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On the existence of infinitely many solutions to a damped sublinear boundary-value problem *

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Abstract

We prove the existence of infinitely many solutions (with prescribed nodal properties) to a damped sublinear boundary-value problem. The proofs are performed by means of an abstract continuation theorem and the time-map technique for strongly nonlinear operators.

1 Introduction

We study the existence and multiplicity of solutions to the boundary-value problem

$$\begin{gathered}(r^{(k-1)}u')' + r^{(k-1)}a(u')f(r,u) = r^{(k-1)}h(r,u,u'),\\ u'(0) = 0 = u(R)\end{gathered} \tag{1.1}$$

($k > 1$). As it is well-known, solutions to (1.1) are radially symmetric solutions to the following elliptic boundary-value problem on a ball $\mathcal{B} = B(0,R)$

$$\begin{gathered}\nabla\cdot(\nabla u) + a(|\nabla u|)f(|x|,u) = h(|x|,u,|\nabla u|) \quad \text{in } \mathcal{B},\\ u = 0 \quad \text{on } \partial\mathcal{B}.\end{gathered} \tag{1.2}$$

We deal with a so-called "sublinear" problem. More precisely, we assume that $a(\xi) = a_0 + |\xi|^q$ ($a_0 > 0$, $0 < q < 2$) and that the following conditions are satisfied:

(H_f) The function $f : [0,R]\times[-\varepsilon_0,\varepsilon_0] \to \mathbb{R}$ is continuous and such that

$$f(r,0) \equiv 0$$

and

$$\lim_{s\to 0}\frac{f(r,s)}{s} = +\infty \text{ uniformly in } r \in [0,R].$$

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(H_F) For $F(r,s) := \int_0^s f(r,x)\,dx$, F is differentiable with respect to $r \in [0,R]$ and there exists a continuous positive function $\alpha : [0,R] \to (0,+\infty)$ such that for all $r \in [0,R]$, all $s \in [-\varepsilon_0, \varepsilon_0]$,

$$\left|\frac{\partial F}{\partial r}(r,s)\right| \le \alpha(r)F(r,s).$$

(H_h) The function $h : [0,R] \times [-\varepsilon_0, \varepsilon_0] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists $H > 0$ such that for all $(r,s,\xi) \in [0,R] \times [-\varepsilon_0, \varepsilon_0] \times \mathbb{R}$,

$$|h(r,s,\xi)| \le H|\xi|.$$

Moreover, there exists a continuous function $C : \mathbb{R} \to (0,+\infty)$ such that

$$\lim_{s\to 0} \frac{h(r,s,\xi)}{s} = C(\xi) \text{ uniformly in } r \in [0,R]. \tag{1.3}$$

We point out that problem (1.1) can be considered "singular" in a two-fold sense. Indeed, on one hand, under condition (H_f) the uniqueness of the solutions to initial value problems associated to (1.1) must be guaranteed by (H_F); on the other hand, a singularity in the r-variable arises because of the boundary condition in zero. For more comments on (H_F) we refer to [3, 4, 6, 16].

Our main result is the following (cf. Theorem 4.1).

Theorem A *Assume (H_f)-(H_F)-(H_h) and let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ problem* (1.1) *has at least two solutions u_n and v_n with $u_n(0) > 0$ and $v_n(0) < 0$, both having exactly n zeros in $[0,R)$. Moreover, we have*

$$\lim_{n\to+\infty} |u_n(r)| + |u_n'(r)| = 0 = \lim_{n\to+\infty} |v_n(r)| + |v_n'(r)|, \text{ uniformly in } r \in [0,R].$$

Multiplicity results for a boundary-value problem of the form (1.1) can be found e.g. in [1], [2], [4], [6], [8], [12]. However, apart from [6] where additional regularity conditions are imposed, in those papers the authors considered the case $a \equiv 1$ and/or $h \equiv 0$. In some of the above quoted papers, the differential operator under consideration is strongly nonlinear. We refer to [4] for a more comprehensive list of references.

We work in the framework of topological degree methods and use some of the ideas developed in [4] (see also [5], [9]). In this situation, two main tasks have to be accomplished. First, one has to study an autonomous problem

$$\begin{gathered} u'' + a(u')g(u) = 0, \\ u'(0) = 0 = u(R), \end{gathered} \tag{1.4}$$

where $g : [-\varepsilon, \varepsilon] \to \mathbb{R}, \varepsilon > 0$, is a continuous function such that

$$\lim_{s\to 0} \frac{g(s)}{s} = +\infty.$$

Secondly, suitable estimates on the (possible) solutions to a family of parameter-dependent problems (cf. (P_λ)) have to be established.

In this paper (cf. Section 2) we overcome the first difficulty by studying problem (1.4) in the equivalent form

$$\begin{gathered}(\phi(u'))' + g(u) = 0,\\ u'(0) = 0 = u(R),\end{gathered} \tag{1.5}$$

where ϕ is an odd increasing homeomorphism defined through a. In this way, we can use the time-map technique for equations containing the ϕ-Laplacian (see [4], [7], [8], [10], [11], [12], [13]) and establish a multiplicity result for (1.4) (cf. Theorem 2.2 and Theorem 5.4 in [4]).

Then, in order to show that the (nodal) properties of the solutions to (1.4) can be "continued" to problem (1.1), some estimates on the number of zeros of the (possible) solutions to the associated parameter-dependent boundary-value problem (cf. (P_λ)) have to be established. To this end, we argue on the lines of [4]; however, some technical difficulties due to the presence in (1.1) of the functions a and h have to be overcome (see in particular the proofs of Lemma 3.1, Lemma 3.3 and Claim 2 in Theorem 4.1).

We end this introductory section by observing that a result analogous to Theorem A can be performed for a more general strongly nonlinear boundary-value problem

$$\begin{gathered}(r^{(k-1)}\psi(u'))' + r^{(k-1)}a(u')f(r,u) = r^{(k-1)}h(r,u,u'),\\ u'(0) = 0 = u(R),\end{gathered} \tag{1.6}$$

where ψ is an odd increasing homeomorphism satisfying suitable assumptions.

