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Asymptotic behaviour of the solvability set for pendulum-type equations with linear damping and homogeneous Dirichlet conditions *

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Abstract

We show some results on the asymptotic behavior of the solvability set for a nonlinear resonance boundary-value problem, with linear damping, periodic nonlinearity and homogeneous Dirichlet boundary conditions. Our treatment of the problem depends on a multi-dimensional generalization of the Riemann-Lebesgue lemma.

1 Introduction

Solvability of the nonlinear boundary-value problem

$$\begin{gathered} -u''(x) - \alpha u'(x) - \lambda_1(\alpha)u(x) + g(u(x)) = h(x), \quad x \in [0,\pi], \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.1}$$

has been studied by several authors under the following set of hypotheses.

[H] α is a given real number, $\lambda_1(\alpha) = 1 + \alpha^2/4$ is the first eigenvalue of the eigenvalue problem

$$\begin{gathered} -u''(x) - \alpha u'(x) = \lambda u(x), \quad x \in [0,\pi] \\ u(0) = u(\pi) = 0\,, \end{gathered} \tag{1.2}$$

$g : \mathbb{R} \to \mathbb{R}$ is a continuous and T-periodic function with zero mean value, and $h \in L^1[0,\pi]$.

The case $\alpha = 0$ can be found in [1, 4, 9, 10], while the case $\alpha \neq 0$ has been recently treated in [2]. These type of problems, with periodic nonlinearity, are important in applications and (1.1) models, for example, the motion of a

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pendulum clock ([6, 8]). If g is not identically zero and $\psi(x) = \exp(\alpha x/2)\sin(x)$ is the principal positive eigenfunction of the adjoint problem to (1.2) for $\lambda = \lambda_1(\alpha)$, it was proven in ([2]) that for a given $\tilde{h} \in L^1[0,\pi]$, with $\int_0^\pi \tilde{h}(x)\psi(x)\,dx = 0$, there exist real numbers $a_1(\tilde{h}) < 0 < a_2(\tilde{h})$, such that (1.1), with h given by $h(x) = a\psi(x) + \tilde{h}(x)$, $(a \in \mathbb{R})$, has solution if, and only if, $a \in [a_1(\tilde{h}), a_2(\tilde{h})]$. However, very little is known on the behavior of the functionals a_1 and a_2. In this paper we deal with their asymptotic behavior. More precisely, we shall show that if

$$\tilde{L}^1[0,\pi] = \Big\{h \in L^1[0,\pi] : \int_0^\pi h(x)\psi(x)\,dx = 0\Big\},$$

then there exist a subset $F \subset \tilde{L}^1[0,\pi]$, (which will be explicitly described) of first category in $\tilde{L}^1[0,\pi]$ in the sense of Baire, such that for each $\tilde{h} \in \tilde{L}^1[0,\pi] \setminus F$,

$$\lim_{|\lambda|\to\infty} a_1(\lambda\tilde{h}) = \lim_{|\lambda|\to\infty} a_2(\lambda\tilde{h}) = 0\,. \tag{1.3}$$

As a trivial consequence, the set of functions $\tilde{h} \in \tilde{L}^1[0,\pi]$ for which (1.3) is true, is a dense and second category subset of $\tilde{L}^1[0,\pi]$. In the final remarks we briefly comment why this result cannot be strengthened very much, since, under hypotheses [H], it may happens that (1.3) does not occur also for a dense subset of $\tilde{L}^1[0,\pi]$ (see [3]).

Let us point out that related results for the case of periodic boundary conditions and $\alpha = 0$ can be found in [7]. However, to the best of our knowledge, properties like (1.3) for the problem (1.1) and periodic nonlinearity g, have not been previously treated in the literature, even for the case $\alpha = 0$. In the proofs we use the Liapunov-Schmidt reduction, The Baire's category theorem, some notions on measure theory and the multi-dimensional version of the Riemann-Lebesgue lemma developed in Lemma 3.1 (see [4, 7, 11] for the classical one-dimensional version).

