USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 155–171.
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Double resonant problems which are locally non-quadratic at infinity *

Marcelo F. Furtado & Elves A. B. Silva

Abstract

We establish the existence of a nontrivial solution for a double resonant elliptic problem under a local non-quadraticity condition at infinity and pointwise limits. We also study the existence of a nonzero solution when there is resonance at the first eigenvalue. The first result is obtained as an application of an abstract theorem that establishes the existence of a nontrivial critical point for functionals of class C^2 on real Hilbert spaces.

1 Introduction

The main goal of this article is to establish sufficient conditions for the existence of a nontrivial solution for a double resonant semi-linear elliptic problem under a local non-quadraticity condition and pointwise limits. We also study the existence and the multiplicity of solutions when there is resonance at the first eigenvalue.

To achieve our main objective, we prove a generalization of a critical point theorem due to Lazer-Solimini [14]. Their theorem establishes the existence of a nontrivial critical point for a functional of class C^2 defined on a real Hilbert space, under the hypotheses of Rabinowitz's saddle point theorem [19].

As it is well known, minimax theorems and related results are based on the existence of a linking structure and on deformation lemmas [2, 3, 20, 17, 29, 6, 26]. In general, to be able to derive such deformation results, it is supposed that the functional satisfies a compactness condition. In this article, we assume the (SCe) condition introduced by Silva in [25] and defined below (see Definition 2.1).

Denoting by $m(I,u)$ $[\overline{m}(I,u)]$ the Morse index [augmented Morse index] of the functional $I \in C^2(E,\mathbb{R})$ at the point u, we prove the following result:

* *Mathematics Subject Classifications:* 35J65, 58E05.
Key words: Resonance, non-quadratic at infinity, Morse index.

Published January 8, 2001.
The first author was supported by CAPES/Brazil.
The second author was partially supported by CNPq/Brazil.

Theorem 1.1 *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^2(E, \mathbb{R})$ satisfies* (SCe) *and*

(I_1) *there exists $\beta \in \mathbb{R}$ such that $I(v) \leq \beta$, for all v in V,*

(I_2) *there exists $\gamma \in \mathbb{R}$ such that $I(w) \geq \gamma$, for all w in W,*

(I_3) *the origin is a critical point of I, $I''(0)$ is a Fredholm operator and either $\dim V < m(I,0)$ or $\overline{m}(I,0) < \dim V$.*

Then I possesses a nonzero critical point.

The above result is a generalization of the Theorem 1.1 in [14] since the condition (I_1) does not imply the anti-coercivity of I on the subspace V. Theorem 1.1 also complements a recent result by Perera-Schechter [18]. Note that in [18] it is assumed the Palais-Smale compactness condition (PS) which is stronger than condition (SCe) and may not be true under the hypotheses of our application.

As in [14], our proof of Theorem 1.1 is based on the infinite dimensional Morse theory. To compensate the lack of anticoercivity of I on V, we use a deformation result, due to Silva [25] (see also [23, 24]), that sends $V \cap \partial B_R(0)$, for $R > 0$ sufficiently large, below the level surface γ of I, preserving the linking between $V \cap \partial B_R(0)$ and the subspace W.

As observed above, Theorem 1.1 is motivated by the semilinear elliptic problem,

$$-\Delta u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$ and the nonlinearity $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $f(x,0) \equiv 0$ and the subcritical growth condition:

(f_1) there are constants $a_1, a_2 > 0$ such that

$$|f_s(x,s)| \leq a_1|s|^{\sigma-2} + a_2,$$

for all $x \in \Omega$, $s \in \mathbb{R}$ where $\sigma > 2$ $(2 < \sigma < 2N(N-2)^{-1}$ if $N \geq 3)$.

Standard arguments show that the associated functional $I : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx, \tag{1.2}$$

with $F(x,s) = \int_0^s f(x,t)\,dt$, is of class C^2 and that the classical solutions of (1.1) are the critical points of I. We also note that $u \equiv 0$ is a (trivial) solution for the problem (1.1). Thus, our first objective is to establish the existence of a nonzero critical point for I.

Denoting by λ_k the k^{th} eigenvalue of $-\Delta$ on Ω with zero boundary conditions and considering the limits

$$L(x) = \liminf_{|s|\to\infty} \frac{2F(x,s)}{s^2} \quad \text{and} \quad K(x) = \limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2}, \tag{1.3}$$

taken in the pointwise sense, we suppose

(F_1) $\lambda_j \leq L(x) \leq K(x) \leq \lambda_{j+1}$, a.e. $x \in \Omega$. Furthermore, if $L(x) \equiv \lambda_j$, there exists $D_+ \in L^1(\Omega)$ such that

$$F(x,s) \geq \frac{\lambda_j}{2} s^2 + D_+(x), \quad \forall\, s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \tag{1.4}$$

and

(F_2) there exist $q > 1$ ($q \geq \frac{N}{2}$, if $N \geq 3$), $A \in L^q(\Omega)$ and $B \in L^1(\Omega)$ such that

$$|F(x,s)| \leq A(x)s^2 + B(x), \quad \forall\, s \in \mathbb{R}, \text{ a.e } x \in \Omega.$$

We note that (F_1) characterizes (1.1) as a double resonant problem (see [7, 8, 9, 4, 5]) and that the conditions (F_1) and (F_2) are related to the geometry of the saddle point theorem.

The necessary Morse index estimates for the functional I at $u \equiv 0$ are provided by a local condition for $f(x,s)$ at the origin:

(f_2) $f_s(x,0) \lneq \lambda_j$ or $\lambda_{j+1} \lneq f_s(x,0)$.

Here, we write $f_s(x,0) \lneq (\gneq)\lambda_k$ to indicate that $f_s(x,0) \leq (\geq)\lambda_k$ with strict inequality holding on a set of positive measure.