Furthermore, on the lines of [4], one could prove the existence of an *additional* double sequence of solutions to (1.1) (whose norm tends to infinity) provided that g has a "superlinear" behaviour at infinity and assumption (H_h) is modified accordingly.

This paper is organized as follows. In Section 2 we study the autonomous problem (1.4). In Section 3 we introduce a parameter-dependent non-autonomous problem and develop some estimates on its solutions. In Section 4 we recall an abstract continuation theorem which is then applied for the proof of the main result.

In what follows, for any Banach space X, for any linear compact operator $L : X \to X$ and for any subset $\Omega \subset X$ we will denote by $\deg(I-L,\Omega)$ the Leray-Schauder degree of $I-L$ (if defined). The space $C^1([0,R])$ of the continuously differentiable real functions u on $[0,R]$ will be equipped with the norm

$$||u||_1 = \max\left\{\sqrt{|u(t)|^2 + |u'(t)|^2} : t \in [0,R]\right\}.$$

Finally, $C^1_\#([0,R])$ denotes the space of functions $u \in C^1([0,R])$ satisfying the boundary condition $u'(0) = 0 = u(R)$.

2 An autonomous problem

Let us consider the second order ODE

$$\begin{aligned} u'' + a(u')g(u) = 0, \\ u'(0) = 0 = u(R), \end{aligned} \tag{2.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is defined by $a(\xi) := a_0 + |\xi|^q, 0 < q < 2, a_0 > 0$. Set

$$\phi(s) = \int_0^s \frac{1}{a(x)}\, dx. \tag{2.2}$$

We assume that $g : [-\varepsilon, \varepsilon] \to \mathbb{R}$ is continuous ($\varepsilon > 0$) and such that

$$\lim_{s\to 0} \frac{g(s)}{\phi(s)} = +\infty. \tag{2.3}$$

We shall also assume (without loss of generality) that $g(s)s > 0$ for all $s \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and we set $G(s) = \int_0^s g(\xi)\, d\xi$.

We observe that problem (2.1) can be written in the form

$$\begin{aligned} (\phi(u'))' + g(u) = 0, \\ u'(0) = 0 = u(R). \end{aligned}$$

It is not difficult to check that ϕ is an odd increasing homeomorphism. Then, as in [10], it is possible to study (2.1) with the time-map technique by means of the system

$$\begin{aligned} u' = \phi^{-1}(y), \\ y' = -g(u). \end{aligned} \tag{2.4}$$

More precisely, for

$$\mathcal{L}(\xi) = \int_0^\xi \frac{x}{a(x)}\, dx, \tag{2.5}$$

we shall use the fact that if u is a solution of (2.4), then $E(r, u(r), u'(r)) := G(u(r)) + \mathcal{L}(u'(r))$ is constant. Observe that our assumptions on g ensure that the orbits of (2.1) are closed curves on the phase-plane. Then, denoting by $\mathcal{L}^{-1}$ the inverse of the restriction to $\mathbb{R}^+$ of the function $\mathcal{L}$, we can introduce the function $T_1 : (0, \varepsilon) \to (0, +\infty)$ by

$$T_1(\alpha) = \int_0^\alpha \frac{dx}{\mathcal{L}^{-1}(G(\alpha) - G(x))}. \tag{2.6}$$

It is straightforward to check that $T_1(\alpha)$ represents the time needed for a rotation along the orbit of "energy" $G(\alpha)$ in the upper (resp. lower) half plane from the point $(0, \mathcal{L}^{-1}(G(\alpha)))$ to the point $(\alpha, 0)$ (resp. from $(\alpha, 0)$ to $(0, -\mathcal{L}^{-1}(G(\alpha)))$). Analogously, for $\alpha_1 < 0$ s.t. $G(\alpha_1) = G(\alpha)$, the function $T_2 : (0, \varepsilon) \to (0, +\infty)$ defined by

$$T_2(\alpha) = \int_{\alpha_1}^0 \frac{dx}{\mathcal{L}^{-1}(G(\alpha) - G(x))} \tag{2.7}$$

is the time needed for a rotation along the orbit of "energy" $G(\alpha)$ from the point $(\alpha_1, 0)$ to the point $(0, \mathcal{L}^{-1}(G(\alpha)))$ (resp. from $(0, -\mathcal{L}^{-1}(G(\alpha)))$ to $(\alpha_1, 0)$). For a classical reference on this topic, the reader can consult [15]. See also [7].

For the completion of the study of the autonomous case, we need the following result.

Proposition 2.1 *Let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$ and ϕ given by (2.2). Assume $g : [-\varepsilon, \varepsilon] \to \mathbb{R}$ is a continuous function such that $g(s)s > 0$ for all $s \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and satisfying (2.3). Then the functions $T_1(\alpha)$ and $T_2(\alpha)$ defined by (2.6) and (2.7) are such that for $i = 1, 2$ we have*

$$\lim_{\alpha \to 0} T_i(\alpha) = 0.$$

Proof. Observe that $\mathcal{L}(s) = (\Phi_* \circ \phi)(s)$ with $\Phi_*(s) = \int_0^s \phi^{-1}(x)\, dx$. The proof follows the same arguments as in Lemma 2.1 in [11] and Theorem 3.2 in [10] where the assumptions on the function g, as well as the result on the asymptotic behaviour of the time-maps, are relative to a neighbourhood of infinity. For a more detailed proof, one can also see Theorem 2.2.8 in [7]. □

Once the time-maps are defined, we can introduce the "generalized Fučik spectrum" as in [3], [4], [7] in order to get a characterization of the existence of solutions with a fixed number of zeros. Indeed, using Proposition 2.1, one gets the following multiplicity result for the autonomous problem (2.1).

Theorem 2.2 *[4, Th. 5.4] Let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$ and ϕ given by (2.2). Assume $g : [-\varepsilon, \varepsilon] \to \mathbb{R}$ is a continuous function such that $g(s)s > 0$ for all $s \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and satisfying (2.3). Then there exists $k_0 \in \mathbb{N}$ such that for every $k \geq 2k_0$ problem* (2.1) *has at least two solutions u_k and v_k with $u_k(0) > 0$ and $v_k(0) < 0$, both having exactly k zeros in* $[0, R)$.

