

First moments of energy and convergence to equilibrium *

Jérôme Busca

Abstract

A basic question is to establish convergence to equilibrium for globally defined solutions to evolution problems. The purpose here is to emphasize the role of symmetry. In particular, it is proved that in some cases the *first moments of energy* are constant on the ω -limit set of the solution. This key property is used to prove convergence in two model evolution problems. This communication is based on two joint works with P. Felmer [3] and M.A. Jendoubi, P. Polacik [4].

1 Introduction and Main Results

A basic question in the study of evolution problems is the following: do globally defined in time solutions converge to an equilibrium? In case the problem is *dissipative*, one can typically prove that the ω -limit set (i.e. the set of all accumulation points of the solution u) $\omega(u)$ is included in the set of the solutions of some limiting stationary equation (*steady states*). This is usually done thanks to some appropriate Lyapunov energy functional.

If the set of steady states contains a continuum, then the convergence issue is whether the solution actually selects one of them as $t \rightarrow +\infty$, that is, whether $\omega(u)$ is a singleton.

In this generality, or even if one specializes to nonlinear parabolic evolution problems for instance, the question is still open, and appears to be surprisingly difficult. Counterexamples in the non-autonomous case suggest that limitations do exist (see [21]). Partial results are available for instance in the analytic setting [12] [14] [22], in one dimension [18] [23], or under assumptions on the linearized operator, either explicitly stated as such, or resulting from the nature of the specific problem [10] [11] [17].

There is a large literature devoted to these questions, and I will not attempt to give any review of the results. For a very clear account on this, I refer to [5] and [13].

* *Mathematics Subject Classifications*: 35B50, 35A05.

Key words: Parabolic Equations, Equilibrium, Convergence.

©2001 Southwest Texas State University.

Published January 8, 2001.

Partially supported by ECOS/CONICYT project C99E06

It is my purpose here to show that when the solutions enjoy some symmetry, the convergence question can be solved. The simplest nontrivial case is probably when the problem is posed in the whole space, is translation and rotationally invariant, and the set of positive steady states is made of all translates of a single radial solution. In this case, proving convergence of positive solutions of the evolution problem is tantamount to proving that $\omega(u)$ cannot contain more than one of such translates. To this purpose, I intend to introduce a method that makes use of *first moments of the energy*, a tool which appears to be new. These moments will be shown to assume constant values on the ω -limit set, just as energy does. However, unlike the latter, they are able to discriminate, meaning taking different values on, distinct translates.

For the interested reader I mention references [19] [20] where a thorough investigation of the links between symmetry and convergence is given, in the context of stable equilibria (different from the one I address here).

Rather than elaborating on this in full generality, let me select two simple instances where one can easily highlight the main underlying idea. These examples are taken from joint works with P. Felmer [3] and M.A. Jendoubi, P. Polacik [4]. These two evolution problems will turn out to share the same stationary equation, namely the so-called scalar field equation

$$\begin{aligned} \Delta w - w + w^p &= 0 \quad \text{in } \mathbb{R}^N \\ w > 0, \quad w(x) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty \end{aligned} \tag{1.1}$$

with p subcritical, i.e. $1 < p < (N + 2)/(N - 2)$, $N \geq 3$, an assumption that I make throughout this paper. By the well-known results of Berestycki-Lions [1], Gidas, Ni and Nirenberg [9] and Kwong [15] we know that the set of solutions to (1.1) is made of all translates of a unique positive radial solution. Note that the *ground state* condition

$$w(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \tag{1.2}$$

is an essential piece of information here.

I now turn to the description of the two model problems.

1) A dissipative case: a nonlinear parabolic equation. Let us consider a globally defined in time weak positive solution $u = u(x, t) \in C([0, +\infty), H^1(\mathbb{R}^N))$ to the following problem:

$$\begin{aligned} u_t &= \Delta u - u + u^p \quad \text{in } \mathbb{R}^N \times (0, +\infty) \\ u(\cdot, 0) &= u_0(\cdot) \in C_0^\infty(\mathbb{R}^N) \end{aligned} \tag{1.3}$$

We have the following convergence result:

Theorem 1.1 ([4, 7, 8]) *Under the above assumptions, $u(\cdot, t)$ converges (in the $H^1(\mathbb{R}^N)$ sense) either to zero or to a solution of (1.1).*