In this article, we also consider a local non-quadraticity condition at infinity on the primitive F. More specifically, setting $H(x,s) = f(x,s)s - 2F(x,s)$, we suppose

$(NQ)_+$ there exist $\Omega_0 \subset \Omega$ with positive measure and $C_+ \in L^1(\Omega)$ such that

(i) $\lim_{|s|\to\infty} H(x,s) = \infty$, a.e. $x \in \Omega_0$,

(ii) $H(x,s) \geq C_+(x)$, $\forall\, s \in \mathbb{R}$, a.e. $x \in \Omega$.

We note that condition $(NQ)_+$ with $\Omega_0 = \Omega$ has been assumed in the works [7, 5] (see also [21, 22]). In this article we show that the hypothesis $(NQ)_+$, for Ω_0 sufficiently large, and weaker versions of (F_1) and (F_2) provide the necessary compactness for the functional I.

Now, denoting by $\mu(U)$ the Lebesgue measure of a measurable set $U \subset \mathbb{R}^N$, we state our application of Theorem 1.1:

Theorem 1.2 *There exists $0 < \alpha < \mu(\Omega)$ such that, if $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $f(x,0) \equiv 0$, (f_1), (f_2), (F_1), (F_2) and $(NQ)_+$ with $\mu(\Omega_0) > \alpha$, then the problem (1.1) possesses a nonzero solution.*

It is worthwhile mentioning that in [7, 9, 4, 5] the authors establish the existence of solution for (1.1) (without supposing $f(x,0) \equiv 0$). Actually, applying a version of the saddle point theorem due to Silva [25] and the technical results proved in this article, we are able to study (1.1) in this setting when $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and satisfies

$(\hat{f}_1)$ there are constants $a_3, a_4 > 0$ such that

$$|f(x,s)| \leq a_3|s|^{\sigma-1} + a_4,$$

for all $x \in \Omega$, $s \in \mathbb{R}$ where $\sigma > 1$ ($1 < \sigma < 2N(N-2)^{-1}$ if $N \geq 3$).

Noting that on this case the functional I is of class C^1, and that the critical points of I are weak solutions of (1.1), we obtain

Theorem 1.3 *There exists $0 < \alpha < \mu(\Omega)$ such that, if $f \in C(\overline{\Omega}\times\mathbb{R}, \mathbb{R})$ satisfies $(\hat{f}_1)$, (F_1), (F_2) and $(NQ)_+$ with $\mu(\Omega_0) > \alpha$, then the problem (1.1) possesses a weak solution.*

In this work, we also present versions of Theorems 1.2 and 1.3 under conditions that are dual to $(NQ)_+$ and (F_1) (see Theorems 3.7 and 4.1).

In our final result, we study the existence of a nonzero solution for the problem (1.1) under a resonant condition at the first eigenvalue. On this case, we may replace the condition (F_2) by

$(\hat{F}_2)$ there exist $q > 1$ ($q \geq \frac{N}{2}$ if $N \geq 3$), $A \in L^q(\Omega)$ and $B \in L^1(\Omega)$ such that

$$F(x,s) \leq A(x)s^2 + B(x), \quad \forall\, s \in \mathbb{R}, \text{ a.e } x \in \Omega,$$

and consider the following local condition for the primitive F

(F_3) there exists $r_1 > 0$ such that

$$F(x,s) \geq 0, \quad \forall\, 0 < s < r_1, \text{ a.e. } x \in \Omega.$$

Setting

$$L_0(x) = \liminf_{s\to 0^+} \frac{2F(x,s)}{s^2},$$

with the limit taken in the pointwise sense, we also suppose the following generalization of the local condition (f_2):

$(\hat{f}_2)$ $L_0 \in L^1(\Omega)$, $L_0(x) \geq \lambda_1$, a.e. $x \in \Omega$. Furthermore, if $L_0(x) \equiv \lambda_1$, there exists $r_2 > 0$ such that

$$F(x,s) \geq \frac{\lambda_1}{2}s^2, \quad \forall\, 0 < s < r_2, \text{ a.e. } x \in \Omega. \tag{1.5}$$

On this case, we may assume $(NQ)_+$ without any restriction on the measure of the set Ω_0 obtaining

Theorem 1.4 *Suppose $f \in C(\overline{\Omega}\times\mathbb{R}, \mathbb{R})$ satisfies $(\hat{f}_1)$, $(\hat{f}_2)$, $(\hat{F}_2)$, (F_3), $K(x) \leq \lambda_1$ and $(NQ)_+$. Then the problem (1.1) possesses a nonzero weak solution.*

There is a vast literature for the resonance at the first eigenvalue. Here, we cite [12, 11, 10] which are closely related to the above result. We also observe that our argument provides the possibility of considering the nonautonomous problem with pointwise limits in those articles.

Double resonant problems were treated first by Figueiredo-Berestick [4]. In [9], Costa-Oliveira assumed (F_1) and a related double resonant condition for f. Later, Costa–Magalhães [7] replaced this last hypothesis by the non-quadraticity condition $(NQ)_+$ with $\Omega_0 = \Omega$. We note that in [9, 7] the authors do not allow $L(x) \equiv \lambda_j$ and they also assume uniform limits on (1.3) with respect to Ω. In [5], Carrião–Gonçalves–Pádua assumed the non-quadraticity on Ω and pointwise limits on (1.3) with some extra hypotheses. We should also mention the articles by Silva [25] and Costa–Magalhães [8] where it is considered, under the non-quadraticity condition, the problem of existence of critical points for strongly indefinite functionals and applications. Finally, we observe that in [18] an abstract theorem related to ours is used to study the existence of a nontrivial solution for problem (1.1) under a nonresonant condition and pointwise limits.

The article has the following structure: in Section 2, after presenting some preliminary results, we prove Theorem 1.1. There, we also state the version of Silva's result [25] that we use to prove Theorem 1.3. In Section 3, we prove Theorem 1.2. In Section 4, we present the proofs of Theorems 1.3 and 1.4.