We end this section by giving two important properties of $\mathcal{L}$, which will be crucial in the sequel.

Proposition 2.3 *Let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$ and $\mathcal{L}$ given by (2.5). Then for all $\xi \in \mathbb{R}$, we have*

$$\frac{\xi^2}{a(\xi)} \leq 2\mathcal{L}(\xi).$$

Proof. An easy computation gives $(\frac{s^2}{a(s)})' \leq 2\mathcal{L}'(s)$, for all $s \geq 0$. Then by integration, we get

$$\frac{\xi^2}{a(\xi)} \leq 2\mathcal{L}(\xi)\,.$$

Proposition 2.4 *Let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$ and $\mathcal{L}$ given by (2.5). Then for any $c_1 > 1$, $c \geq c_1^2 + 1$ and $\xi > 0$ small enough, we have*

$$c_1\mathcal{L}^{-1}(\xi) \leq \mathcal{L}^{-1}(c\xi). \tag{2.8}$$

Proof. Notice that since $\lim_{x\to 0}\frac{\mathcal{L}(c_1x)}{\mathcal{L}(x)}=c_1^2$, then for any $c\geq c_1^2+1$ and $x>0$ small enough we have $\mathcal{L}(c_1x)\leq c\mathcal{L}(x)$. As $\mathcal{L}^{-1}:\mathbb{R}^+\to\mathbb{R}^+$ is continuous, we have for $\xi>0$ small enough

$$\mathcal{L}(c_1\mathcal{L}^{-1}(\xi))\leq c\mathcal{L}(\mathcal{L}^{-1}(\xi))=c\xi$$

and since $\mathcal{L}^{-1}$ is increasing $c_1\mathcal{L}^{-1}(\xi)\leq\mathcal{L}^{-1}(c\xi)$. □

Remark 2.5 *In general, if one sets $\Phi_*(s)=\int_0^s\phi^{-1}(x)\,dx$, where ϕ is an odd increasing homeomorphism, an inequality like (2.8) can be proved separately for the functions Φ_*^{-1} and ϕ^{-1}. This is done in [4] under the "lower σ-condition"*

$$\liminf_{s\to 0}\frac{\phi(\sigma s)}{\phi(s)}>1,\quad\forall\sigma>1.$$

In our situation, we observe that $\mathcal{L}(s)=(\Phi_*\circ\phi)(s)$, so we could have proved Proposition 2.4 by combining the inequalities for Φ_*^{-1} and ϕ^{-1}. A direct proof of Proposition 2.4 *is simpler thanks to the fact that we can explicitly use the function $\mathcal{L}$ and its properties.*

3 Preliminary results

We consider the boundary-value problem

$$\begin{gathered}(r^{(k-1)}u')'+r^{(k-1)}a(u')f(r,u)=r^{(k-1)}h(r,u,u'),\\ u'(0)=0=u(R),\end{gathered}\tag{3.1}$$

where $k>1$, $a:\mathbb{R}\to\mathbb{R}$ is defined by $a(\xi):=a_0+|\xi|^q$ with $0<q<2$, $a_0>0$ and for a fixed $\varepsilon_0>0$, the functions f and h satisfy the following properties.

(H_f) The function $f:[0,R]\times[-\varepsilon_0,\varepsilon_0]\to\mathbb{R}$ is continuous and such that

$$f(r,0)=0$$

and

$$\lim_{s\to 0}\frac{f(r,s)}{s}=+\infty\text{ uniformly in }r\in[0,R].$$

(H_F) For $F(r,s):=\int_0^s f(r,x)\,dx$, F is differentiable with respect to $r\in[0,R]$ and there exists a continuous positive function $\alpha:[0,R]\to(0,+\infty)$ such that for all $r\in[0,R]$, all $s\in[-\varepsilon_0,\varepsilon_0]$,

$$\left|\frac{\partial F}{\partial r}(r,s)\right|\leq\alpha(r)F(r,s).$$

(H_h) The function $h:[0,R]\times[-\varepsilon_0,\varepsilon_0]\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists $H>0$ such that for all $(r,s,\xi)\in[0,R]\times[-\varepsilon_0,\varepsilon_0]\times\mathbb{R}$,

$$|h(r,s,\xi)|\leq H|\xi|.$$

Moreover, there exists a continuous function $C : \mathbb{R} \to (0, +\infty)$ such that

$$\lim_{s\to 0} \frac{h(r,s,\xi)}{s} = C(\xi) \text{ uniformly in } r \in [0,R]. \tag{3.2}$$

A typical example for the function h is $h(r,s,\xi) = \eta(s)|\xi|^\beta$ with $\beta > 1$ for $|\xi| < 1$, $0 < \beta < 1$ for $|\xi| \geq 1$ and $\eta(s) \sim s$ for $s \to 0$. □

Following a degree approach, problem (3.1) will be treated by means of the parameter-dependent family of problems ($\lambda \in [0,1]$)

$$\begin{gathered} (r^{\lambda(k-1)}u')' + r^{\lambda(k-1)}a(u')f_\lambda(r,u) = \lambda r^{\lambda(k-1)}h(r,u,u'), \\ u'(0) = 0 = u(R), \end{gathered} \tag{P_λ}$$

where $f_\lambda : [0,R] \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ is defined by

$$f_\lambda(r,s) = \lambda f(r,s) + (1-\lambda)g(s), \tag{3.3}$$

and $g : [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}$ is a continuous nondecreasing function such that

$$\lim_{s\to 0} \frac{g(s)}{s} = +\infty. \tag{H_g}$$

We shall also assume (without loss of generality) that $g(s)s > 0$, for every $s \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$. Note that our assumptions on g guarantee that condition (2.3) is satisfied.

