USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 243–255.
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On a fourth order superlinear elliptic problem *

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Abstract

We prove the existence of a nonzero solution for the fourth order elliptic equation

$$\Delta^2 u = \mu u + a(x)g(u)$$

with boundary conditions $u = \Delta u = 0$. Here, μ is a real parameter, g is superlinear both at zero and infinity and $a(x)$ changes sign in Ω. The proof uses a variational argument based on the argument by Bahri-Lions [3].

1 Introduction

We consider the fourth order problem

$$\begin{gathered} \Delta^2 u = \mu u + a(x)g(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary ($\partial\Omega \in C^{3,1}$ for example), μ is a real parameter, $a \in C^1(\overline{\Omega}; \mathbb{R})$ and $g \in C^1(\mathbb{R}; \mathbb{R})$ is subcritical and has a superlinear behavior both at zero and at infinity. Precisely, we shall assume that, for some $\ell > 0$ and $2 < p < 2N/(N-4)$ ($2 < p < \infty$ if $1 \leq N \leq 4$), it holds

H1) $g(0) = 0 = g'(0)$ and $\lim_{|u|\to\infty} \frac{g'(u)}{|u|^{p-2}} = \ell$.

Problems of this type with $a(x) \equiv c > 0$ and $p \leq 2N/(N-4)$ were studied (along with other boundary conditions and more general operators) in [5, 7, 12, 15, 16, 17]. Here, the main feature in (1.1) will be the fact that we assume a changes sign in Ω. Thus we extend for the biharmonic operator results that were recently obtained for the corresponding second order problem, involving the operator $(-\Delta, H_0^1(\Omega))$. Precisely, our main result is inspired by the work in [1, 4, 6, 13]. We refer the reader to [6] for a more complete discussion and bibliography on the subject.

In the following we denote by $\nu(x)$ the unit outward normal of Ω at the point $x \in \partial\Omega$ and by $\langle \cdot, \cdot \rangle$ the inner product in $\mathbb{R}^N$. We assume that the function a and the domain Ω are related in the following way:

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* *Mathematics Subject Classifications:* 35J25, 35J20, 58E05.
Key words: Superlinear elliptic problems, Morse index, biharmonic operator.

Published January 8, 2001.

H2) $\nabla a(x) \neq 0$ for all $x \in \overline{\Omega}$ such that $a(x) = 0$

H3) $\langle \nabla a(x), \nu(x) \rangle = 0$ for all $x \in \partial\Omega$ such that $a(x) = 0$.

The condition in (H2) is a non-degeneracy assumption [4] while the condition in (H3) arises in connection with some integral identities of Pohožaev type (see section 2). Of course, (H3) is trivially satisfied if a does not vanish on the boundary of Ω. On the other hand, both (H2) and (H3) are satisfied if, for example, a is a linear projection and Ω is a ball (or an annulus domain).

Our main result is as follows.

Theorem 1.1. *Assume a changes sign in Ω, that* H1, H2, H3 *hold, and that μ is not an eigenvalue of the operator $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$. Then problem (1.1) has a nonzero solution $u \in C^4(\Omega; \mathbb{R}) \cap C^3(\overline{\Omega}; \mathbb{R})$.*

The rest of the text is devoted to the proof of Theorem 1.1. We mention that the theorem can probably be extended in the lines of [6, Th.1] where, in contrast with assumption (H2), the authors let a and ∇a vanish simultaneously in $\overline{\Omega}$. However, as explained in [6], it remains an open question to fully understand to what extend can assumptions (H2)-(H3) be relaxed, even for the corresponding second order problem.

2 Proofs

We introduce the functional

$$J(u) = \frac{1}{2} \int_\Omega [(\Delta u)^2 - \mu u^2] - \int_\Omega a(x) G(u), \quad u \in H^2(\Omega) \cap H_0^1(\Omega),$$

where we denote $G(u) = \int_0^u g(s)\, ds$. It is known that critical points of J are strong solutions of (1.1) (see e.g. [16]). However, our assumptions do not seem to imply suitable compactness properties for J (namely, the so called Palais Smale condition). Moreover, due to the absence of sign in the nonlinear term, it is not clear whether the geometric structure of such functional falls into one of the usual schemes used in critical point theory.