2 Abstract Result

Let E be a real Hilbert space with norm $\|\cdot\|$ derived from the inner product $\langle \cdot, \cdot \rangle$. This symbol will also represent the pairing between E and its dual E^*.

Given a functional I of class C^1 on E, and γ, β, $c \in \mathbb{R}$, we set $I^\beta = \{u \in E \mid I(u) \leq \beta\}$, $I_\gamma = \{u \in E \mid I(u) \geq \gamma\}$, $K = \{u \in E \mid I'(u) = 0\}$ and $K_c = \{u \in K \mid I(u) = c\}$.

Throughout this article we use a generalization of the classical Palais-Smale condition which has been introduced by Silva [25]:

Definition 2.1 The functional $I \in C^1(E, \mathbb{R})$ satisfies the strong Cerami condition at the level $c \in \mathbb{R}$ [$(\mathrm{SCe})_c$] if for any sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ we have either

(i) (u_n) is bounded and possesses a convergent subsequence, or

(ii) for every $(u_{n_j}) \subset (u_n)$ such that $\|u_{n_j}\| \to \infty$, we have

$$\lim_{n_j \to \infty} \|I'(u_{n_j})\| \|u_{n_j}\| \to \infty.$$

If I satisfies $(\mathrm{SCe})_c$ for every $c \in [a, b]$ or for every $c \in \mathbb{R}$, we say that I satisfies (SCe) on $[a, b]$ or (SCe), respectively.

Since the condition (SCe) is stronger than the Cerami condition we have (see [27]) the following version of the deformation lemma due to Chang [6]:

Lemma 2.2 *Suppose $I \in C^1(E,\mathbb{R})$ satisfies* (SCe) *on $[a,b]$, a is the only critical value of I on $[a,b)$ and K_a have only pointwise connected components. Then I^a is a strong deformation retract of $I^b \setminus K_b$.*

Remark 2.3 We may have $b = \infty$ in Lemma 2.2. On this case, I^a is a strong deformation retract of E.

For the proof of Theorem 1.1 we also use a technical result proved in [25]:

Lemma 2.4 *Let $E = V \oplus W$ be a real Hilbert space with $W = V^\perp$. Suppose $I \in C^1(E,\mathbb{R})$ satisfies (I_1), (I_2) and* (SCe) *on $[\gamma,\beta]$. Then, given $\overline{\varepsilon} > 0$, there exist $\varepsilon \in (0,\overline{\varepsilon})$, $R > 0$ and a continuous function $\eta : [0,1] \times E \to E$ such that*

(η_1) *$\eta(t,u) = u$, for every $u \in W$,*

(η_2) *$\eta(t,\cdot) : E \to E$ is a homeomorfism, for every $t \in [0,1]$,*

(η_3) *$\eta(1,u) \in I^{\gamma-\varepsilon}$, for every $u \in V \cap \partial B_R(0)$,*

(η_4) *$I(\eta(t,u)) \leq I(u)$, for every $u \in E$, $t \in [0,1]$.*

Given two topological spaces A and X, with A a subspace of X, let $H_n(X,A)$ denote the n^{th} relative singular homology group with coefficient group equal to the real numbers. The next result is based on [14].

Proposition 2.5 *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^\perp$. Suppose $I \in C^2(E,\mathbb{R})$ satisfies (I_1), (I_2),* (SCe) *and has only a finite number of critical points, all of which are nondegenerate. Then I possesses a critical point u such that $m(I,u) = \dim V$.*

Proof. Let $k = \dim V$. When $k = 0$, the condition (I_2) implies that I is bounded from below and consenquently, by (SCe), the infimum of I is attained at a point $u \in E$ (see [3, 20]). Since u is nondegenerate, $m(I,u) = 0$.

Now, let us consider $k \geq 1$. Arguing by contradiction, we suppose that I has no critical point with Morse index equals to k.

First, we choose $d_0 < \gamma$ such that I has no critical points on I^{d_0}. Noting that I satisfies (I_2) with γ replaced by d_0, we may apply Lemma 2.4 to obtain $\varepsilon \in (0,1)$, $R > 0$ and $\eta \in C([0,1] \times E, E)$ satisfying (η_1), (η_2), (η_4) and

$$\tilde{S} = \eta(1,S) \subset I^{d_0-\varepsilon} \subset I^{d_0} \subset E \setminus W, \tag{2.1}$$

where $S = V \cap \partial B_R(0)$.

Moreover, by (η_1) and (η_2), we have that $\eta(1, E\setminus W) = E \setminus W$. Thus, considering the inclusions $i : \tilde{S} \to E \setminus W$ and $l : S \to E \setminus W$, we obtain the following commutative diagram of homomorphisms

$$\begin{array}{ccc} H_*(S) & \xrightarrow{l_*} & H_*(E\setminus W) \\ \psi_{1*} \downarrow & & \downarrow \psi_{2*} \\ H_*(\tilde{S}) & \xrightarrow{i_*} & H_*(E\setminus W), \end{array}$$

where $\psi_1 = \eta(1,\cdot)|_S$ and $\psi_2 = \eta(1,\cdot)|_{E\setminus W}$. Invoking (η_2) one more time, we have that ψ_{1*} and ψ_{2*} are isomorphisms. Therefore, since S is a strong deformation retract of $E \setminus W$, we conclude that

$$i_* : H_*(\tilde{S}) \to H_*(E \setminus W)$$

is an isomorphism.