Through this paper, $\langle\cdot,\cdot\rangle$ will stand for the Euclidean inner product in $\mathbb{R}^N$, while for any $x \in \mathbb{R}^N$, $\|x\| := \sqrt{\langle x, x\rangle}$ will denote its associated norm and $x_1, \ldots, x_N$ its components. We will write as $\|\cdot\|_1$ and $\|\cdot\|_\infty$ the usual norms in $L^1[0,\pi]$ and $L_\infty[0,\pi]$ respectively. A function $h \in L^1[0,\pi]$ will be called a *step function* if there exists a partition $0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = \pi$ of the interval $[0,\pi]$, and constants c_i, $1 \le i \le m$ such that $h|_{(x_{i-1},x_i)} \equiv c_i$, $1 \le i \le m$. If, furthermore, all constants c_i, $i : 1, \ldots, m$, are not zero, h will be called a *non-vanishing step function*. Finally, for every measurable set $I \subset [0,\pi]$, we will denote by χ_I its characteristic function, and by $\operatorname{meas} I$ its one-dimensional Lebesgue measure.

2 Liapunov-Schmidt reduction

Let any $h \in L^1[0,\pi]$ be written in the form $h(x) = a\psi(x) + \tilde{h}(x)$, $a \in \mathbb{R}$, $\int_0^\pi \tilde{h}(x)\psi(x)\,dx = 0$. Let $W_0^{2,1}[0,\pi] = \{u \in W^{2,1}[0,\pi],\ u(0) = u(\pi) = 0\}$ be the usual Sobolev space with the usual $W_0^{2,1}[0,\pi]$ norm, and define the operators

$$L : W_0^{2,1}[0,\pi] \to L^1[0,\pi], \quad Lu = -u'' - \alpha u' - \lambda_1(\alpha)u,$$
$$N : W_0^{2,1}[0,\pi] \to L^1[0,\pi], \quad (Nu)(x) = a\psi(x) + \tilde{h}(x) - g(u(x))\,.$$

Then (1.1) is equivalent to the operator equation

$$Lu = Nu\,. \tag{2.1}$$

Let $\varphi(x) = \exp(\frac{-\alpha}{2}x)\sin(x)$ be the principal eigenfunction associated with $\lambda = \lambda_1(\alpha)$ of the eigenvalue problem (1.2). Each $u \in W_0^{2,1}[0,\pi]$ can be written in the form $u(x) = c\varphi(x) + \tilde{u}(x)$, $c \in \mathbb{R}$, $\int_0^\pi \tilde{u}(x)\varphi(x)\,dx = 0$. Consider the linear, continuous projections

$$P : W_0^{2,1}[0,\pi] \to W_0^{2,1}[0,\pi], \quad c\varphi + \tilde{u} \mapsto c\varphi,$$
$$Q : L^1[0,\pi] \to L^1[0,\pi], \quad a\psi + \tilde{h} \mapsto a\psi\,.$$

(so that $\operatorname{im} P = \ker L$, $\operatorname{im} L = \ker Q = \tilde{L}^1[0,\pi]$), and let $K : \ker Q \to \ker P$ be the inverse of the mapping $L : \ker P \to \ker Q$. With this notation, (2.1) is equivalent to the system

$$\tilde{u} = K(I-Q)N(c\varphi + \tilde{u}) \tag{2.2}$$

$$a = \frac{1}{\int_0^\pi (\psi(x))^2\,dx} \int_0^\pi g(c\varphi(x) + \tilde{u}(x))\psi(x)\,dx \tag{2.3}$$

(auxiliary and bifurcation equation, respectively). Since the natural embedding of $W_0^{2,1}[0,\pi]$ into $C[0,\pi]$ is compact, we get that for any fixed $c \in \mathbb{R}$, there exists at least one solution $\tilde{u} \in \ker P$ of (2.2) ([4], [5]). Denote by Σ the solution set of equation (2.2), i.e.,

$$\Sigma = \{(c,\tilde{u}) \in \mathbb{R} \times \ker P : \tilde{u} = K(I-Q)N(c\varphi + \tilde{u})\}$$

and let $\Gamma : \Sigma \to \mathbb{R}$, be defined by

$$\Gamma(c,\tilde{u}) = \frac{1}{\int_0^\pi (\psi(x))^2\,dx} \int_0^\pi g(c\varphi(x) + \tilde{u}(x))\psi(x)\,dx \tag{2.4}$$

Hence, for a given $\tilde{h}$, BVP (1.1), with $h(x) = a\psi(x) + \tilde{h}(x)$ has solution, if and only if, a belongs to the set $\Gamma(\Sigma)$. The next Theorem, which describes the solvability of (1.1) may be seen in [2].