Set $F_\lambda(r,s) := \int_0^s f_\lambda(r,x)\,dx$. It is immediate to remark that in the situation described above we have

(H_{F_λ}) $F_\lambda(r,s)$ is differentiable with respect to $r \in [0,R]$ and there exists a continuous positive function $\alpha : [0,R] \to (0,+\infty)$ such that for all $r \in [0,R]$, all $s \in [-\varepsilon_0, \varepsilon_0]$,

$$\left|\frac{\partial F_\lambda}{\partial r}(r,s)\right| \leq \alpha(r) F_\lambda(r,s).$$

Moreover, using (H_f) and (H_g), we have

$$\lim_{s\to 0} \frac{f_\lambda(r,s)}{s} = +\infty \quad \text{uniformly in } \lambda \in [0,1]$$

and, by (H_{F_λ}), for all $r \in [0,R]$, all $s \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ and all $\lambda \in [0,1]$,

$$F_\lambda(r,s) > 0. \tag{3.4}$$

In our main result we will prove the existence of infinitely many solutions of (P_1) using an abstract continuation theorem. To this end, we need the following lemma concerning the Cauchy problem

$$\begin{gathered} (r^{\lambda(k-1)}u')' + r^{\lambda(k-1)}a(u')f_\lambda(r,u) = \lambda r^{\lambda(k-1)}h(r,u,u'), \\ u(0) = d, \ u'(0) = 0. \end{gathered} \tag{3.5}$$

Lemma 3.1 *For all $\varepsilon \in (0, \varepsilon_0]$, if u is a (local) solution of problem (3.5) with d small enough, then u can be defined on $[0, R]$ and $||u||_1 \le \varepsilon$.*

Proof. Let $\varepsilon > 0$ be fixed and u be a solution of (3.5). Assume that there exists $\rho \in (0, R]$ such that for all $r \in [0, \rho]$,

$$|u(r)| \le \varepsilon \qquad \text{and} \qquad |u'(r)| \le \varepsilon.$$

Let

$$E_\lambda(r, s, \xi) := F_\lambda(r, s) + \mathcal{L}(\xi)$$

where $\mathcal{L}$ is given in (2.5) and for all $r \in [0, \rho]$, we consider the function

$$v_\lambda(r) := E_\lambda(r, u(r), u'(r)). \tag{3.6}$$

We have, using (H_h), (H_{F_λ}) and Proposition 2.3

$$\begin{aligned} v'_\lambda(r) &= \frac{\partial F_\lambda}{\partial r}(r, u(r)) + \frac{u'(r)}{a(u'(r))}\left(\lambda h(r, u(r), u'(r)) - \frac{\lambda(k-1)}{r}u'(r)\right) \\ &\le \frac{\partial F_\lambda}{\partial r}(r, u(r)) + \lambda h(r, u(r), u'(r))\frac{u'(r)}{a(u'(r))} \\ &\le \frac{\partial F_\lambda}{\partial r}(r, u(r)) + H\frac{(u'(r))^2}{a(u'(r))} \\ &\le \alpha(r)F_\lambda(r, u(r)) + 2H\mathcal{L}(u'(r)) \\ &\le \tilde{\alpha}(r)v_\lambda(r), \end{aligned}$$

where $\tilde{\alpha} : [0, R] \to (0, +\infty)$ is a continuous function. Integrating on $(0, r)$, we get

$$v_\lambda(r) \le v_\lambda(0)e^{\int_0^r \tilde{\alpha}(s)\,ds} = F_\lambda(0, d)e^{\int_0^r \tilde{\alpha}(s)\,ds}$$

and by definition of v_λ, we have

$$\mathcal{L}(u'(r)) \le v_\lambda(r) \le F_\lambda(0, d)e^{\int_0^r \tilde{\alpha}(s)\,ds}. \tag{3.7}$$

For the rest of the proof, we argue as in the proof of Lemma 2.3 in [4]; however, we give the details for the reader's convenience.

Consider $(a_1, a_2) \in (0, 1)^2$ such that

$$a_1 + Ra_2 \le \frac{1}{2} \quad \text{and} \quad a_2 \le \frac{1}{2} \tag{3.8}$$

(observe that, for every $R > 0$, a similar choice of a_1 and a_2 is always possible). Since $\lim_{d\to 0} \mathcal{L}^{-1}(F_\lambda(0, d)e^{\int_0^r \tilde{\alpha}(s)\,ds}) = 0$ uniformly in $\lambda \in [0, 1]$, for every $\varepsilon \le \varepsilon_0$ there exists $d_\varepsilon > 0$ such that $d_\varepsilon \le a_1\varepsilon$ and for all $0 < |d| \le d_\varepsilon$, all $\lambda \in [0, 1]$,

$$\mathcal{L}^{-1}(F_\lambda(0, d)e^{\int_0^r \tilde{\alpha}(s)\,ds}) \le a_2\varepsilon.$$

Then, for $0 < |d| \le d_\varepsilon$, we deduce from (3.7) that for all $r \in [0, \rho]$,

$$|u'(r)| \le \mathcal{L}^{-1}(F_\lambda(0, d)e^{\int_0^r \tilde{\alpha}(s)\,ds}) \le a_2\varepsilon \le \frac{\varepsilon}{2}. \tag{3.9}$$

The above estimate implies that for all $r \in [0, \rho]$,

$$\begin{aligned} |u(r)| &\le d + \int_0^r |u'(s)|\,ds \le d + R\mathcal{L}^{-1}(F_\lambda(0,d)e^{\int_0^r \tilde{\alpha}(s)\,ds}) \\ &\le a_1\varepsilon + Ra_2\varepsilon = (a_1 + Ra_2)\varepsilon \le \tfrac{\varepsilon}{2}. \end{aligned} \tag{3.10}$$

Since (3.9) and (3.10) hold independently on ρ, we can extend u on $[0, R]$ as a C^1-function. Finally, (3.9) and (3.10) imply that

$$||u||_1 = \max_{r\in[0,R]} \sqrt{|u(r)|^2 + |u'(r)|^2} \le \varepsilon.$$

Remark 3.2 *We deduce from Lemma 3.1 that if u is a solution of (3.5) with d small enough then u' is bounded. Hence condition (3.2) in (H_h) implies that there exists $\tilde{\delta} > 0$ such that for all $r \in [0, R]$,*

$$0 < |u(r)| \le \tilde{\delta} \Longrightarrow |h(r, u(r), u'(r))| \le C|u(r)|,$$

with C independent of u'.