Now, let $j : \tilde{S} \to I^{d_0}$ and $h : I^{d_0} \to H \setminus W$ be the inclusion maps. Since, by (2.1), $i = h \circ j$, we have that $i_* = h_* \circ j_*$. Hence, h_* is surjective and

$$\dim H_{k-1}(I^{d_0}) \geq \dim H_{k-1}(E \setminus W) = \dim H_{k-1}(S).$$

Therefore,

$$\dim H_{k-1}(I^{d_0}) \geq \begin{cases} 1, & \text{if } k > 1, \\ 2, & \text{if } k = 1. \end{cases} \tag{2.2}$$

Now, consider

$$c_1 < c_2 < \cdots < c_m,$$

the possible critical levels of I, and take $d_1, \ldots, d_m$ real numbers such that $d_{j-1} < c_j < d_j$, for $j = 1, \ldots, m$. If I does not possess critical values, set $d_1 = d_m > d_0$. Recalling that $\dim H_k(I^{d_j}, I^{d_{j-1}})$ is equal to the number of critical points of I with Morse index k at the critical level c_j (see [17]), we have

$$\dim H_k(I^{d_j}, I^{d_{j-1}}) = 0, \quad \forall\, j = 1, \ldots, m. \tag{2.3}$$

The exactness of the sequence

$$\cdots \longrightarrow H_k\left(I^{d_{j-1}}, I^{d_0}\right) \longrightarrow H_k\left(I^{d_j}, I^{d_0}\right) \longrightarrow H_k\left(I^{d_j}, I^{d_{j-1}}\right) \longrightarrow \cdots$$

implies that

$$\dim H_k\left(I^{d_j}, I^{d_0}\right) \leq \dim H_k\left(I^{d_j}, I^{d_{j-1}}\right) + \dim H_k\left(I^{d_{j-1}}, I^{d_0}\right).$$

Consequently, by the above inequality and (2.3), we obtain

$$\dim H_k(I^{d_m}, I^{d_0}) \leq \sum_{j=1}^{m} \dim H_k(I^{d_j}, I^{d_{j-1}}) = 0. \tag{2.4}$$

Considering the exact sequence of the pair (I^{d_m}, I^{d_0}), we conclude that

$$\dim H_{k-1}(I^{d_0}) \leq \dim H_k(I^{d_m}, I^{d_0}) + \dim H_{k-1}(I^{d_m}).$$

Since I^{d_m} is a strong deformation retract of E (see Remark 2.3), by (2.4) and the above expression, we have

$$\dim H_{k-1}(I^{d_0}) \leq \dim H_{k-1}(E) = \begin{cases} 0, & \text{se } k > 1, \\ 1, & \text{se } k = 1, \end{cases}$$

which contradicts (2.2) and concludes the proof of Proposition 2.5. $\diamondsuit$

Proposition 2.5 is a key ingredient for the proof of Theorem 1.1. We also need the following version of a result by Marino-Prodi [15] (see also [28]).

Lemma 2.6 *Suppose $I \in C^2(E,\mathbb{R})$ satisfies* (SCe)*, u_0 is an isolated critical point of I and $I''(u_0)$ is a Fredholm operator. Then, given $\varepsilon > 0$, there exists $J \in C^2(E,\mathbb{R})$ such that J satisfies* (SCe)*, $I(u) = J(u)$ for $\|u - u_0\| \geq \varepsilon$, J has only a finite number of critical points all of which are nondegenerate in the open ball $B_\varepsilon(u_0)$, and $\left\|I^{(j)}(u) - J^{(j)}(u)\right\| < \varepsilon$, for $j = 0, 1, 2$ and $u \in E$.*

Remark 2.7 In [15], the authors consider the above lemma with the hypothesis that I satisfies the Palais-Smale condition. Noting that for bounded sequences the (SCe) condition is equivalent to the Palais-Smale condition and that the functionals I and J are equals on the complement of $B_\varepsilon(u_0)$, we see easily that the result in [15] holds with the weaker condition (SCe).

We are ready to prove our abstract theorem. The following argument is due to Lazer–Solimini [14] and it will be presented for the sake of completness.

Proof of Theorem 1.1. Without loss of generality we may assume that 0 is an isolated critical point. Since $I''(0)$ is a Fredholm operator, we can use the spectral theory to obtain a constant $b > 0$ and an orthogonal decomposition $E = E^0 \oplus E^+ \oplus E^-$, where E^0 is the kernel of $I''(0)$, E^+ and E^- are closed and invariants under $I''(0)$, and

$$\langle I''(0)u, u\rangle \geq b\|u\|^2, \quad \forall\, u \in E^+, \tag{2.5}$$

$$\langle I''(0)u, u\rangle \leq -b\|u\|^2, \quad \forall\, u \in E^-. \tag{2.6}$$

Now, let J be the functional given by Lemma 2.6, with $\varepsilon > 0$ such that $0 < \varepsilon < \frac{b}{3}$ and $\|I''(u) - I''(0)\| < \frac{b}{3}$, if $\|u\| < \varepsilon$. It is clear that J also satisfies (I_1) and (I_2) for appropriated constants.

To prove Theorem 1.1 it suffices to show that J possesses a critical point on the complement of $B_\varepsilon(0)$. Suppose, by contradiction, that this is not the case. Then, by Proposition 2.5, there exists a critical point $\overline{u}$ of J such that $m(J,\overline{u}) = \dim V$.

By our choice of ε and Lema 2.6, we obtain

$$\|J''(\overline{u}) - I''(0)\| \leq \|J''(\overline{u}) - I''(\overline{u})\| + \|I''(\overline{u}) - I''(0)\| < \frac{2b}{3}.$$

Hence, by (2.5) and (2.6), we get

$$\langle J''(\overline{u})u, u\rangle \geq \frac{b}{3}\|u\|^2, \quad \forall\, u \in E^+,$$

$$\langle J''(\overline{u})u, u\rangle \leq -\frac{b}{3}\|u\|^2, \quad \forall\, u \in E^-.$$

Consequently,

$$m(I,0) = \dim E^- \leq m(J,\overline{u}) \leq \dim\left(E^0 \oplus E^-\right) = \overline{m}(I,0).$$

The last expression and the hypotheses of Theorem 1.1 show that $m(J,\overline{u}) \neq \dim V$. This concludes the proof of Theorem 1.1. ♢

By the above argument and repeated applications of Lemma 2.6, we get

Proposition 2.8 *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^2(E, \mathbb{R})$ satisfies* (SCe), (I_1), (I_2) *and*

$(\hat{I_3})$ *I possesses a finite number of critical points $\{u_j\}_{j=1}^m$ such that $I''(u_j)$ is a Fredholm operator and either $\dim V < m(I, u_j)$ or $\overline{m}(I, u_j) < \dim V$, for $j = 1, \ldots, m$.*

Then there exists a critical point u of I with $u \neq u_j$, $j = 1, \ldots, m$.