To overcome these dificulties, we use a truncation argument introduced in [13] and subsequently developed in [6]. Precisely, we fix any sequences $a_j \to +\infty$ and $p_j \in]2, p[$, $p_j \to p$, and define

$$g_j(u) := \begin{cases} \tilde{A}_j |u|^{p_j - 2} u + \tilde{B}_j, & \text{for } u \leq -a_j; \\ g(u), & \text{for } |u| \leq a_j; \\ A_j |u|^{p_j - 2} u + B_j, & \text{for } u \geq a_j, \end{cases}$$

in such a way that g_j is C^1. Next we consider the modified problem

$$\begin{gathered} \Delta^2 u = \mu u + a^+(x) g(u) - a^-(x) g_j(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $a^{\pm} := \max\{\pm a, 0\}$. Then minor changes in the proof of [6, Th.1] show that (2.1) has indeed a nonzero solution u_j, for any $j \in \mathbb{N}$ (here, a unique continuation principle for the biharmonic operator is needed; it can be found in [10, Th.6.3 and Rem.6.8]). These solutions u_j are found by means of the so called "local linking" theorem; in particular (see [6, Prop.2]), it follows that their Morse indexes are bounded. Denoting by J_j the energy functional associated to (2.1), this means that the second derivative $D^2 J_j$ cannot be negative definite in subspaces with dimension larger than some fixed number (which depens only on μ, not on j). We use this fact to show that (u_j) is bounded in $C(\overline{\Omega})$ and this proves, of course, Theorem 1.1.

So, in the remaining of the proof we assume that $||u_j||_{L^\infty(\Omega)} \to \infty$ and try to reach a contradiction, thus proving Theorem 1.1.

As in [4, 6, 13], we use assumption (H1) to perform a blow-up scaling. Since this procedure is explained in great detail in [13], here we only mention that (2.1) can be written as a system, through

$$\begin{aligned} -\Delta u &= v, \\ -\Delta v &= \mu u + a^+(x) g(u) - a^-(x) g_j(u), \end{aligned}$$

so that standard elliptic estimates can be applied. As a conclusion of these arguments, it follows easily that the solutions (u_j) converge (up to a suitable scaling) to a nonzero bounded function $u \in C^4(\omega) \cap C^3(\overline{\omega})$, defined in an unbounded domain of the form

$$\omega = \{x \in \mathbb{R}^N : \langle x, \overline{x} \rangle < d\}, \tag{2.2}$$

with $d \in]-\infty, +\infty]$, $\overline{x} \in \mathbb{R}^N$, $|\overline{x}| = 1$; the function u is a solution of the problem

$$\begin{aligned} \Delta^2 u &= (\beta^+(x) - L\beta^-(x))|u|^{p-2}u \quad \text{in } \omega, \\ u &= \Delta u = 0 \quad \text{on } \partial\omega, \end{aligned} \tag{2.3}$$

where $L \in [0, 1]$ and β is a nonzero affine function

$$\beta(x) = \langle \ell, x \rangle + c, \tag{2.4}$$

with $c \in \mathbb{R}$, $\ell \in \mathbb{R}^N$. In fact, either $\ell = 0$ or else $\ell = \nabla a(x_0)$ and $\overline{x} = \nu(x_0)$ for some $x_0 \in \partial\Omega$ such that $a(x_0) = 0$ so that, in any case (see (H3)),

$$\langle \ell, \overline{x} \rangle = 0. \tag{2.5}$$

We stress that all this follows exactly as in [6, 13]. As a final information on u, we mention that, as a consequence of the boundedness of the Morse indexes, u has *finite index* in the sense of [3]. This means that there exists some number $R_0 > 0$ such that, for any $\varphi \in H^2 \cap H_0^1(\omega \setminus B_{R_0}(0))$ with compact support, it holds

$$J''(u)\varphi, \varphi := \int_\omega (\Delta\varphi)^2 - (p-1) \int_\omega (\beta^+(x) - L\beta^-(x))|u|^{p-2}\varphi^2 \geq 0. \tag{2.6}$$

As a final step in our proof, we state below some Liouville type theorems implying that, under the present conditions, $u = 0$ (see Proposition 2.5), which is a contradiction and completes the proof of Theorem 1.1.