Lemma 3.3 *There exists $\bar{\delta} > 0$ such that if u is a solution of (P_λ) with $|u(0)| = \bar{d}$ small enough then for all $r \in [0, R]$,*

$$|u(r)|^2 + |u'(r)|^2 \ge \bar{\delta}.$$

Proof. **Step 1 -** Let $0 < \tilde{\varepsilon} < a_0$ be fixed and $u(\cdot, d) = u(\cdot)$ be a solution of (3.5). Integrating the equation in (3.5) from 0 to r, we have

$$-u'(r) = r^{-\lambda(k-1)} \int_0^r s^{\lambda(k-1)}(f_\lambda(s, u(s))a(u'(s)) - \lambda h(s, u(s), u'(s)))\,ds.$$

Let $\tilde{\delta}$ be given by Remark 3.2. We deduce from (H_f) that there exists $0 < \delta < \tilde{\delta}$ such that for all $r \in [0, R]$,

$$0 < |u(r)| \le \delta \Longrightarrow |f(r, u(r))| \ge \frac{C}{\tilde{\varepsilon}}|u(r)| \text{ and } f(r, u(r))u(r) > 0. \tag{3.11}$$

Hence if $d > 0$ and r are small enough, for every $s \in [0, r]$ we have $0 < u(s) \le \delta$ and

$$\begin{aligned} f_\lambda(s, u(s))a(u'(s)) - \lambda h(s, u(s), u'(s)) &\ge \lambda\frac{C}{\tilde{\varepsilon}}u(s)a(u'(s)) - \lambda C u(s) \\ &= \lambda C u(s)\left(\frac{a(u'(s))}{\tilde{\varepsilon}} - 1\right) > 0\,, \end{aligned}$$

which proves that u is decreasing in $[0, r]$. In the same way, if $d < 0$ is small enough then

$$f_\lambda(s, u(s))a(u'(s)) - \lambda h(s, u(s), u'(s)) \le \lambda C u(s)\left(\frac{a(u'(s))}{\tilde{\varepsilon}} - 1\right) < 0$$

and u is increasing in $[0,r]$. Arguing as in [5], for every $\theta \in (0,1)$ we can consider the first point $r_\theta(d)$ such that

$$u(r_\theta(d);d) = \theta d.$$

Moreover, we denote by $r_0(d)$ the first zero of $u(\cdot;d)$.

Step 2 - There exists $\bar{\delta} > 0$ such that if u is a solution of (P_λ) with $|u(0)| = \bar{d}$ small enough then for all $r \in [0,R]$ we have $|u(r)|^2 + |u'(r)|^2 \geq \bar{\delta}$.

Let u be a solution of (P_λ) with $|u(0)| = \bar{d}$ sufficiently small. To prove this Step we will need the following two claims.

Claim 1 : For every $\theta \in (0,1)$ and for every $\lambda \in (0,1)$, we have

$$r_\theta(\bar{d}) \geq \sqrt{\frac{2\bar{d}(1-\theta)(1+\lambda(k-1))}{(\hat{f}(\bar{d})+g(\bar{d}))a(\varepsilon_0)+H\varepsilon_0}} =: \beta(\bar{d}) > 0,$$

where $\hat{f}$ is defined by

$$\hat{f}(s) = \begin{cases} \sup\{f(r,x),\, r\in[0,R],\, x\in[0,s]\} & \text{if } 0 < s \leq \varepsilon_0, \\ \inf\{f(r,x),\, r\in[0,R],\, x\in[s,0]\} & \text{if } -\varepsilon_0 \leq s < 0\,. \end{cases}$$

An analogous result holds for $r_\theta(-\bar{d})$.

Assume $u(0) = \bar{d} > 0$. If $\bar{d}$ is small enough, we have for every $s \in [0, r_0(\bar{d})]$ that $0 < u(s) \leq \delta$ and we deduce from Lemma 3.1 that $|u'(s)| \leq \varepsilon_0$. Moreover, using (H_h), we have for all $s \in [0, r_0(\bar{d})]$,

$$f_\lambda(s,u(s))a(u'(s)) - \lambda h(s,u(s),u'(s)) \leq (\hat{f}(\bar{d})+g(\bar{d}))a(\varepsilon_0)+H\varepsilon_0.$$

Hence for $r \in [0, r_0(\bar{d})]$ we obtain

$$\begin{aligned} -u'(r) &= r^{-\lambda(k-1)}\int_0^r s^{\lambda(k-1)}(f_\lambda(s,u(s))a(u'(s)) - \lambda h(s,u(s),u'(s)))\,ds \\ &\leq \frac{r}{\lambda(k-1)+1}((\hat{f}(\bar{d})+g(\bar{d}))a(\varepsilon_0)+H\varepsilon_0) \end{aligned}$$

and integrating from 0 to $r_\theta(\bar{d})$ we get

$$\theta\bar{d} - \bar{d} = u(r_\theta(\bar{d})) - u(0) \geq -\frac{((\hat{f}(\bar{d})+g(\bar{d}))a(\varepsilon_0)+H\varepsilon_0)(r_\theta(\bar{d}))^2}{2(\lambda(k-1)+1)}$$

which implies

$$r_\theta(\bar{d}) \geq \sqrt{\frac{2\bar{d}(1-\theta)(1+\lambda(k-1))}{(\hat{f}(\bar{d})+g(\bar{d}))a(\varepsilon_0)+H\varepsilon_0}}$$

and the Claim is proved.

A similar computation holds if $u(0) = -\bar{d}$.

Claim 2 : There exists $\delta_0 > 0$ such that if u is a solution of (P_λ) with $|u(0)| = \bar{d}$ sufficiently small, we have $E_\lambda(r,u(r),u'(r)) \geq \delta_0$ for every $r \in [0,R]$.

