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An abstract existence result and its applications *

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Abstract

By means of Borsuk's theorem and continuation through an admissible homotopy, we establish an existence theorem for operator equation with homogeneous nonlinearity. We illustrate our theorem by considering a perturbed functional differential equation under periodic boundary conditions.

1 Introduction

Continuation theorems have been used to derive periodic solutions for differential systems with perturbations. In particular, in [1], existence criteria for ω-periodic solutions are given for the equation

$$x' = g(x) + e(t,x)$$

by means of 'continuation' through an admissible homotopy carrying the given problem to the equation

$$x' = g(x),$$

which admits only the trivial ω-periodic solution (see [1, pp. 101-103]).

In this note, we are interested in the study of a similar problem for the perturbed functional differential system

$$x' = g(t, x_t) + h(t, x_t), \quad 0 \le t \le \omega,$$

with solutions that satisfy the periodic boundary condition

$$x(0) = x(\omega)\,.$$

This will be achieved by first proving an abstract existence theorem utilizing Borsuk's theorem and continuation through an admissible homotopy carrying our given problem to the equation

$$x' = g(t, x_t),$$

which admits only the trivial periodic solution.

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2 Main Results

Let X, Y be real normed spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $L : \mathrm{dom}(L) \subseteq X \to Y$ be a linear Fredholm mapping of index zero, and let Ω be an open and bounded subset of X. It is well known [1, Section 2.2] that there exist projections $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}\, P = \ker L, \ker Q = \mathrm{Im}\, L$ and $X = \ker L \oplus \ker P, Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q$. Suppose $F : \mathrm{dom}(L) \cap \overline{\Omega} \to Y$ has the form $F = L - N$ where $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$ and satisfies the condition $0 \notin F(\mathrm{dom}(L) \cap \partial\Omega)$. Then a coincidence degree $D_L(F, \Omega)$ can be defined which satisfies the properties listed in [1, Section 2.3]. As mentioned above, we will need the following Borsuk's Theorem: Suppose Ω is an open, bounded subset of X which is symmetric with respect to the origin and suppose further that the function F mentioned above satisfies the additional condition that $F(-x) = -F(x)$ for every $x \in \mathrm{dom}(L) \cap \partial\Omega$, then the coincidence degree $D_L(F, \Omega)$ is odd. We remark that there are a number of studies which are concerned with the existence of periodic solutions of differential equations by means of coincidence theory, see for examples [2-6].

Lemma 2.1 *Let $\overline{\Omega} = \{x \in X | \ \|x\|_X \leq 1\}$. Let $N_2 : X \to Y$ be a continuous mapping which maps bounded sets into bounded sets and satisfies*

$$\lim_{\|x\|_X \to \infty} \frac{\|N_2 x\|_Y}{\|x\|_X^\beta} = 0 \tag{2.1}$$

for some $\beta \in (0, 1]$. Suppose $H : \overline{\Omega} \times [0, 1] \to Y$ is defined by

$$H(x, \mu) = \begin{cases} \mu^\beta N_2(\mu^{-\beta} x) & \text{if } \mu \in (0, 1] \\ 0 & \text{if } \mu = 0\,. \end{cases}$$

Then H is continuous and bounded on $\overline{\Omega} \times [0, 1]$.

Proof. To show that H is continuous, it suffices to show that H is continuous at $(x, 0)$ where $x \in \overline{\Omega}$. For any $\varepsilon \in (0, 1)$, in view of assumption (2.1), we see that there exists a constant $\rho > 0$ such that for arbitrary $x \in X$ which satisfies $\|x\|_X > \rho, \|N_2 x\|_Y \leq \varepsilon \|x\|_X^\beta$. Since N_2 maps bounded sets into bounded sets, hence

$$M = \sup \{\|N_2 x\|_Y : \ \|x\|_X \leq \rho < \infty\} > 0.$$

Let $\mu_0 = \left(\frac{\varepsilon}{M+1}\right)^{1/\beta}$. Clearly,

$$0 < \mu_0 < \left(\frac{1}{M+1}\right)^{1/\beta}.$$

For every positive $\mu \leq \mu_0$ and every $x \in \overline{\Omega}$, we assert that $\|H(x, \mu)\|_Y < \varepsilon$. In fact, if $\mu^{-\beta} \|x\|_X > \rho$, then

$$\|H(x, \mu)\|_Y \leq \mu^\beta \left\|N_2(\mu^{-\beta} x)\right\|_Y$$

$$\begin{aligned} &\leq \mu^{\beta}\varepsilon \left\|\mu^{-\beta}x\right\|_X^{\beta} \\ &\leq \mu^{\beta}\varepsilon\mu^{-\beta^2} \|x\|_X^{\beta} \\ &\leq \mu_0^{\beta(1-\beta)}\varepsilon \\ &< (\frac{1}{M+1})^{1-\beta}\varepsilon < \varepsilon, \end{aligned}$$

and if $\mu^{-\beta}\|x\|_X \leq \rho$, then

$$\|H(x,\mu)\|_Y \leq \mu^{\beta}\left\|N_2(\mu^{-\beta}x)\right\|_Y \leq \mu^{\beta}M \leq \frac{\varepsilon}{M+1}M < \varepsilon.$$

Thus we have shown that H is continuous at $(x,0) \in \overline{\Omega} \times [0,1]$.