Theorem 2.1 *Let us assume the hypotheses [H] with g not identically zero. Then for each $\tilde{h} \in \tilde{L}^1[0,\pi]$, there exist real numbers $a_1(\tilde{h}) < 0 < a_2(\tilde{h})$ such that (1.1) with $h(x) = a\psi(x) + \tilde{h}(x)$ has a solution if, and only if, $a \in [a_1(\tilde{h}), a_2(\tilde{h})]$.*

In the next section we deal with the asymptotic behavior of the functionals a_1 and a_2 as $\tilde{h}$ becomes 'large'.

3 Asymptotic behavior of the solvability set

In what follows, choose one of the functionals $a_1,\ a_2$, and denote it simply by a. Let $\tilde{h} \in \tilde{L}^1[0,\pi]$ be given. Taking into account the results of the previous section, we obtain that, for each $\lambda \in \mathbb{R}$, there is $(c_\lambda, u_\lambda) \in \mathbb{R} \times \ker P$ such that

$$u_\lambda = K(I-Q)N_\lambda(c_\lambda\varphi + u_\lambda) \tag{3.1}$$

$$a(\lambda\tilde{h}) = \frac{1}{\int_0^\pi (\psi(x))^2\,dx} \int_0^\pi g(c_\lambda\varphi(x) + u_\lambda(x))\psi(x)\,dx \tag{3.2}$$

where $N_\lambda u(x) = a\psi(x) + \lambda\tilde{h}(x) - g(u(x)) = \lambda\tilde{h}(x) + N_0u(x)$. Therefore, equation (3.2) becomes

$$\begin{gathered} a(\lambda\tilde{h}) \int_0^\pi (\psi(x))^2\,dx = \\ \int_0^\pi g(c_\lambda\varphi(x) + \lambda K\tilde{h} + K(I-Q)N_0(c_\lambda\varphi + u_\lambda))\psi(x)\,dx \end{gathered} \tag{3.3}$$

Since $K(I-Q)N_0(c_\lambda\varphi + u_\lambda) = K(I-Q)(-g(c_\lambda\varphi(.) + u_\lambda(.))$, there is a constant $M > 0$ independent of $\lambda \in \mathbb{R}$, such that

$$\begin{gathered} |K(I-Q)N_0(c_\lambda\varphi + u_\lambda)(x)| \le M,\ \forall\ x \in [0,\pi], \\ |(K(I-Q)N_0(c_\lambda\varphi + u_\lambda))'(x)| \le M,\ \forall\ x \in [0,\pi] \end{gathered} \tag{3.4}$$

Previous discussion motivates the next multidimensional generalization of the Riemann-Lebesgue lemma.

Lemma 3.1 *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and T-periodic function with zero mean value and let $u_1, \ldots, u_N \in C^1[0,\pi]$ be given functions satisfying the following property:*

[P] *If $\rho_1, \ldots, \rho_N$ are real numbers such that*

$$\text{meas}\left\{x \in [0,\pi] : \sum_{i=1}^N \rho_i u_i'(x) = 0\right\} > 0\,,$$

then $\rho_1 = \ldots = \rho_N = 0$.

Let $B \subset C^1[0,\pi]$ be such that the set $\{b',\ b \in B\}$ is uniformly bounded in $C[0,\pi]$. Then, for any given function $r \in L^1[0,\pi]$, we have

$$\lim_{\|\rho\|\to\infty} \int_0^\pi g\left(\sum_{i=1}^N \rho_i u_i(x) + b(x)\right) r(x)\,dx = 0 \tag{3.5}$$

uniformly with respect to $b \in B$.