Finally, we state the version of the saddle point theorem, due to Silva [25] (see also [23, 24]), that will be used in the proof of Theorem 1.3.

Theorem 2.9 *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^{\perp}$. Suppose $I \in C^1(E, \mathbb{R})$ satisfies (I_1), (I_2) and $(\mathrm{SCe})_b$ for every $b \geq \gamma$. Then I possesses a critical value $b \in [\gamma, \beta]$.*

3 Proof of Theorem 1.2

We denote by $|\cdot|_p$ the $L^p(\Omega)$-norm $(1 \leq p \leq \infty)$ and by $\|\cdot\|$ the norm in $E = H_0^1(\Omega)$ induced by the inner product

$$\langle u, v\rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \forall\, u,\, v \in H_0^1(\Omega).$$

As mentioned previously, the condition (f_1) implies that the functional I given by (1.2) is of class C^2, and the classical solutions of (1.1) are the critical points of I. A standard argument [20, 6] shows that for $\varphi, \psi \in H$,

$$I''(0)(\varphi, \psi) = \int_{\Omega} \nabla\varphi \cdot \nabla\psi \, dx - \int_{\Omega} f_s(x, 0)\varphi\psi \, dx \,. \tag{3.1}$$

We denote by E_k the subspace of E spanned by $\{\phi_1, \ldots, \phi_k\}$, where $\{\phi_1, \phi_2, \ldots\}$ is the orthonormal basis of E formed by the eigenfunctions associated to the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$ of $-\Delta$ on Ω with zero boundary conditions.

Given $N \geq 3$, consider $2^* = \frac{2N}{N-2}$ and set

$$S = S(N) = \inf_{u \in E,\, u \not\equiv 0} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\left(\int_{\Omega} |u|^{2^*} \, dx\right)^{2/2^*}} > 0,$$

the best constant for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. Note that S depends only on the dimension N [29].

We start with a technical result

Lemma 3.1 *Suppose $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $K(x) \leq \lambda_{j+1}$. Then, given $\varepsilon > 0$, there exists $\tilde{\Omega} \subset \Omega$ and a constant $M_1 = M_1(\varepsilon) > 0$ such that $\mu(\Omega \backslash \tilde{\Omega}) < \varepsilon$ and*

$$F(x, s) \leq \frac{1}{2}(\varepsilon + \lambda_{j+1})s^2 + M_1, \quad \forall\, s \in \mathbb{R},\ a.e.\ x \in \tilde{\Omega}. \tag{3.2}$$

Proof. Effectively, given $\varepsilon > 0$, we apply the Egorov's Theorem for the sequence

$$G_n(x) = \sup\left\{\frac{2F(x,s)}{s^2} \mid |s| \geq n\right\},$$

to obtain $\tilde{\Omega} \subset \Omega$ such that $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$ and

$$\limsup_{|s|\to\infty} \frac{2F(x,s)}{s^2} \leq \lambda_{j+1}, \quad \text{unif. for a.e. } x \in \tilde{\Omega}.$$

The above expression provide $r = r(\varepsilon) > 0$ such that

$$F(x,s) \leq \frac{1}{2}(\varepsilon + \lambda_{j+1})s^2, \quad \forall\, |s| \geq r, \text{ a.e. } x \in \tilde{\Omega}. \tag{3.3}$$

Now, by the continuity of $F(x,s)$ on $\overline{\Omega} \times [-r,r]$,

$$F(x,s) \leq M_1, \quad \forall\, |s| \leq r, \text{ a.e. } x \in \Omega.$$

The above expression and (3.3) prove the statement (3.2). ♢

Remark 3.2 If we suppose that for every $r > 0$ there exists $B_r \in L^1(\Omega)$ such that

$$F(x,s) \leq B_r(x), \quad \forall\, |s| \leq r, \text{ a.e. } x \in \Omega,$$

we obtain a version of the above result with $M_1 \in L^1(\Omega)$ without to assume the continuity of f.

Now, we prove the compactness condition for the functional I.

Lemma 3.3 *There exists $0 < \alpha < \mu(\Omega)$ such that, if $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $(\hat{f}_1)$, $(\hat{F}_2)$, $K(x) \leq \lambda_{j+1}$ and $(NQ)_+$ with $\mu(\Omega_0) > \alpha$, then the functional I satisfies* (SCe).

Proof. Let $(u_n) \subset E$ be such that $I(u_n) \to c \in \mathbb{R}$ and $I'(u_n) \to 0$. If (u_n) is bounded, hypothesis $(\hat{f}_1)$ implies that (u_n) possesses a convergent subsequence. Consequently, by Definition 2.1, it suffices to verify that $\|I'(u_{n_j})\|\|u_{n_j}\| \to \infty$, for every sequence $(u_{n_j}) \subset (u_n)$ such that $\|u_{n_j}\| \to \infty$.