By arguments similar to those just described, we may show by means of the continuity of H at $(x,0) \in \overline{\Omega} \times [0,1]$ that there exists a constant $\delta > 0$ and a real number M_1 such that for $(x,\mu) \in \overline{\Omega} \times [0,\delta]$, $\|H(x,\mu)\|_Y \leq M_1$. Since N_2 maps bounded sets into bounded sets, there exists a number M_2 such that $\|H(x,\mu)\|_Y \leq M_2$ for $(x,\mu) \in \overline{\Omega} \times [\delta,1]$. Thus H is bounded on $\overline{\Omega} \times [0,1]$. The proof is complete. Let us now consider the operator equation

$$Lx = N_1x + N_2x, x \in X, \tag{2.2}$$

where

H1) L is a linear Fredholm mapping of index zero,

H2) $N_1 : X \to Y$ is a continuous mapping which satisfies $N_1(\lambda x) = \lambda N(x)$ for $\lambda \in (-\infty,\infty)$ and $x \in X$,

H3) $N_2 : X \to Y$ is a continuous mapping which maps bounded sets into bounded sets and satisfies (2.1) for some $\beta \in (0,1]$,

H4) N_1, N_2 are L-completely continuous.

Theorem 2.2 *Suppose the conditions H1-H4 hold. Suppose further that*

$$Lx = N_1x \tag{2.3}$$

admits only the trivial solution. Then (2.2) has a nontrivial solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Proof. Let $\Omega = \{x \in X | \ \|x\|_X \leq 1\}$. Let $T : \overline{\Omega} \times [0,1] \to Y$ be defined by

$$T(x,\mu) = \begin{cases} N_1x + \mu^{\beta}N_2(\mu^{-\beta}x) & \text{if } \mu \in (0,1] \\ N_1x & \text{if } \mu = 0\,. \end{cases} \tag{2.4}$$

Then

$$T(x,1) = N_1x + N_2x, x \in \overline{\Omega},$$

furthermore, in view of Lemma 2.1, T is continuous and bounded on $\overline{\Omega} \times [0,1]$. Since N_1 and N_2 are L-completely continuous, it is also easy to see that T is L-compact on $\overline{\Omega} \times [0,1]$.

Note that, in view of the assumption that (2.3) admits only the trivial solution, for any $x \in \partial\Omega, (x,0)$ cannot be a solution of

$$Lx = T(x,\mu). \tag{2.5}$$

Note further that if $(x,\mu) \in \partial\Omega \times (0,1]$ is a nontrivial solution of (2.5), then in view of (2.4) and (H2), $\mu^{-\beta}x$ will be a nontrivial solution of (2.2).

Let $\tilde{F} = L - T$. Suppose to the contrary that the operator equation (2.2) does not have any nontrivial solutions, then in view of the above discussions, $0 \notin \tilde{F}((\mathrm{dom}(L) \cap \partial\Omega) \times [0,1])$. Thus the degree $D_L(\tilde{F}(\cdot,\mu),\Omega)$ can be defined for arbitrary $\mu \in [0,1]$, and it takes constant on $[0,1]$. But since

$$\begin{aligned}\tilde{F}(-x,0) &= -Lx - T(-x,0) = -Lx - N_1(-x) \\ &= -Lx + N_1 x = -Lx + T(x,0) = -\tilde{F}(x,0)\end{aligned}$$

for all $x \in X$, by Borsuk's Theorem stated above, we see that $D_L(\tilde{F}(\cdot,0),\Omega)$, and (hence) $D_L(\tilde{F}(\cdot,1),\Omega)$ are odd. But this is contrary to the existence property of the coincidence degree. The proof is complete.

Let us now turn back to the perturbed functional differential equation

$$x' = g(t,x_t) + h(t,x_t), \quad 0 \le t \le \omega, \tag{2.6}$$

under the periodic boundary condition

$$x(0) = x(\omega), \tag{2.7}$$

where $x(t) \in C(R,R^n)$, $x_t \in BC(R,R^n)$ are given by $x_t(s) = x(t+s)$, and $g, h : [0,\omega] \times BC(R,R^n) \to R^n$ are continuous mappings that take bounded sets into bounded sets. Here $BC(R,R^n)$ is the linear normed space of all continuous and bounded functions from R into R^n endowed with the usual supremum norm.