Proof. Let $r \in L^1[0,\pi]$ be a given function and let $\{\rho^n,\ n \in \mathbb{N}\} \subset \mathbb{R}^N$ and $\{b^n,\ n \in \mathbb{N}\} \subset B$ be given sequences with $\|\rho^n\| \to \infty$. If we define $\mu^n = \rho^n/\|\rho^n\|$, we have, at least for a subsequence, that $\mu^n \to \mu$ for some $\mu \in \mathbb{R}^N$ with $\mu_1^2 + \ldots + \mu_N^2 = 1$. If $u = (u_1, \ldots, u_N)$, then by hypothesis, $\operatorname{meas}(Z) = 0$, where $Z = \{x \in [0,\pi] : \langle \mu, u'(x)\rangle = 0\}$. This implies that the linear span of the set

$$S = \{\langle \mu, u'\rangle \chi_I :\ I \text{ is any compact subinterval of } [0,\pi], I \cap Z = \emptyset\} \tag{3.6}$$

is a dense set in $L^1[0,\pi]$. To see this, let us define

$$S_1 = \{\chi_I :\ I \text{ is any compact subinterval of } [0,\pi], I \cap Z = \emptyset\} \tag{3.7}$$

Then, for any open subset $A \subset [0,\pi]$ (in particular, for any open subinterval of $[0,\pi]$), $\operatorname{meas}(A \setminus Z) = \operatorname{meas}(A)$. Since $A \setminus Z$ is also open, there exists an at most countable collection $\{I_i,\ i \in \mathbb{N}\}$ of pairwise disjoint open intervals such that $A \setminus Z = \cup_{i\in\mathbb{N}} I_i$ and $\operatorname{meas}(A \setminus Z) = \sum_{i\in\mathbb{N}} \operatorname{meas}(I_i)$. Consequently, the linear span of the set S_1 is a dense set in the set of step functions and therefore in $L^1[0,\pi]$.

Now, let χ_I be a given element of S_1. Write $w = \langle \mu, u'\rangle$ and $m = \inf_I\ |w|$ $(m > 0)$. Finally, fix $\epsilon > 0$. Choose a partition of $I = [a,b]$, $a = a_0 < a_1 < \ldots < a_{m-1} < a_m = b$ such that if $x, y \in J_i = [a_{i-1}, a_i]$, $1 \le i \le m$, then $|w(x) - w(y)| \le \epsilon$. Then, for any $x \in I$, there is some i, $1 \le i \le m$, such that $x \in J_i$ and

$$\left|\chi_I(x) - \sum_{i=1}^{m} \frac{w\chi_{J_i}(x)}{w(a_i)}\right| = \left|\frac{w(a_i) - w(x)}{w(a_i)}\right| \le \epsilon/m,$$

so that

$$\left\|\chi_I - \sum_{i=1}^{m} \frac{w\chi_{J_i}}{w(a_i)}\right\|_1 \le \epsilon\pi/m\,.$$

Consequently, we deduce that the linear span of S is dense in S_1 and therefore in $L^1[0,\pi]$.

On the other hand, if l^∞ denotes the space of bounded sequences of real numbers with the usual norm, the linear operator $T : L^1[0,\pi] \to l^\infty$, $s \to \{(Ts)^n,\ n \in \mathbb{N}\}$, defined by

$$(Ts)^n = \int_0^\pi g(\langle \rho^n, u(x)\rangle + b^n(x))s(x)\ dx,\ \forall\ s \in L^1[0,\pi],\ \forall\ n \in \mathbb{N},$$

is trivially continuous. Recall that our purpose is to prove that $T(L^1[0,\pi]) \subset l_0$, the closed subspace of l^∞ of all sequences which converge to zero. Since T is continuous and l_0 is closed, to prove the lemma it is sufficient to demonstrate that $T(S) \subset l_0$, i.e.,

$$\lim_{n\to\infty} \int_I g(\langle \rho^n, u(x)\rangle + b^n(x))(\langle \mu, u'(x)\rangle)\, dx = 0, \tag{3.8}$$

for any compact subinterval I of $[0,\pi]$ such that $I\bigcap Z=\emptyset$. But, if $v^n, v: I\to\mathbb{R}$ are defined as