For second order problems, theorems of this kind were first proved in [3] (corresponding to the case where $\beta(x) = c$, see also [9] for a related situation) and, subsequently, in [6, 13] (for an affine or quadratic function β). Here we combine the arguments in [12, 13] to extend these theorems to the fourth order case. We mention that the case where β is constant probably follows also from the main result in [2], where the authors study systems of the form $-\Delta u = |v|^{q-2}v$, $-\Delta v = |u|^{p-2}u$ with $p, q > 2$; however, at least with our method, the case where β vanishes at some points of ω demands more involved arguments than the case where β is a (nonzero) constant.

In what follows, we denote by ω an open set of the form indicated in (2.2); in case d is finite, the function u in (2.3) is assumed to vanish, together with Δu, on the boundary $\partial\omega$. We also let $B_R := B_R(0) \subset \mathbb{R}^N$ and write φ_R for any function in $\mathcal{D}(\mathbb{R}^N)$ such that $\varphi_R = 1$ in B_R and $||\nabla\varphi_R||_{L^\infty} \leq CR^{-1}$, $||D^\alpha\varphi_R||_{L^\infty} \leq CR^{-2}$ for every $|\alpha| = 2$.

Our first lemma is a modification of a result in [12] for bounded domains.

Lemma 2.1. *Let $u \in C^4(\omega) \cap C^3(\overline{\omega})$ be such that, for some $p \neq 2N/(N-4)$, $p > 2$,*

$$\Delta^2 u = |u|^{p-2}u \ \text{ in } \omega, \quad u = \Delta u = 0 \ \text{ on } \partial\omega. \tag{2.7}$$

Suppose that

$$\int_\omega (\Delta u)^2 < \infty \quad \text{and} \quad \int_{\omega\cap B_R} u^2 \leq CR^4 \tag{2.8}$$

for some sequence $R \to \infty$ and some constant C (independent of R). Then $u = 0$.

Proof. 1) For simplicity, we drop the subscript in φ_R and simply write φ. All integrals are taken over ω or over subsets of ω. We remark that, up to a translation, we may assume that $d = +\infty$ or else $d = 0$ in (2.2). We also note that (2.8) implies (see [8, pp. 238-239])

$$\sum_{i,j=1}^N \int u_{ij}^2 < \infty \quad \text{and} \quad \int_{B_R} |\nabla u|^2 \leq CR^2. \tag{2.9}$$

Finally, we recall the following identity in [12]:

$$\begin{aligned} &\operatorname{div}(w\langle\nabla a, \nabla v\rangle\nabla u + w\langle\nabla a, \nabla u\rangle\nabla v - w\langle\nabla u, \nabla v\rangle\nabla a) \\ = \;& 2w\sum_{i,j} a_{ij}u_jv_i + w\langle\nabla a, \nabla v\rangle\Delta u + w\langle\nabla a, \nabla u\rangle\Delta v - w\langle\nabla u, \nabla v\rangle\Delta a \\ &+\langle\nabla a, \nabla v\rangle\langle\nabla u, \nabla w\rangle + \langle\nabla a, \nabla u\rangle\langle\nabla v, \nabla w\rangle - \langle\nabla u, \nabla v\rangle\langle\nabla a, \nabla w\rangle, \end{aligned}$$

which holds for arbitrary smooth functions u, v, a and w, provided one of them has compact support.