First, we observe that, by (H_{F_λ}), there is a constant γ such that for all $r \in (0, R]$, all $s \in [-\epsilon_0, \epsilon_0]$ and all $\lambda \in [0,1]$,

$$\frac{\partial F_\lambda}{\partial r}(r,s) + \frac{\gamma}{r} F_\lambda(r,s) \geq 0. \tag{3.12}$$

Recall that $E_\lambda(r,s,\xi) := F_\lambda(r,s) + \mathcal{L}(\xi)$. Using (H_h) and Proposition 2.3,

$$\begin{aligned}
&\frac{d}{dr}E_\lambda(r,u(r),u'(r)) + \frac{\gamma}{r}E_\lambda(r,u(r),u'(r)) \\
&= \frac{\partial F_\lambda}{\partial r}(r,u(r)) + \frac{\gamma}{r}F_\lambda(r,u(r)) + \frac{u'(r)}{a(u'(r))}\lambda h(r,u(r),u'(r)) + \frac{\gamma}{r}\mathcal{L}(u'(r)) \\
&\quad -\lambda\frac{(k-1)}{r}\frac{(u'(r))^2}{a(u'(r))} \\
&\geq \frac{u'(r)}{a(u'(r))}\lambda h(r,u(r),u'(r)) + \frac{\gamma}{r}\mathcal{L}(u'(r)) - \frac{2(k-1)}{r}\mathcal{L}(u'(r)) \\
&\geq -H\frac{(u'(r))^2}{a(u'(r))} + \frac{\gamma}{r}\mathcal{L}(u'(r)) - \frac{2(k-1)}{r}\mathcal{L}(u'(r)) \\
&\geq \mathcal{L}(u'(r))\left(-2H + \frac{\gamma}{r} - \frac{2(k-1)}{r}\right) \geq 0
\end{aligned}$$

if $\gamma \geq 2HR + 2(k-1)$. Multiplying by r^γ and integrating from $r_\theta(\bar{d})$ to r, we obtain

$$E_\lambda(r,u(r),u'(r))r^\gamma - E_\lambda(r_\theta(\bar{d}),u(r_\theta(\bar{d})),u'(r_\theta(\bar{d})))r_\theta(\bar{d})^\gamma \geq 0$$

and

$$\begin{aligned}
E_\lambda(r,u(r),u'(r)) &\geq E_\lambda(r_\theta(\bar{d}),u(r_\theta(\bar{d})),u'(r_\theta(\bar{d})))r_\theta(\bar{d})^\gamma R^{-\gamma} \\
&= R^{-\gamma}\left(\mathcal{L}(u'(r_\theta(\bar{d}))) + F_\lambda(r_\theta(\bar{d}),u(r_\theta(\bar{d})))\right) r_\theta(\bar{d})^\gamma \\
&\geq R^{-\gamma}F^0(\theta\bar{d})r_\theta(\bar{d})^\gamma \\
&\geq R^{-\gamma}F^0(\theta\bar{d})(\beta(\bar{d}))^\gamma,
\end{aligned}$$

where $F^0(\theta\bar{d}) = \min\{F_\lambda(r,\theta\bar{d}) : r \in [0,R],\ \lambda \in [0,1]\} > 0$. We finish the proof of the Claim by setting $\delta_0 = R^{-\gamma}F^0(\theta\bar{d})(\beta(\bar{d}))^\gamma$.

If the claim in Step 2 were not true then for every $\bar{\delta} > 0$, there exists $\bar{r} \in [0,R]$ such that $|u(\bar{r})|^2 + |u'(\bar{r})|^2 < \bar{\delta}$, which contradicts Claim 2. $\square$

Now, for every $d \in \mathbb{R}_0$ and for every $\lambda \in [0,1]$ we can define

$$\mathbf{n} : S_{d,\lambda} \to \mathbb{N} : u \mapsto \mathbf{n}(u),$$

where

$$S_{d,\lambda} = \{u : u \text{ is a solution of } (P_\lambda) \text{ and } u(0) > d \text{ if } d > 0,\ u(0) < d \text{ if } d < 0\}$$

and $\mathbf{n}(u)$ is the number of zeros of u in $[0,R)$. Arguing as in the proof of Lemma 3.1 and of Lemma 2.5 in [4], it is possible to prove that if u is a solution of the equation in (P_λ) such that $u(r_*) = 0 = u'(r_*)$ (for some $r_* \in (0,R]$) then $u \equiv 0$. This fact guarantees that $\mathbf{n}$ is well defined. Moreover, for d small enough, arguing as in Lemma 3.1 in [12] we can prove that $\mathbf{n}$ is continuous.

4 Main result

In this section we will prove the existence of infinitely many solutions of problem

$$\begin{array}{c}(r^{(k-1)}u')' + r^{(k-1)}a(u')f(r,u) = r^{(k-1)}h(r,u,u'),\\ u'(0) = 0 = u(R)\end{array} \tag{3.1}$$

($k > 1$), using an abstract theorem. Note that solutions to (P_1) are solutions to (3.1), while solutions to (P_0) are solutions to the autonomous problem

$$\begin{array}{c}u'' + a(u')g(u) = 0,\\ u'(0) = 0 = u(R).\end{array}$$

Our main result is the following.

Theorem 4.1 *Assume (H_f)-(H_F)-(H_h) and let $a : \mathbb{R} \to \mathbb{R}$ be defined by $a(\xi) := a_0 + |\xi|^q$ with $0 < q < 2$, $a_0 > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n > n_0$ problem (3.1) has at least two solutions u_n and v_n with $u_n(0) > 0$ and $v_n(0) < 0$, both having exactly n zeros in $[0,R)$. Moreover, we have*

$$\lim_{n\to+\infty} |u_n(r)| + |u_n'(r)| = 0 = \lim_{n\to+\infty} |v_n(r)| + |v_n'(r)|, \text{ uniformly in } r \in [0,R].$$

To prove this theorem, we will need an abstract continuation result. In order to state this theorem, let us consider a Banach space X and a completely continuous operator $\mathcal{N} : \operatorname{dom} \mathcal{N} \subset X \times [0,1] \to X$. Moreover, let A, B be two open sets such that $A \subset \bar{A} \subset B \subset \bar{B}$ and $(\bar{B} \setminus A) \subset \operatorname{dom} \mathcal{N}$.