$$v^n(x)=\langle\mu^n,u(x)\rangle+b^n(x)/\|\rho^n\|,$$
$$v(x)=\langle\mu,u(x)\rangle,\ \forall\, x\in[0,\pi],$$

we trivially have

$$\lim_{n\to\infty}\int_I g(\|\rho^n\|v^n(x))(v'(x)-(v^n)'(x))\,dx=0 \tag{3.9}$$

and

$$\begin{aligned}&\lim_{n\to\infty}\int_I g(\|\rho^n\|v^n(x))(v^n)'(x))\,dx \qquad (3.10)\\ =&\lim_{n\to\infty}\frac{G(\|\rho^n\|v^n(\max I))-G(\|\rho^n\|v^n(\min I))}{\|\rho^n\|}=0\end{aligned}$$

where G is any primitive function of function g. Now, (3.9) and (3.10) imply (3.8).

Remark. It is clear that both the conclusion and the proof of the previous lemma are still true under more general hypotheses on the function g. It is sufficient that g be a continuous and bounded function with bounded primitive G.

Next, we apply the previous lemma to the specific problem of the asymptotic behavior of the functional a whose expression was given in (3.3).

Corollary 3.2 *Let $\tilde h\in\tilde L^1[0,\pi]$ be a given function and suppose that the functions $K\tilde h$ and φ satisfy the following property*

[P1] *If ρ_1,ρ_2 are real numbers such that*

$$\text{meas}\{x\in[0,\pi]:\rho_1(K\tilde h)'(x)+\rho_2\varphi'(x)=0\}>0,$$

then $\rho_1=\rho_2=0$.

Let $B\subset\tilde L^1[0,\pi]$ be any bounded subset. Then

$$\lim_{|\lambda|\to\infty}a(\lambda\tilde h+b)=0, \tag{3.11}$$

uniformly with respect to $b\in B$.

Proof. For each $\lambda\in\mathbb{R}$ and each $b\in B$, there is $(c_{\lambda,b},\tilde u_{\lambda,b})\in\Sigma_{\lambda,b}$ such that

$$\begin{aligned}a(\lambda\tilde h+b)\int_0^\pi(\psi(x))^2\,dx &= \int_0^\pi g(c_{\lambda,b}\varphi(x)+\lambda K\tilde h(x)+Kb(x) \qquad (3.12)\\ &\quad +K(I-Q)N_0(c_{\lambda,b}\varphi+\tilde u_{\lambda,b})(x))\psi(x)\,dx\end{aligned}$$

where $\Sigma_{\lambda,b}$ is the corresponding solution set of the auxiliary equation for $\lambda\tilde{h}+b$. Since the set

$$\{Kb + K(I-Q)N_0(c_{\lambda,b}\varphi + \tilde{u}_{\lambda,b}),\ \lambda \in \mathbb{R},\ b \in B\}$$

is bounded in $C^1[0,\pi]$ (see (3.4)), the conclusion follows from the previous lemma.

The following equivalent version of previous corollary will be very useful for our purposes.

Corollary 3.3 *Let $\tilde{h} \in \tilde{L}^1[0,\pi]$ be a given function and suppose that, for every $\rho \in \mathbb{R}$,*

$$\text{meas}\{x \in [0,\pi]:\ (K\tilde{h})'(x) = \rho\varphi'(x)\} = 0.$$

Let $B \subset \tilde{L}^1[0,\pi]$ be any bounded subset. Then

$$\lim_{|\lambda|\to\infty} a(\lambda\tilde{h}+b) = 0, \tag{3.13}$$

uniformly with respect to $b \in B$.

Now, we state and prove our main result.

Theorem 3.4 *There exists a subset $F \subset \tilde{L}^1[0,\pi]$, of first category in $\tilde{L}^1[0,\pi]$, such that for any given $\tilde{h} \in \tilde{L}^1[0,\pi] \setminus F$, and each given bounded subset $B \subset \tilde{L}^1[0,\pi]$, one obtains*

$$\lim_{|\lambda|\to\infty} a(\lambda\tilde{h}+b) = 0 \tag{3.14}$$

uniformly with respect to $b \in B$.