2) A conservative case: a nonlinear elliptic equation. Let us now consider an entire positive solution $u = u(x)$, $x = (x_1, \dots, x_{N+1})$ to the scalar field equation

$$\Delta_{\mathbb{R}^{N+1}} u - u + u^p = 0$$

in \mathbb{R}^{N+1} , and suppose we are interested in solutions which are not necessarily ground states in the sense of (1.2). The simplest case is to assume that u goes to zero in a cylindrical set of directions. This leaves out one variable (x_{N+1} , say) in the direction of which one wants to study the possible asymptotic behaviour of u . For that reason, it is convenient to think of x_{N+1} as *time*, and recast the problem as:

$$\begin{aligned} u_{tt} + \Delta u - u + u^p &= 0 \quad \text{in } \mathbb{R}^N \times (-\infty, +\infty) \\ u(\cdot, t) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \text{ uniform in } t \in \mathbb{R}, \end{aligned} \tag{1.4}$$

with $(x, t) = (x_1, \dots, x_N, x_{N+1})$ and $\Delta = \sum_{1 \leq i \leq N} \frac{\partial^2}{\partial x_i^2}$, the Laplace operator in \mathbb{R}^N .

Since this problem is conservative, soliton-like solution may exists, so it is natural to assume that

$$u_t(x, t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \text{ for all } x \in \mathbb{R}^N. \tag{1.5}$$

Under these assumptions, one has the following convergence results:

Theorem 1.2 ([3]) *Let u be a bounded weak solution to (1.4) satisfying (1.5). Then $u(\cdot, t)$ converges (in the $H^1(\mathbb{R}^N)$ sense) either to zero or to a solution of (1.1) as $t \rightarrow +\infty$.*

Since time is reversible in (1.4), it is straightforward to get the same result as $t \rightarrow -\infty$ if one assumes the equivalent of (1.5) as $t \rightarrow -\infty$. Moreover, it is an interesting fact that the right- and left-hand limits actually turn out to coincide. I refer to [3] for a proof of this.

Theorem 1.1 has been obtained in earlier independent works by Cortazar, Elgueta, del Pino [7] and Feireisl and Petzeltová [8] by different methods. Theorem 1.1 and Theorem 1.2 are quite particular cases of the results in [4] and [3] respectively. I simplified the setting here in order to draw a parallel between these two results.

In the sequel, I denote by case 1, or parabolic case (resp. 2 or elliptic case) the situation prevailing in Theorem 1.1 (resp. 1.2).

2 Sketch of the proofs

The key role is played by the energy functional

$$E(u(\cdot, t)) = \int \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} dx, \tag{2.1}$$

where $F(s) = \int_0^s f(\zeta) d\zeta = -\frac{1}{2}s^2 + \frac{1}{p+1}s^{p+1}$, $f(\zeta) = -\zeta + \zeta^p$, and its first “moments” (for want of a better name):

$$E_i(u(\cdot, t)) = \int x_i \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} dx, \quad (2.2)$$

$i = 1, \dots, N$. As for the ω -limit set, it is defined as usual by:

$$\omega(u) = \bigcap_{T>0} \overline{\bigcup_{t \geq T} \{u(\cdot, t)\}} \quad (2.3)$$

in the parabolic case. In the elliptic case, some care is needed. One introduces the $\{v(t)\}_{t \in \mathbb{R}}$, defined as $v(t)(x, \tau) = u(x, t + \tau)$, for all $(x, \tau) \in \mathbb{R}^N \times [0, 1]$. The correct notion of (right-hand) ω -limit set turns out to be in this case:

$$\omega(u) = \bigcap_{T>0} \overline{\bigcup_{t \geq T} \{v(t)\}}.$$

Here the closures are taken for instance in the $C^2(\mathbb{R}^N)$ (resp. $C^2(\mathbb{R}^N \times [0, 1])$) topology.

The relevant properties of these functions are summarized in the following result.

Proposition 2.1 *In both cases 1 and 2 we have:*

- a) $\omega(u)$ is either $\{0\}$ or made of positive steady states, i.e. solutions to (1.1)
- b) E is constant on $\omega(u)$
- c) Each function E_1, \dots, E_N assumes a constant value on $\omega(u)$.

Note that in case 2, it is part of result a) that the functions in $\omega(u)$ do not depend on $\tau \in [0, 1]$.