2) For any $R > 0$, denote by $\gamma(R)$ the number

$$\int [\langle \nabla\varphi, \nabla(\Delta u)\rangle\langle \nabla u, x\rangle + \langle \nabla(\Delta u), x\rangle\langle \nabla u, \nabla\varphi\rangle - \langle \nabla(\Delta u), \nabla u\rangle\langle \nabla\varphi, x\rangle]\,.$$

Then

$$\gamma(R) = \mathrm{o}(1) \quad \text{as } R \to \infty. \tag{2.10}$$

Indeed, this follows from the previous identity, replacing u, v, a and w by φ, $|x|^2/2$, u and Δu, respectively. Recalling that $u = \Delta u = 0$ on $\partial\omega$, the divergence theorem implies that $\gamma(R)$ is equal to

$$\begin{aligned}&\textstyle\int[-2\Delta u \sum_{i,j} u_{ij}\varphi_j x_i - \Delta u\Delta\varphi\langle \nabla u, \nabla x\rangle - N\Delta u\langle \nabla u, \nabla\varphi\rangle + (\Delta u)^2\langle \nabla\varphi, x\rangle]\\ &\quad\le C[\left(\int(\Delta u)^2\right)^{\frac12}\sum_{i,j}\left(\int u_{ij}^2\right)^{\frac12} + \left(\int(\Delta u)^2\right)^{\frac12}\left(R^{-2}\int|\nabla u|^2\right)^{\frac12} + \int(\Delta u)^2].\end{aligned}$$

To deduce (2.10), it is then sufficient to observe that each one of the above terms goes to zero as $R \to \infty$, since the integrals are taken over $B_{2R} \setminus B_R$ and taking (2.8), (2.9) into account.
3) Once (2.10) is established, the rest of the proof follows much as in [12]. For the reader's convenience, we shall go into some details. We use again the previous identity with function u in (2.7), and v, a and w replaced with Δu, $|x|^2/2$ and φ, respectively. We obtain

$$\begin{aligned}\int \varphi\Delta^2 u\langle \nabla u, x\rangle &= -2\int \varphi\langle \nabla u, \nabla(\Delta u)\rangle - \int \varphi\Delta u\langle x, \nabla(\Delta u)\rangle\\ &\quad + N\int \varphi\langle \nabla u, \nabla(\Delta u)\rangle - \gamma(R),\end{aligned}$$

hence, using (2.10),

$$\int \varphi\Delta^2 u\langle \nabla u, x\rangle = (N-2)\int \varphi\langle \nabla u, \nabla(\Delta u)\rangle - \int \varphi\Delta u\langle x, \nabla(\Delta u)\rangle + \mathrm{o}(1). \tag{2.11}$$

Observe that, in integrating by parts, the boundary integral does vanish. Indeed, we integrate over $\partial\omega \cap B_{2R}$ the expression

$$\varphi\langle \nabla(\Delta u), x\rangle\langle \nabla u, \overline{x}\rangle + \varphi\langle \nabla u, x\rangle\langle \nabla(\Delta u), \overline{x}\rangle - \varphi\langle \nabla u, \nabla(\Delta u)\rangle\langle x, \overline{x}\rangle,$$

and each one of these terms vanish since, on $\partial\omega$, $u = \Delta u = 0$ and $\langle x, \overline{x}\rangle = 0$. Now, as $\Delta u = 0$ on $\partial\omega$, an integration by parts shows that

$$-\int \varphi(\Delta u)^2 = \int \langle \nabla u, \nabla(\varphi\Delta u)\rangle = \int \varphi\langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{o}(1),$$

thanks to (2.8), (2.9). Similarly,

$$2\int \varphi\Delta u\langle x, \nabla(\Delta u)\rangle + \mathrm{o}(1) = \int \langle \nabla(\varphi(\Delta u)^2), x\rangle = -N\int \varphi(\Delta u)^2.$$

Plugging these two identities in (2.11) yields

$$\frac{N-4}{2}\int \varphi(\Delta u)^2 = -\int \varphi\Delta^2 u\langle\nabla u, x\rangle + \mathrm{o}(1). \tag{2.12}$$