Proof. Let

$$F = \left\{\tilde{h} \in \tilde{L}^1[0,\pi]:\ \exists\ \rho \in \mathbb{R} \text{ s.t. meas } \{x \in [0,\pi] : (K\tilde{h})'(x) = \rho\varphi'(x)\} > 0\right\}$$

Then $F = \cup_{n\in\mathbb{N}} F_n$, where

$$F_n = \left\{\tilde{h} \in \tilde{L}^1[0,\pi]:\ \exists\ \rho \in \mathbb{R} \text{ s.t. meas}\{x \in [0,\pi] : (K\tilde{h})'(x) = \rho\varphi'(x)\} \geq 1/n\right\}$$

Let us prove that each subset F_n is closed and has an empty interior. To see this, let us fix F_n. Then, since $K : \ker Q \to \ker P$ is a topological isomorphism, F_n is a closed subset of $\ker Q$ if and only if $K(F_n) \equiv G_n$ is a closed subset of $\ker P$. Now, it is clear that G_n is the set of functions

$$\{u \in \ker P : \exists\rho \in \mathbb{R} \text{ s.t. meas}\{x \in [0,\pi] : u'(x) = \rho\varphi'(x)\} \geq 1/n\}$$

Let $\{u_m,\ m \in \mathbb{N}\} \subset G_n$ be a sequence such that $\{u_m\} \to u$ in $\ker P$. Then, for any $m \in \mathbb{N}$, we can find $\rho_m \in \mathbb{R}$ such that

$$\text{meas}\{x \in [0,\pi] : u_m'(x) = \rho_m\varphi'(x)\} \geq 1/n$$

Since

$$\text{meas}\{x \in [0,\pi] : \varphi'(x) = 0\} = 0,$$

the sequence $\{\rho_m\}$ must be bounded and, after possibly passing to a subsequence, we can suppose, without loss of generality, that $\{\rho_m\} \to \rho$. Moreover, if we define

$$M_m = \{x \in [0,\pi] : u_m'(x) = \rho_m \varphi'(x)\}$$

then meas $M_m \geq 1/n$, $\forall\, m \in \mathbb{N}$ and meas $(\bigcap_{m=1}^{\infty} [\bigcup_{s=m}^{\infty} M_s]) \geq 1/n$. Finally, let us observe that if $x \in \bigcap_{m=1}^{\infty} [\bigcup_{s=m}^{\infty} M_s]$, then $u'(x) = \rho\varphi'(x)$, so that meas$\{x \in [0,\pi] : u'(x) = \rho\varphi'(x)\} \geq 1/n$ and consequently $u \in G_n$.

Next, we prove that F (and therefore each F_n) has an empty interior. To see this, let us define the function φ_1 as the primitive of φ with zero mean value and φ_2 as the primitive of φ_1 satisfying $\varphi_2(0) = \varphi_2(\pi) = 0$. Then, $\varphi_2 \in W_0^{2,1}[0,\pi]$, $\varphi_2'' = \varphi$ and for any $u \in W_0^{2,1}[0,\pi]$, we have

$$\int_0^{\pi} u\varphi = -\int_0^{\pi} u'\varphi_1 = \int_0^{\pi} u''\varphi_2.$$

As a consequence, the mapping $\Phi : \ker P \to \tilde{L}^1_{\varphi_2}[0,\pi]$, $u \to u''$ is a topological isomorphism, where

$$\tilde{L}^1_{\varphi_2}[0,\pi] = \{h \in L^1[0,\pi] : \int_0^{\pi} h(x)\varphi_2(x)\,dx = 0\}$$

Therefore, F has an empty interior in $\tilde{L}^1[0,\pi]$ provided $\Phi(K(F))$ has an empty interior in $\tilde{L}^1_{\varphi_2}[0,\pi]$. This last result is an easy consequence of the following lemma.