Theorem 1.1 and 1.2 are simple consequences of this proposition. Indeed, suppose for contradiction that $\omega(u)$ were to contain two distinct translates of the radial solution of (1.1), say w_1 and w_2 . Up to a Euclidean change in co-ordinates, $w_1(x) = w_1(|x|)$ and $w_2(x) = w_2(|x - \alpha e_1|)$ with $\alpha \neq 0$. Now $E_1(w_1) = E_1(w_2)$ yields:

$$\begin{aligned} 0 &= E_1(w_1) - E_1(w_2) \\ &= \int x_1 \left\{ \frac{1}{2} |\nabla w_2|^2 - F(w_2) \right\} dx \\ &= \int (x_1 - \alpha) \left\{ \frac{1}{2} |\nabla w_2(|x - \alpha e_1|)|^2 - F(w_2(|x - \alpha e_1|)) \right\} dx + \alpha E(w_2) \\ &= \alpha E(w_2). \end{aligned}$$

Here and in the sequel, unless otherwise specified, all integrals in space range over \mathbb{R}^N .

Multiplying the stationary equation in (1.1) by w_2 and integrating by parts, it is straightforward to see that $E(w_2) \neq 0$, a contradiction. Convergence then follows easily from the fact that $\omega(u)$ is a singleton, by compactness arguments that I do not reproduce here.

Lemma 2.1 $\exists \varepsilon_0 > 0 \exists C > 0$ such that $\forall \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N, \forall k \in \{0, 1\}$ in case 1 ($\{0, 1, 2\}$ in case 2), $|\alpha| + k \leq 2$,

$$|\partial_x^\alpha \partial_t^k u(x, t)| \leq C e^{-\varepsilon_0 |x|} \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}_+ \text{ (resp. } \mathbb{R}).$$

Proof: It results from Corollary 3.1 in [7] and Lemma 2.1 in [3], to which I refer. The proof is based on comparison principles, together with Harnack inequality and blow-up arguments in the parabolic case.

We now turn to the proof of part a) and b) in Proposition 2.1.

Case 1: It is well-known that $t \mapsto E(u(\cdot, t))$ is decreasing since:

$$\frac{d}{dt} \left\{ \int \left\{ \frac{1}{2} |\nabla u|^2 - F(u) \right\} dx \right\} = - \int u_t^2 dx. \quad (2.4)$$

Hence $\int_0^{+\infty} dt \int u_t^2 dx < \infty$. Making use of Lemma 2.1, it is standard to infer part a) and b) in Proposition 2.1.

Case 2: Let us test equation (1.4) with u_t :

$$\int_t^{t'} ds \int \{u_t u_{tt} + \Delta u u_t + f(u) u_t\} dx = 0.$$

Integrating by part (see Lemma 2.1) results in:

$$E(u(\cdot, t')) - E(u(\cdot, t)) = \frac{1}{2} \int u_t^2(x, s) dx \Big|_{s=t}^{t'}.$$

Hence by assumption (1.5) and Lemma 2.1, clearly $\exists \lim_{t \rightarrow +\infty} E(u(\cdot, t))$, Now, testing (1.4) with $\phi \in C_0^\infty(\mathbb{R}^N)$ and integrating by parts results in:

$$\begin{aligned} \int_t^{t+1} ds \int \{u_{tt} + \Delta u + f(u)\} \phi(x) dx &= 0 \\ \int u_t(s, x) \phi(x) dx \Big|_{s=t}^{t+1} + \int_t^{t+1} ds \int \{u \Delta \phi + f(u) \phi\} dx &= 0. \end{aligned}$$

Since the first term in this last expression goes to zero as $t \rightarrow +\infty$ by assumption (1.5) and Lemma 2.1, it is clear from the definition of $\omega(u)$, Lemma 2.1, and (1.5) again that any $v = v(x, \tau) \in \omega(u)$ is actually independent of τ and satisfies $\forall \phi \in C_0^\infty(\mathbb{R}^N) \int \{v \Delta \phi + f(v) \phi\} dx = 0$. Since $v \in C^2$, it implies that v is a steady state. This completes the proof of part a) and b) in Proposition 2.1.

I shall now sketch the proof of part c) in Proposition 2.1 in Case 1 and 2.

Case 1: That the E_i 's are constant on $\omega(u)$ rely on the following lemma:

Lemma 2.2 *i) We have $\frac{d}{dt} E_i(u(\cdot, t)) = - \int_{\mathbb{R}^N} x_i u_t^2 dx$*

ii) There exists $\delta > 0$ such that

$$\int_0^{+\infty} dt \int e^{\delta|x|} u_t^2 dx < +\infty.$$

Hence the limits $\lim_{t \rightarrow +\infty} E_i(u(\cdot, t))$, $i = 1, \dots, N$, are well-defined.