Let Σ be the set of the solutions of the abstract equation $u = \mathcal{N}(u,\lambda)$, i.e.

$$\Sigma = \{(u,\lambda) : u = \mathcal{N}(u,\lambda)\}.$$

For any subset $D \subset X \times [0,1]$, we denote the section of D at $\lambda \in [0,1]$ by $D_\lambda = \{x \in X : (x,\lambda) \in D\}$ and we also set $\mathcal{N}_\lambda = \mathcal{N}(\cdot,\lambda)$. We are now in position to state the following

Theorem 4.2 *[4, Th. 2.1] Let $\mathbf{k} : \Sigma \cap (\bar{B} \setminus A) \to \mathbb{N}$ be a continuous function; suppose that there exists a positive integer n satisfying the following conditions*

$$n \notin \mathbf{k}(\partial(\bar{B} \setminus A)) \tag{4.1}$$

and

$$\mathbf{k}^{-1}(n) \quad \text{is bounded.} \tag{4.2}$$

Then, for an open set U_0^n such that $(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset (\bar{B} \setminus A)_0$ and $\Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0$, the Leray-Schauder degree $\deg(I - \mathcal{N}_0, U_0^n)$ is defined. If

$$\deg(I - \mathcal{N}_0, U_0^n) \neq 0, \tag{4.3}$$

then there is a continuum $C_n \subset \Sigma$ with

$$\{\lambda \in [0,1] : \exists u \in X : (u,\lambda) \in C_n\} = [0,1]$$

and such that

$$(u,\lambda)\in C_n \quad\Longrightarrow\quad (u,\lambda)\in(B\setminus\bar{A}) \quad \textit{and} \quad \mathbf{k}(u,\lambda)=n.$$

In particular there is at least one solution $\tilde{u}\in(B\setminus\bar{A})_1$ *of the operator equation*

$$u=\mathcal{N}(u,1)$$

with

$$\mathbf{k}(\tilde{u},1)=n.$$

Proof of Theorem 4.1. First note that problem (P_λ) can be put into the form $u=\mathcal{N}(u,\lambda)$ where

$$\mathcal{N}:\operatorname{dom}\mathcal{N}\subset C^1_{\#}([0,R])\times[0,1]\to C^1_{\#}([0,R])$$

is a completely continuous operator (see for example [14]). In order to give the appropriate definition for the sets A and B, we need some estimates on $\mathbf{n}$. Let δ be given by (3.11) and $\bar{d}\le\delta$ sufficiently small.

Claim 1 : There exists $n^*\in\mathbb{N}$ such that for any solution u of (P_λ) we have

$$|u(0)|=\bar{d} \quad\Longrightarrow\quad \mathbf{n}(u)<n^*.$$

The proof follows the same lines as the proof of Lemma 3.1 in [12], using Lemma 3.1 and Lemma 3.3.

Claim 2 : For every $N>0$ there exists $d_N>0$, $d_N<\bar{d}$, such that for any solution $u\in S_{d,\lambda}$ (for some d) we have

$$|u(0)|\le d_N \Longrightarrow \quad \mathbf{n}(u)>N.$$

Let us consider $u\in S_{d,\lambda}$. We observe that for every $N>0$ there exists $M(N)>2^{2(2k-1)}$ such that for all $|s|\le\varepsilon_0$

$$\frac{1}{\sqrt{M(N)}}\int_0^s\frac{du}{\mathcal{L}^{-1}(s^2-u^2)}<\frac{1}{N}. \tag{4.4}$$

Let $\tilde{M}(N):=M(N)+\frac{C}{a_0}$. By (H_f), there is $\eta:=\eta_{\tilde{M}(N)}$ such that for all $r\in[0,R]$, all $0<|s|\le\eta$ and all $\lambda\in[0,1]$,

$$|f_\lambda(r,s)|>\tilde{M}(N)|s|. \tag{4.5}$$

Let $\varepsilon_N\le\min\{\eta,\bar{d}\}$. From Lemma 3.1, we can consider $d_N>0$ small enough such that

$$|u(0)|\le d_N \quad\Longrightarrow\quad ||u||_1\le\varepsilon_N.$$

Now, let $(u,\lambda)\in\Sigma$ with $|u(0)|\le d_N$. The equation in (P_λ) can be written as

$$\begin{aligned} u'&=\frac{y}{r^{\lambda(k-1)}},\\ y'&=-r^{\lambda(k-1)}(f_\lambda(r,u)a(u')-\lambda h(r,u,u')). \end{aligned} \tag{4.6}$$

We shall be concerned with the zeros $\{r_i\}_{i=1,\ldots,I}$ of u in the interval $[R/2, R]$. More precisely, we first estimate the distance between two successive zeros r_i and r_{i+1} of u in the case when

$$u'(r_i) > 0, \quad u'(r_{i+1}) < 0, \quad \text{and} \quad u(r) > 0, \forall r \in (r_i, r_{i+1}).$$

From (4.6) we infer that $y'(r) < 0$ for every $r \in (r_i, r_{i+1})$. Since $y(r_i) > 0$ and $y(r_{i+1}) < 0$, we deduce that there exists exactly one point $r^* \in (r_i, r_{i+1})$ such that $y(r^*) = 0$ and again from (4.6) it follows that

$$u'(r) > 0, \quad \forall r \in (r_i, r^*), \quad u'(r) < 0, \quad \forall r \in (r^*, r_{i+1}) \quad \text{and} \quad u'(r^*) = 0.$$

Let $A = (R/2)^{\lambda(k-1)}$ and $B = R^{\lambda(k-1)}$. Using (H_h), (3.11) and (4.5), we observe that

$$f_\lambda(r, u) - \frac{\lambda h(r, u, u')}{a(u')} \geq \left(\tilde{M}(N) - \frac{C}{a_0}\right) u = M(N)u.$$