Arguing by contradiction, we suppose that there exists a subsequence, which we will denote by (u_n), such that $\|u_n\| \to \infty$ and $\|I'(u_n)\|\,\|u_n\|$ is bounded. This assumption and the fact that $I(u_n) \to c$ provide $M \in \mathbb{R}$ such that

$$\liminf \int_\Omega H(x,u_n)\,dx = \liminf\,[2I(u_n) - I'(u_n)u_n] \leq M. \tag{3.4}$$

On the other hand, given $\varepsilon > 0$, for n sufficiently large, we have

$$\frac{1}{2}\|u_n\|^2 \leq (\varepsilon + c) + \int_\Omega F(x,u_n)\,dx. \tag{3.5}$$

Now, we apply Lemma 3.1 to obtain $M_1 > 0$ and $\tilde{\Omega} \subset \Omega$ such that $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$ and

$$F(x, u_n) \le \frac{1}{2}(\varepsilon + \lambda_{j+1})u_n^2 + M_1, \quad \text{a.e. } x \in \tilde{\Omega}.$$

Considering $N \ge 3$, we may use the above expression, (3.5), Hölder inequality, $(\hat{F_2})$ and the Sobolev Embeding Theorem, to obtain

$$\begin{aligned} \frac{1}{2}\|u_n\|^2 &\le (c+\varepsilon) + \frac{1}{2}(\varepsilon + \lambda_{j+1})|u_n|_2^2 + M_1\mu(\Omega) \\ &\quad + \int_{\Omega\setminus\tilde{\Omega}} \left[Au_n^2 + B\right] dx \\ &\le \frac{1}{2}(\varepsilon + \lambda_{j+1})|u_n|_2^2 + S^{-1}|A|_{L^{\frac{N}{2}}(\Omega\setminus\tilde{\Omega})}\|u_n\|^2 + \tilde{M}, \end{aligned}$$

where $\tilde{M} = \tilde{M}(\varepsilon) = c + \varepsilon + M_1\mu(\Omega) + |B|_1$, or equivalently

$$\left(\frac{1}{2} - S^{-1}|A|_{L^{\frac{N}{2}}(\Omega\setminus\tilde{\Omega})}\right)\|u_n\|^2 \le \frac{1}{2}(\varepsilon + \lambda_{j+1})|u_n|_2^2 + \tilde{M}. \tag{3.6}$$

Defining $\tilde{u}_n = \frac{u_n}{\|u_n\|}$ we may suppose that $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in E, $\tilde{u}_n \to \tilde{u}$ strongly in $L^2(\Omega)$ and $\tilde{u}_n(x) \to \tilde{u}(x)$ for almost everywhere $x \in \Omega$. Thus, dividing (3.6) by $\|u_n\|^2$, taking $n \to \infty$, $\varepsilon \to 0$, we conclude that

$$1 \le \lambda_{j+1}|\tilde{u}|_2^2. \tag{3.7}$$

At this point, we claim that there exists a measurable set $\Omega_1 \subset \Omega_0$ with positive measure such that

$$\tilde{u}(x) \neq 0, \quad \text{a.e. } x \in \Omega_1. \tag{3.8}$$

Supposing the claim, we use $(NQ)_+$ and Fatou's lemma to conclude that

$$\liminf \int_\Omega H(x, u_n)\, dx \ge \int_\Omega \liminf H(x, u_n)\, dx = \infty,$$

which contradicts (3.4).

To prove the claim we set

$$\alpha = \mu(\Omega) - \left(\frac{S}{\lambda_{j+1}}\right)^{N/2} > 0,$$

and suppose, by contradiction, that the claim is false. Then, by (3.7), we have

$$\begin{aligned} 1 &\le \lambda_{j+1}\int_\Omega |\tilde{u}(x)|^2\, dx \le \lambda_{j+1}\left[\int_{\Omega\setminus\Omega_0} |\tilde{u}(x)|^{2^*}\, dx\right]^{2/2^*} \mu(\Omega\setminus\Omega_0)^{2/N} \\ &\le \lambda_{j+1}S^{-1}\|\tilde{u}\|^2\mu(\Omega\setminus\Omega_0)^{2/N} \le \lambda_{j+1}S^{-1}\mu(\Omega\setminus\Omega_0)^{2/N} < 1. \end{aligned}$$

For $N = 1$ or $N = 2$, we use the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$, $1 < r < \infty$, and the above argument for $\mu(\Omega \setminus \Omega_0)$ sufficiently small. $\diamondsuit$

Remark 3.4 We note that for $K(x) \leq \lambda_1$ the above result is valid with $\alpha = 0$. Effectively, on this case, we verify (3.8) with $\Omega_1 = \Omega_0$ using (3.7) and the Poincaré's inequality

$$1 \leq \lambda_1 |\tilde{u}|_2^2 \leq \|\tilde{u}\|^2 \leq 1.$$

The last expression shows that $\tilde{u}$ is a λ_1-eigenfunction and therefore $\tilde{u}(x) \neq 0$ a.e. $x \in \Omega$.

The following lemma is a version of a technical result due to Costa-Magalhães [7].

Lemma 3.5 *Suppose f satisfies $(NQ)_+$ and $K(x) \leq \lambda_{j+1}$. Then*

$$F(x,s) - \frac{1}{2}K(x)s^2 \leq -\frac{C_+(x)}{2}, \quad \forall\, s \in \mathbb{R},\ a.e\ x \in \Omega.$$

Proof. Let $g(x,s) = f(x,s) - K(x)s$ and $G(x,s) = F(x,s) - \frac{1}{2}K(x)s^2$. Then we have

$$g(x,s)s - 2G(x,s) = f(x,s)s - 2F(x,s)\,.$$

Taking $s > 0$ and using $(NQ)_+$, we obtain

$$\frac{d}{ds}\left[\frac{G(x,s)}{s^2}\right] = \frac{g(x,s)s - 2G(x,s)}{s^3} \geq \frac{C_+(x)}{s^3}. \tag{3.9}$$

Integrating the above expression on $[s,t] \subset (0,\infty)$, we get

$$\frac{G(x,s)}{s^2} \leq \frac{G(x,t)}{t^2} - \frac{C_+(x)}{2}\left[\frac{1}{s^2} - \frac{1}{t^2}\right].$$

Since $K(x) \leq \lambda_{j+1}$, we have that $\limsup_{t\to\infty} t^{-2}G(x,t) \leq 0$ for almost everywhere $x \in \Omega$, and therefore

$$G(x,s) \leq -\frac{C_+(x)}{2}, \quad \forall\, s > 0,\ \text{a.e. } x \in \Omega.$$

The proof for $s < 0$ is similar. $\diamondsuit$

The above lemma is used to verify the condition (I_2) of Theorem 1.1. The next result is a version of Proposition 2 in [9] (see also [16]). It provides estimates on subsets of Ω and it will be used to verify the hypothesis (I_1) of Theorem 1.1.