Proof of i) Denoting by u_i the derivative of u with respect to x_i one has:

$$\begin{aligned} \frac{d}{dt} E_i(u(\cdot, t)) &= - \int u_t \{ \nabla \cdot (x_i \nabla u) + x_i f(u) \} dx \\ &= - \int x_i u_t^2 dx - \int u_t u_i dx \end{aligned}$$

$$\begin{aligned} \int u_t u_i dx &= \int \{ \Delta u + f(u) \} u_i dx \\ &= - \int \nabla u \cdot \nabla u_i dx + \int f(u) u_i dx = 0. \end{aligned}$$

Proof of ii) Taking polar co-ordinates $x = (r, \theta)$, define for $r > 0$:

$$H^T(r) = \frac{1}{2} \int_0^T dt \int_{S^{N-1}} u_t^2(r, \theta, t) d\theta.$$

Denoting by $\Delta_r = \frac{\partial^2}{\partial r^2} + \frac{(N-1)}{r} \frac{\partial}{\partial r}$ the radial Laplace operator, we have: We have:

$$\begin{aligned} \Delta_r H^T &= \int_0^T dt \int_{S^{N-1}} u_t \Delta_r u_t d\theta + \int_0^T dt \int_{S^{N-1}} |\nabla_r u_t|^2 d\theta \\ &\geq -\frac{1}{2} \int_{S^{N-1}} u_t^2(0, r, \theta) d\theta - \int_0^T dt \int_{S^{N-1}} f'(u) u_t^2 d\theta, \end{aligned}$$

hence:

$$\Delta_r H^T(r) \geq -\psi(r) + \alpha H^T(r) \quad \forall r \geq R_0,$$

for some positive constants α, R_0 and some $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \geq 0$.

Now a simple comparison argument with the solutions of $\Delta_r g_0 - \alpha g_0 = -\psi$, $g_0(r_0) = 0$, $g_0(r) \rightarrow 0$ as $r \rightarrow \infty$ and $\Delta_r g_1 - \alpha g_1 = 0$, $g_1(r_0) = 1$, $g_1(r) \rightarrow 0$ as $r \rightarrow \infty$ implies $\forall r_0 \geq R_0 \exists \delta > 0 \exists C > 0 \forall T > 0 \forall r \geq r_0$ $0 \leq H^T(r) \leq C(1 + H^T(r_0))e^{-2\delta r}$. Since $\int_0^{+\infty} dt \int u_t^2 dx < +\infty$ by Fubini's Theorem $H^T(r_0) \rightarrow H^\infty(r_0) < \infty$ as $T \rightarrow \infty$ for a.e. r_0 . Hence $H^\infty(r) \leq C e^{-2\delta r} \forall r \geq r_0$.

Case 2: Let us define

$$\widetilde{E}_i = E_i - \frac{1}{2} \int x_i u_i^2 dx.$$

Differentiating \widetilde{E}_i results in:

$$\frac{d}{dt}\widetilde{E}_i(u) = - \int u_t \{ \nabla \cdot (x_i \nabla u) + x_i f(u) \} dx - \int x_i u_t u_{tt} dx$$

integrating by parts:

$$\frac{d}{dt}\widetilde{E}_i(u) = \int u_t u_i dx \tag{2.5}$$

differentiating once more:

$$\begin{aligned} \frac{d^2}{dt^2}\widetilde{E}_i(u) &= \int \{ \Delta u + f(u) \} u_i dx - \int u_t u_{it} dx \\ &= - \int \nabla u \cdot \nabla u_i - f(u) u_i + u_t u_i \\ &= - \int \frac{\partial}{\partial x_i} \left\{ \frac{1}{2} |\nabla u|^2 - F(u) + \frac{u_i^2}{2} \right\} dx = 0, \end{aligned}$$

thanks to Fubini's Lemma. Note that I have repeatedly used Lemma 2.1 here. Hence $\widetilde{E}_i(u) = \alpha t + \beta$ for constants α, β . Now (2.5) together with assumption (1.5) imply $\lim_{t \rightarrow \infty} \frac{d}{dt} \widetilde{E}_i(u(\cdot, t)) = 0$. Thus $\alpha = 0$, hence $t \mapsto \widetilde{E}_i(u(\cdot, t))$ is constant. Hence E_i is constant on $\omega(u)$ by (1.5). This completes the proof of part c) in Proposition 2.1, hence that of Theorem 1.2.