4) Next we use the equation in (2.7). Multiply the equation by $u\varphi$ and integrate by parts to obtain

$$\int \varphi|u|^p = \int \varphi(\Delta u)^2 + \mathrm{o}(1). \tag{2.13}$$

Similarly,

$$\int |u|^p\langle x, \nabla\varphi\rangle = \int \Delta u\Delta(u\langle x, \nabla\varphi\rangle) = \mathrm{o}(1). \tag{2.14}$$

Finally, using (2.12), (2.13), (2.14) and the divergence theorem applied to

$$\mathrm{div}(\varphi\frac{|u|^p}{p}x) = \varphi|u|^{p-2}u\langle\nabla u, x\rangle + \frac{|u|^p}{p}\langle\nabla\varphi, x\rangle + \frac{N}{p}|u|^p\varphi,$$

we deduce that

$$(\frac{N}{p} - \frac{N-4}{2})\int \varphi(\Delta u)^2 = \mathrm{o}(1).$$

Since $\varphi = 1$ in B_R, this implies $\Delta u = 0$, hence $u = 0$. □

Suppose now that u satisfies (2.7) and that u has finite index. In particular, (cf. (2.6)), this implies

$$J''(u)u\varphi, u\varphi \geq 0 \tag{2.15}$$

for any function φ as above (with the extra restriction that $\varphi = 0$ in B_{R_0} and $\varphi = 1$ in $B_R \setminus B_{2R_0}$, say). Combining (2.7) and (2.15) and replacing φ by φ^2 one immediately gets

$$\int_\omega |u|^p\varphi^4 + \int_\omega (\Delta u)^2\varphi^4 \leq C(1 + R^{-4}\int_{B_{2R}\cap\omega} u^2 + R^{-2}\int_\omega |\nabla u|^2\varphi^2).$$

Using interpolation, this in turn implies

$$\int_\omega |u|^p\varphi^4 + \int_\omega (\Delta u)^2\varphi^4 \leq C(1 + R^{-4}\int_{B_{2R}\cap\omega} u^2). \tag{2.16}$$

This allows us to prove the following.

Proposition 2.2. *Let $u \in C^4(\omega) \cap C^3(\overline{\omega})$ be such that, for some $2 < p < 2N/(N-4)$,*

$$\Delta^2 u = |u|^{p-2}u \ \text{ in } \omega, \quad u = \Delta u = 0 \ \text{ on } \partial\omega.$$

If u is bounded and has finite index then $u = 0$.

Proof. Again we may assume that $d = +\infty$ or else $d = 0$ in (2.2). In case $\int_{B_R\cap\omega} u^2 \leq R^4$ for some sequence $R \to \infty$, (2.16) and Lemma 2.1 imply $u = 0$. Thus, to prove the proposition it is enough to show that the condition

$$\int_{B_R\cap\omega} u^2 > R^4, \quad \forall R \geq R_1, \tag{2.17}$$

leads to a contradiction. Now, (2.16) and (2.17) imply

$$\int_{B_R\cap\omega} |u|^p \le CR^{-4}\int_{B_{2R}\cap\omega} u^2. \tag{2.18}$$

On the other hand, since u is bounded, there exist $C>0$ and a sequence $R\to\infty$ such that

$$\int_{B_{2R}\cap\omega} u^2 \le C\int_{B_R\cap\omega} u^2. \tag{2.19}$$

Using (2.18), (2.19) and Hölder inequality, we conclude that, for some sequence $R\to\infty$,

$$\int_{B_R\cap\omega} |u|^p \le CR^{-4}\int_{B_R\cap\omega} u^2 \le C\left(\int_{B_R\cap\omega} |u|^p\right)^{2/p} R^{N(1-\frac{2}{p})-4}.$$

Since $N(1-\frac{2}{p})-4<0$, this implies $u=0$, contradicting (2.17) and proving the proposition. □

Remark. 1) An examination of the proof shows that the conclusion of Proposition 2.2 still holds without the assumption that u is bounded.

2) With a simpler proof we obtain a similar result for the equation $\Delta^2 u = -|u|^{p-2}u$.

We need the following extension of Proposition 2.2.