Lemma 3.6 *Suppose (F_1) and $L(x) \not\geq \lambda_j$. Then there exist $\delta_1, \varepsilon > 0$ such that, for every subset $\tilde{\Omega} \subset \Omega$ with $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$, we have*

$$\|u\|^2 - \int_{\tilde{\Omega}} L(x)u^2\,dx \leq -\delta_1 \|u\|^2, \quad \forall\, u \in E_j.$$

Proof. By Proposition 2 in [9], the lemma holds for $\tilde{\Omega} = \Omega$ and $\hat{\delta}_1 > 0$. Let $u \in E_j$, $\tilde{\Omega} \subset \Omega$. Supposing $N \geq 3$, using Hölder inequality and the Sobolev Embeding Theorem, we obtain

$$\begin{aligned} \|u\|^2 - \int_{\tilde{\Omega}} L(x)u^2\,dx &= \|u\|^2 - \int_{\Omega} L(x)u^2\,dx + \int_{\Omega\setminus\tilde{\Omega}} L(x)u^2\,dx \\ &\leq -\hat{\delta}_1 \|u\|^2 + |L|_\infty S^{-1}\mu(\Omega\setminus\tilde{\Omega})^{\frac{2}{N}} \|u\|^2 \\ &= \|u\|^2 \left(-\hat{\delta}_1 + |L|_\infty S^{-1}\mu(\Omega\setminus\tilde{\Omega})^{\frac{2}{N}}\right). \end{aligned}$$

Hence, the lemma holds for $\delta_1 = \hat{\delta}_1/2$ and $\varepsilon > 0$ sufficiently small. The argument for the cases $N = 1$ or $N = 2$ is similar. ♢

Proof of Theorem 1.2 Without loss of generality, we may suppose that the origin is an isolated critical point of I. Taking $V = E_k$ and $W = V^\perp$, by Lemma 3.5, we have

$$I(w) = \frac{1}{2}\big(\|w\|^2 - \int_\Omega K(x)w^2\,dx\big) - \int_\Omega \big[F(x,w) - \frac{K(x)}{2}w^2\big]\,dx \geq -\frac{|C_+|_1}{2},$$

for every $w \in W$. Consenquently, I satisfies (I_2). To verify (I_1), we first suppose that $L(x) \equiv \lambda_j$. Then, using (1.4), we have

$$I(v) \leq \frac{1}{2}\left(\|v\|^2 - \lambda_j|v|_2^2\right) + |D_+|_1 \leq |D_+|_1, \quad \forall\, v \in V.$$

Thus, I satisfies (I_1) on this case.

Now, let us consider $L(x) \not\geq \lambda_j$. Consider $\delta_1, \varepsilon > 0$ given by Lemma 3.6. Using the same argument employed in the proof of Lemma 3.1, we obtain $M_1 = M_1(\varepsilon) > 0$ such that

$$2F(x,v) \geq (L(x) - \varepsilon)v^2 - M_1, \quad \forall\, v \in V, \text{ a.e. } x \in \tilde{\Omega},$$

where $\tilde{\Omega} \subset \Omega$ satisfies $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$. Taking $N \geq 3$, the above expression, Lemma 3.6, (F_2), Hölder inequality and the Sobolev Embeding Theorem imply

$$\begin{aligned} 2I(v) &\leq \|v\|^2 - \int_{\tilde{\Omega}} L(x)v^2\,dx + \varepsilon|v|_2^2 + M_1\mu(\Omega) + 2\int_{\Omega\setminus\tilde{\Omega}} \left[Av^2 + B\right]\,dx \\ &\leq \left(-\delta_1 + \frac{\varepsilon}{\lambda_1} + 2S^{-1}|A|_{L^{\frac{N}{2}}(\Omega\setminus\tilde{\Omega})}\right)\|v\|^2 + M_1\mu(\Omega) + 2|B|_1, \end{aligned}$$

for every $v \in V$. Considering ε smaller if necessary, we conclude that $I(v) \to -\infty$, when $\|v\| \to \infty$. Thus, (I_1) also holds on this case. The proof for $N = 1$ and $N = 2$ is similar.

Since f satifies (f_1), $I''(0)$ is a Fredholm operator. To establish the Morse Index estimates, we suppose first that $f_s(x,0) \not\geq \lambda_{j+1}$. Using (3.1) and the same argument employed on the proof of the Lemma 3.6, we obtain $\delta_1 > 0$ such that

$$I''(0)(u,u) = \|u\|^2 - \int_\Omega f_s(x,0)u^2\,dx \leq -\delta_1 \|u\|^2, \quad \forall\, u \in E_{j+1}.$$

Consequently, $m(I,0) \geq j+1 > \dim V$. An analogous argument shows that, $\overline{m}(I,0) \leq j-1 < \dim V$, whenever $f_s(x,0) \not\leq \lambda_j$. Hence, I satisfies (I_3). Now, we invoke Lemma 3.3 and Theorem 1.1 to obtain a nonzero solution for (1.1). $\diamondsuit$

We also observe that the problem (1.1) can be considered under conditions that are dual to $(NQ)_+$ and (F_1). More specifically, assuming