Lemma 3.5 *Let us denote by A the subset of $L^1[0,\pi]$ given by all the step functions and by B the subset of $L^1[0,\pi]$ given by all the non-vanishing step functions. Then,*

1. *The set $A \cap \tilde{L}^1_{\varphi_2}[0,\pi]$ is dense in $\tilde{L}^1_{\varphi_2}[0,\pi]$.*
2. *The set $B \cap \tilde{L}^1_{\varphi_2}[0,\pi]$ is dense in $\tilde{L}^1_{\varphi_2}[0,\pi]$.*
3. *$B \cap \Phi(K(F)) = \emptyset$*

Proof.

1. Let us choose any $h \in \tilde{L}^1_{\varphi_2}[0,\pi]$ and $\epsilon > 0$. Then, there exists $s \in A$ such that $\|h - s\|_1 < \min\,\{\epsilon/2\pi,\ \frac{\|\varphi_2\|_1}{\|\varphi_2\|_\infty}\}$. Now, the function $\tilde{s} = s + \frac{\int_0^{\pi} s\varphi_2}{\|\varphi_2\|_1}$ is again a step function which belongs to $\tilde{L}^1_{\varphi_2}[0,\pi]$ and such that $\|h - \tilde{s}\|_1 < \epsilon$.

2. Let us demonstrate that $B \cap \tilde{L}^1_{\varphi_2}[0,\pi]$ is dense in $A \cap \tilde{L}^1_{\varphi_2}[0,\pi]$. To see this, let us take $u \in A \cap \tilde{L}^1_{\varphi_2}[0,\pi]$. If $a, b \in \mathbb{R}$, define the function $u_{a,b} = u + a\chi_{[0,\pi/2]} + b\chi_{[\pi/2,\pi]}$. The condition for $u_{a,b}$ to belong to $\tilde{L}^1_{\varphi_2}[0,\pi]$ is

$$a\int_0^{\pi/2} \varphi_2 + b\int_{\pi/2}^{\pi} \varphi_2 = 0$$

Since $\int_0^{\pi/2} \varphi_2 < 0$ and $\int_{\pi/2}^{\pi} \varphi_2 < 0$, (think that, by the maximum principle, $\varphi_2 < 0$ in $(0, \pi)$), it is clear that we may choose a and b both different from zero but as small as we want in absolute value such that $u_{a,b} \in B \cap \tilde{L}^1_{\varphi_2}[0, \pi]$.

3. If $s \in B \cap \Phi(K(F))$, then there is $\tilde{h} \in F$ such that $K(\tilde{h}) = u$, $\Phi(u) = u'' = s$. Since $\tilde{h} \in F$, there exists $\rho \in \mathbb{R}$ such that meas $\{x \in [0, \pi] : u'(x) = \rho\varphi'(x)\} > 0$. Choose some nontrivial compact interval $I \subset [0, \pi]$ satisfying $s|_I \equiv c \neq 0$ and such that meas $\{x \in I : u'(x) = \rho\varphi'(x)\} > 0$. This implies that meas $\{x \in I : c = u''(x) = \rho\varphi''(x)\} > 0$, which is a contradiction with the form of the function φ.

Final Remark. Under the hypotheses [H], it is possible to show that, in many cases, the set of functions $\tilde{h} \in \tilde{L}^1[0, \pi]$ for which $\lim_{|\lambda| \to \infty} a(\lambda\tilde{h})$ is not zero, is also dense in $\tilde{L}^1[0, \pi]$. For example, this is true for the oscillating function δg in the place of g provided that $|\delta|$ is small enough. In this case, it may be proved that the previous limit is not zero if the function $u = K(\tilde{h})$ belongs to the set of functions in $\ker P$ for which there exists a partition $0 = x_0 < x_1 < \ldots < x_{p-1} < x_p = \pi$ and $1 \leq i_0 \leq p$ and constants $\mu \neq 0$, $c \neq 0$, such that

i) $u''_{[x_{i-1}, x_i]}$ is a constant function, for any $1 \leq i \leq p$, $i \neq i_0$.

ii) $u(x) = \mu\varphi(x) + c$, $\forall\, x \in [x_{i_0 - 1}, x_{i_0}]$.

After this, it may be proved that this set is dense in $\ker P$. The detailed proof may be found in [3].

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