Theorem 2.3 *Assume that*

$$g(t,\lambda x) = \lambda g(t,x), \lambda, t \in R; x \in BC(R,R^n), \tag{2.8}$$

and there exists $\beta \in (0,1]$ such that

$$\lim_{\|x\|\to\infty} \frac{|h(t,x)|}{\|x\|^\beta} = 0 \text{ uniformly in } t \in [0,\omega]. \tag{2.9}$$

Suppose further that the boundary value problem

$$\begin{gathered} x' = g(t,x_t) \quad t \in [0,\omega] \\ x(0) = x(\omega) \\ x(t) = x(0) \quad t \in (-\infty,0] \cup [\omega,\infty) \end{gathered} \tag{2.10}$$

admits only the trivial solution. Then (2.6) has a nontrivial solution x that satisfies (2.7).

Proof. Let

$$X = \{x \in C(R, R^n)|\, x(0) = x(\omega), x(t) = x(0), t \in (-\infty, 0] \cup [\omega, \infty)\},$$

and $Y = C([0, \omega], R^n)$. Then X is a closed subset in $BC(R, R^n)$, and therefore it is a Banach space. Let $\mathrm{dom}(L) = \{x \in X|\, x' \text{ is continuous on } [0, \omega]\}$, let $L : \mathrm{dom}(L) \cap X \to Y$ be defined by $(Lx)(t) = x'(t)$ for $t \in R$, and let $N : X \to Y$ be defined by

$$(Nx)(t) = (N_1x)(t) + (N_2x)(t), t \in R,$$

where $(N_1x)(t) = g(t, x_t), (N_2x)(t) = h(t, x_t)$ for $t \in R$. Then it is easy to show that the kernel of L is

$$\ker L = \{x \in X|\, x = c \in R^n\},$$

the image of L is

$$\mathrm{Im}\, L = \left\{ y \in Y|\, \frac{1}{\omega}\int_0^\omega y(s)ds = 0 \right\},$$

and $\dim \ker L = \mathrm{codim}\, \mathrm{Im}\, L = n$. Furthermore, if we define the projections $P : X \to X$ and $Q : Y \to Y$ by

$$(Px)(t) = x(0), t \in R,$$

and

$$(Qy)(t) = \frac{1}{\omega}\int_0^\omega y(s)ds, t \in R,$$

respectively, then $\ker L = \mathrm{Im}\, P$ and $\ker Q = \mathrm{Im}\, L$. Thus, L is a Fredholm operator with index zero, and the generalized inverse $K_P : \mathrm{Im}\, L \to \ker P \cap \mathrm{dom}(L)$ of L is given by

$$(K_P y)(t) = \begin{cases} \int_0^t y(s)ds & \text{if } 0 \le t \le \omega \\ 0 & \text{if } t \in (-\infty, 0] \cup [\omega, \infty), \end{cases}$$

and is compact. Since

$$(QN)(x) = \frac{1}{\omega}\int_0^\omega (g(s, x_s) + h(s, x_s))ds,$$

we easily see that $QN(\overline{\Omega})$ is bounded, furthermore, by the Arzela-Ascoli theorem, it is also easily seen that $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. As a consequence, N is L-compact on $\overline{\Omega}$.

Note that the conditions (H2) and (H3) follow (2.8) and (2.9) respectively, and that $Lx = N_1x$ admits only the trivial solution. By Theorem 2.2, (2.6) will have a nontrivial solution which satisfies (2.7). The proof is complete.

As an example, consider the boundary value problem

$$x' = p(t)x(t - \tau) + p(t)\left(-x^{1/2}(t - \tau) + a\right), 0 \le t \le \omega,$$
$$x(0) = x(\omega),$$

where a, τ, ω are real numbers which satisfy $0 < \omega < \tau$ and $a \le 1/4$. The function $p \in C(R, R)$ is bounded and

$$\int_0^{\omega} p(s)ds \neq 0.$$

Let $\beta = 3/4$. Then

$$\lim_{|x|\to\infty} \frac{\left|p(t)\left(-x^{1/2}+a\right)\right|}{|x|^{\beta}} \le \lim_{|x|\to\infty} \frac{\max|p(t)|\left(|x|^{1/2}+|a|\right)}{|x|^{3/4}} = 0.$$

Furthermore, since $x(t-\tau) = x(0)$ for $0 \le t \le \omega$, $x \equiv 0$ is the unique solution of the periodic boundary problem

$$\begin{gathered} x' = p(t)x(t-\tau) \quad t \in [0,\omega] \\ x(0) = x(\omega) \\ x(t) = x(0) \quad -\tau \le t \le 0 \end{gathered}$$

By Theorem 2.3, there will be a nontrivial solution of our boundary value problem. In fact,

$$x(t) = \big(\frac{1+\sqrt{1-4a}}{2}\big)^{1/2}, \quad -\tau \le t \le \omega,$$

is one of its nontrivial solutions.

We remark that similar results can be obtained for boundary-value problems involving infinite delay, or problems of the form

$$\begin{gathered} x^{(m)}(t) = g\left(t, x_t', ..., x_t^{(m-1)}\right) + h\left(t, x_t', ..., x_t^{(m-1)}\right), \quad 0 \le t \le T, \\ x^{(i)}(0) = x^{(i)}(T), i = 0, 1, ..., m-1. \end{gathered}$$

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