Acknowledgments It is my pleasure to thank the organizers of the USA-Chile Congress on Nonlinear Analysis for their invitation. This paper is based on a contribution to the Nonlinear Analysis 2000 meeting (Courant Institute of Mathematical Sciences, New York, May 2000).

References

- [1] Berestycki H., Lions P.-L. Non linear scalar field equations I. Existence of a ground state. *Arch. Rational Mech. Anal.* **82**, 313-345 (1983).
- [2] Busca J. Symmetry and nonexistence results for Emden-Fowler equations in cones. To Appear in *Diff. Int. Eq.*
- [3] Busca J., Felmer, P. Qualitative properties of some bounded positive solutions to scalar field equations. To Appear in *Calc. Var.*
- [4] Busca J., Jendoubi M.A., Polacik P. Convergence to equilibrium for semi-linear parabolic problems in \mathbb{R}^N . In preparation.
- [5] Cazenave T., Haraux A. Introduction aux problèmes d'évolution semi-linéaires. *Coll. Mathématiques et Applications*, Vol 1, Ellipses, Paris 1990.
- [6] Coffman C.V. Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions, *Arch. Rational Mech. Anal.* **46** (1972), 81-95 .

- [7] Cortazar C., del Pino M., Elgueta M. The problem of uniqueness of the limit in a semilinear heat equation. *Comm. Part. Diff. Eq.* **24**, no. 11-12 (1999), 2147-2172.
- [8] Feireisl E., Petzeltová H. Convergence to a ground state as a threshold phenomenon in nonlinear parabolic equations. *Diff. and Int. Eq.* **10**, no 1 (1997), 181-196 .
- [9] Gidas B., Ni W.-M., Nirenberg L. Symmetry of positive solutions of nonlinear elliptic equations in R^n . *Mathematical Analysis and Applications Part A*, Adv. Math. Suppl. Studies 7A, Academic Press, New York(1981), 369-402.
- [10] Hale J.K., Raugel G. Convergence in gradient-like systems with applications to P.D.E. *Z. Angew. Math. Phys.* **43** (1992), 63-124.
- [11] Haraux A., Polacik P. Convergence to a positive equilibrium for some nonlinear evolution equations in a ball. *Acta Math. Univ. Comenianae* **2** (1992), 129-141.
- [12] Haraux, A., Jendoubi, M. A. Convergence of solutions of second-order gradient-like systems with analytic nonlinearities. *J. Diff. Eq.* **144**, no. 2 (1998), 313–320.
- [13] Jendoubi, M.A. Convergence vers un équilibre de divers systèmes-gradient multidimensionnels. Ph.D. Dissertation, Paris VI University, 1997.
- [14] Jendoubi, M.A. A simple unified approach to some convergence theorems of L. Simon. *J. Funct. Anal.* **153**, no. 1 (1998), 187–202.
- [15] Kwong M.K. Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n . *Arch. Rational Mech. Anal.* **105**, 243-266 (1989).
- [16] Licois J.R., Véron L. A class of nonlinear conservative elliptic equations in cylinders. *Ann. Scuola Norm. Sup. Pisa* **26**, no. 2, 249-283 (1998).
- [17] Lions P.-L. Structure of the set of the steady-state solutions and asymptotic behavior of semilinear heat equations, *J. Diff. Eq* **53** (1984), 362-386.
- [18] Matano H. Convergence of solutions of one-dimensional semilinear heat equation *J. Math. Kyoto Univ.* **18** (1978), 221-227.
- [19] Ogiwara T., Matano H. Stability analysis in order-preserving systems in the presence of symmetry. *Proc. Roy. Soc. Edin. Sect. A* 129 (1999), no. 2, 395-438.
- [20] Ogiwara, T., Matano, H. Monotonicity and convergence results in order-preserving systems in the presence of symmetry. *Discrete Contin. Dynam. Systems* **5** (1999), no. 1, 1-34.

- [21] Polacik P., Rybakowski K.P. Nonconvergent bounded trajectories in semi-linear heat equations, *J. Diff. Eq.* **124** (1996), 472-494.
- [22] Simon L. Asymptotics for a class of non-linear evolution equations with applications to geometric problems, *Annals of Maths.* **118** (1983), 525-571.
- [23] Zelenyak T.J. Stabilization of solutions of boundary value problems for a second-order parabolic equation with one space variable *Differentsial'nye Uravneniya* **4** (1968), 17-22.

JÉRÔME BUSCA

Laboratoire de Mathématiques et de Physique Théorique

Université François Rabelais

Parc de Grandmont

37200 Tours, France

email: busca@univ-tours.fr