Hence for $r \in (r_i, r^*) \subset [R/2, R]$, we get

$$\begin{aligned} u' &\leq \tfrac{y}{A}, \\ y' &\leq -AM(N)ua\left(\tfrac{y}{B}\right). \end{aligned} \tag{4.7}$$

Multiplying the first inequality in (4.7) by $\frac{A}{B}M(N)u$ and the second one by $\frac{y}{ABa(\frac{y}{B})}$ and adding up, we obtain

$$\frac{A}{B}M(N)uu' + \frac{yy'}{a(\frac{y}{B})AB} \leq \frac{M(N)uy}{B} - \frac{M(N)uy}{B} = 0.$$

This means that the function $\frac{A}{B}M(N)\frac{u^2}{2} + \frac{B}{A}\mathcal{L}(\frac{y}{B})$ is non-increasing in (r_i, r^*). Hence, setting $u^* := u(r^*)$, we obtain

$$\frac{A}{B}M(N)\frac{u(r)^2}{2} + \frac{B}{A}\mathcal{L}\left(\frac{y(r)}{B}\right) \geq \frac{A}{B}M(N)\frac{(u^*)^2}{2}$$

which implies for all $r \in (r_i, r^*)$,

$$\begin{aligned} u'(r) &\geq \frac{B}{r^{\lambda(k-1)}}\mathcal{L}^{-1}\left(\frac{M(N)A^2}{2B^2}((u^*)^2 - u^2(r))\right) \\ &\geq \mathcal{L}^{-1}\left(\frac{M(N)A^2}{2B^2}((u^*)^2 - u^2(r))\right). \end{aligned}$$

Finally, using Proposition 2.4 with $c = \frac{M(N)A^2}{2B^2}$, $c_1 = (1/2)^{2k-1}\sqrt{M(N)}$, we have for all $r \in (r_i, r^*)$,

$$u'(r) \geq (1/2)^{2k-1}\sqrt{M(N)}\mathcal{L}^{-1}((u^*)^2 - u^2(r)))$$

(notice that with the above choices $c_1 > 1$ and $c > c_1^2 + 1$). Integrating from r_i to r^*, we get

$$\int_{r_i}^{r^*} \frac{u'(r)}{(1/2)^{2k-1}\sqrt{M(N)}\mathcal{L}^{-1}((u^*)^2 - u(r)^2)}\, dr \geq r^* - r_i.$$

If we set $u(r) = u$, then using (4.4) we obtain

$$r^* - r_i \leq \int_0^{u^*} \frac{du}{(1/2)^{2k-1}\sqrt{M(N)}\mathcal{L}^{-1}((u^*)^2 - u^2)} < \frac{2^{2k-1}}{N}.$$

For the completion of the proof of the Claim, it is now sufficient to observe that a computation analogous to the one developed above can be performed if we consider the interval (r^*, r_{i+1}) or an interval (r_i, r_{i+1}) where u is negative.

Now, let $n_0 = \max(n^*, 2k_0)$ (for the definition of k_0, see Theorem 2.2). Next, let us consider $n > n_0$ and the number d_n arising from Claim 2. In order to prove the existence of the solutions with exactly n zeros by an application of Theorem 4.2, we introduce the sets

$$B = \{(u, \lambda) \in \text{dom } \mathcal{N} : u(0) < \bar{d}\}$$

($\bar{d}$ as in Lemma 3.3 and Claim 1) and

$$A_n = \{(u, \lambda) \in \text{dom } \mathcal{N} : u(0) < d_n\}.$$

Moreover, the functional

$$\mathbf{k} : \Sigma \cap (\bar{B} \setminus A_n) \to \mathbb{N}$$

will be defined by

$$\mathbf{k}(u, \lambda) = \mathbf{n}(u).$$

Let us now prove that conditions (4.1) and (4.2) are satisfied. We observe that

$$\partial(\bar{B} \setminus A_n) = \{(u, \lambda) : u(0) = \bar{d}\} \cup \{(u, \lambda) : u(0) = d_n\}.$$

If $(u, \lambda) \in \Sigma$ and $u(0) = d_n$ then, by Claim 2, we get $\mathbf{n}(u) > n$; on the other hand, if $(u, \lambda) \in \Sigma$ and $u(0) = \bar{d}$ then, by Claim 1, we have $\mathbf{n}(u) < n^*$. Hence, being $n^* < n$, condition (4.1) is satisfied.

As far as the boundedness of $\mathbf{k}^{-1}(n)$ is concerned, if $(u, \lambda) \in \mathbf{k}^{-1}(n) \subset \Sigma \cap (\bar{B} \setminus A_n)$, then $u(0) < \bar{d}$ and we deduce from Lemma 3.1 that u is bounded, hence (4.2) is fulfilled.

Now we have to choose an open set on which to compute the degree. From Theorem 2.2, we know that problem (2.1) has solutions with exactly n zeros in $[0, R)$. These solutions enable us to determine an open bounded set Ω_0 such that

$$(\mathbf{k}^{-1}(n))_0 \subset \Omega_0.$$

We define

$$U_0^n = \Omega_0 \cap (\bar{B} \setminus A_n).$$

Arguing as in [4], we can prove that the degree $\deg(I - \mathcal{N}_0, U_0^n)$ is well defined and $\deg(I - \mathcal{N}_0, U_0^n) \neq 0$.

Hence, an application of Theorem 4.2 provides the existence of a solution u_n of problem (3.1) with

$$\mathbf{n}(u_n) = n \quad \text{and} \quad u_n(0) > 0.$$

Moreover, this solution u_n is such that $||u_n||_1 \leq \varepsilon_0$.

A similar argument, considering the sets

$$B = \{(u, \lambda) \in \text{dom } N : u(0) > -\bar{d}\}$$

and

$$A_n = \{(u, \lambda) \in \text{dom } N : u(0) > -d_n\}$$

shows that there exists at least one solution v_n of (3.1) such that

$$\mathbf{n}(v_n) = n \quad \text{and} \quad v_n(0) < 0.$$

The last statement in Theorem 4.1 follows from the properties of $\mathbf{n}$. $\square$

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