case of *jumping nonlinearities*. They establish sufficient conditions on g for (8) to have three solutions when

$$g(x,t) = h(t) - s\theta(x) + p(x), \ \lim_{t\to-\infty} h'(t), \lim_{t\to\infty} h'(t) \in (\lambda_{N-1}, \lambda_{N+1}),$$

where p is orthogonal to θ, and θ is and eigenfunction correponding to the eigenvalue λ_1 of (11).

Conclusion. Over thirty five years Professor Lazer has provided the Nonlinear Analysis community with fundamental new ideas and has left for others to ellaborate. AL, please keep on giving us exciting food for thought.

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