$(\hat{F}_1)$ $\lambda_j \leq L(x) \leq K(x) \leq \lambda_{j+1}$, a.e. $x \in \Omega$. Furthermore, if $K(x) \equiv \lambda_{j+1}$, there exist $D_- \in L^1(\Omega)$ such that

$$F(x,s) \leq \frac{\lambda_{j+1}}{2} s^2 + D_-(x), \quad \forall\, s \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

and

$(NQ)_-$ there exist $\Omega_0 \subset \Omega$ with positive measure and $C_- \in L^1(\Omega)$ such that

$$\text{(i)} \quad \lim_{|s|\to\infty} H(x,s) = -\infty, \quad \text{a.e. } x \in \Omega_0,$$
$$\text{(ii)} \quad H(x,s) \leq C_-(x), \qquad \forall\, s \in \mathbb{R}, \text{a.e. } x \in \Omega,$$

an argument similar to the one employed in the proof of Theorem 1.2 provides the following theorem.

Theorem 3.7 *There exists* $0 < \alpha < \mu(\Omega)$ *such that, if* $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *satisfies* $f(x,0) \equiv 0$, (f_1), (f_2), $(\hat{F}_1)$, (F_2) *and* $(NQ)_-$ *with* $\mu(\Omega_0) > \alpha$*, then the problem (1.1) possesses a nonzero solution.*

4 Proofs of Theorems 1.3 and 1.4

We start by observing that, under hypothesis $(\hat{f}_1)$, the functional I given by (1.2) is of class C^1 in $E = H_0^1(\Omega)$ and that the weak solutions of (1.1) are the critical points of I.

Proof of Theorem 1.3 The same arguments employed in the proof of Theorem 1.2 show that I satisfies (I_1) and (I_2). Thus, we may invoke Lemma 3.3 and Theorem 2.9 to derive the existence of a critical point for I. $\diamondsuit$

Considering the dual conditions $(NQ)_-$ and $(\hat{F}_1)$, we may apply the above argument to prove

Theorem 4.1 *There exist* $0 < \alpha < \mu(\Omega)$ *such that, if* $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ *satisfies* $(\hat{f}_1)$, $(\hat{F}_1)$, (F_2) *and* $(NQ)_-$ *with* $\mu(\Omega_0) > \alpha$*, then the problem (1.1) possesses a weak solution.*

Finally, let us establish the multiplicity of solutions for the resonance at the first eigenvalue.

Proof of Theorem 1.4 Using $(\hat{f}_1)$, $(\hat{F}_2)$, $K(x) \leq \lambda_1$ and Remark 3.4, we conclude that I satisfies (SCe). The Lemma 3.5 and the argument used in the proof of Theorem 1.2 imply that I is bounded from bellow. Thus, by (SCe), the infimun of I is attained at a critical point $u \in E$. Hence, to prove Theorem 1.4, it suffices to verify that $I(t\phi_1) \leq 0$, for some $t > 0$, where ϕ_1 is the first eigenfunction of $-\Delta$ on Ω with zero boundary conditions.

Suppose first that $L_0(x) \equiv \lambda_1$. The regularity of ϕ_1 implies that, for $t > 0$ sufficiently small, we have $0 < t\phi_1(x) < r_2$ for a.e. $x \in \Omega$, with r_2 given by $(\hat{f}_2)$. Thus, we may use (1.5) to obtain the desired inequality

$$I(t\phi_1) = \frac{1}{2}\int_\Omega \lambda_1 (t\phi_1)^2\, dx - \int_\Omega F(x, t\phi_1)\, dx \leq 0\,.$$

Now, let us consider $L_0(x) \not\geq \lambda_1$. Then, given $\varepsilon > 0$, we may use the definition of $L_0(x)$ and the same argument of the proof of Lemma 3.1, to obtain $\tilde{\Omega} \subset \Omega$ such that $\mu(\Omega \setminus \tilde{\Omega}) < \varepsilon$ and

$$F(x, s) \geq \frac{1}{2}(L_0(x) - \varepsilon)s^2, \quad 0 < s < r, \text{ a.e. } x \in \tilde{\Omega}, \tag{4.1}$$

for some constant $r = r(\varepsilon) > 0$. Furthermore, we may suppose that ε and r are so that

$$\int_{\tilde{\Omega}} (L_0(x) - \lambda_1)\phi_1^2\, dx \geq \alpha > 0, \tag{4.2}$$

and $r < r_1$, with r_1 given by (F_3). This last relation implies that

$$m = \inf_{0<s<r} F(x, s) \geq 0. \tag{4.3}$$

Now, using the regularity of ϕ_1 one more time, we have that $0 < t\phi_1(x) < r$, for $t > 0$ sufficiently small and for almost everywhere $x \in \Omega$. Considering (4.1)–(4.3), we obtain

$$\begin{aligned} 2I(t\phi_1) \quad &\leq \quad \int_\Omega \lambda_1 (t\phi_1)^2\, dx - \int_{\tilde{\Omega}} (L_0(x) - \varepsilon)(t\phi_1)^2\, dx \\ &\quad\quad - \int_{\Omega\setminus\tilde{\Omega}} F(x, t\phi_1)\, dx \\ &\leq \quad -t^2 \left[\alpha - \varepsilon \left(|\phi_1|_2^2 + \lambda_1 |\phi_1|_\infty^2\right)\right]. \end{aligned}$$

Taking ε smaller, if necessary, we conclude that $I(t\phi_1) < 0$, for every $t > 0$ sufficiently small. This concludes the proof of Theorem 1.4. $\diamondsuit$

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Marcelo F. Furtado
Dpto. Matemática, Universidade de Brasília
70910 Brasilia, DF Brazil
e-mail: furtado@mat.unb.br

Elves A. B. Silva
Dpto. Matemática, Universidade de Brasília
70910 Brasilia, DF Brazil
e-mail: elves@mat.unb.br