Proposition 2.3. *Let ω and β be given by* (2.2) *and* (2.4) *and assume (in case $d<\infty$) that $\ell\notin \mathrm{span}\{\overline{x}\}$. Given $2<p<2N/(N-4)$, suppose that $u\in C^4(\omega)\cap C^3(\overline{\omega})$ satisfies*

$$\Delta^2 u = \beta(x)|u|^{p-2}u \ \text{ in } \omega, \quad u=\Delta u = 0 \ \text{ on } \partial\omega.$$

If u is bounded and has finite index then $u=0$.

Proof. 1) By translation, we may assume that $c=0$ in (2.4). Let x_1 be the projection of $\overline{x}$ in the orthogonal space to ℓ; using a translation along x_1 we see that we can also assume $d=0$ (or else $d=+\infty$) in (2.2).
2) Suppose first that $\int_{B_R\cap\omega} u^2 \le R^4$ along a sequence $R\to\infty$. It follows precisely as in (2.16) that $\int_\omega (\Delta u)^2$ is finite. Since β is homogeneous, also the argument in Lemma 2.1 applies, yielding that $u=0$. Thus, to conclude the proof it is enough to show that, again, (2.17) leads to a contradiction. We will do that by exploiting the compact injection of H^2 into L^p in bounded sets (see [8]).
3) Assuming (2.17), fix a sequence $\alpha_j\to\infty$ such that

$$\int_{B_{8\alpha_j}\cap\omega} |u|^p \le C\int_{B_{\alpha_j}\cap\omega} |u|^p, \tag{2.20}$$

for some $C>0$ (compare with (2.19)). Define $u_j(x)=\beta_j u(\alpha_j x)$, where $\beta_j>0$ is such that

$$\int_{B_1\cap\omega}|u_j|^p=1, \tag{2.21}$$

that is, $\beta_j^p=\alpha_j^N(\int_{B_{\alpha_j}\cap\omega}|u|^p)^{-1}\le C\alpha_j^N$. Then u_j satisfies

$$\Delta^2 u_j=\mu_j\beta(x)|u_j|^{p-2}u_j \ \text{ in } \omega, \quad u_j=\Delta u_j=0 \ \text{ on } \partial\omega, \tag{2.22}$$

where $\mu_j=\alpha_j^5\beta_j^{2-p}\ge\alpha_j\alpha_j^{4-N(p-2)/p}\to\infty$. Since (see (2.16))

$$\int_{B_{4R}\cap\omega}(\Delta u)^2\le CR^{-4}\int_{B_{8R}\cap\omega}u^2, \tag{2.23}$$

this, together with (2.20) and (2.21), implies that the sequence $||u_j||_{L^2(B_4\cap\omega)}+||\Delta u_j||_{L^2(B_4\cap\omega)}$ is bounded. By interpolation, (u_j) is bounded in $H^2(B_2\cap\omega)$. Thus, up to a subsequence, $u_j\to v$ weakly in $H^2(B_2\cap\omega)$ and strongly in $L^p(B_2\cap\omega)$. In particular, it follows from (2.21) that $v\ne 0$ in $B_1\cap\omega$. Now we multiply the equation in the statement of the proposition by $\beta u_j\varphi$, where $\varphi\in\mathcal{D}(B_2)$ is such that $\varphi=1$ in B_1; integrating by parts yields

$$\mu_j\int_{B_1\cap\omega}\beta^2|u_j|^p\le\int_{B_2\cap\omega}|\Delta u|\,|\Delta(\beta u_j\varphi)|\le C.$$

Since $\mu_j\to\infty$, this implies

$$\int_{B_1\cap\omega}\beta^2|v|^p=0. \tag{2.24}$$

Thus $v=0$ in $B_1\cap\omega$, which is a contradiction and ends the proof of the proposition. □

An inspection of the above proof shows that the conclusion still holds when $\beta(x)$ is replaced with $\beta^+(x)-L\beta^-(x)$, provided L is positive. The case where $L=0$ requires some care, since the above argument allows to conclude (using the notation in (2.24)) that $v=0$ in $B_1\cap\omega\cap\{\beta>0\}$ only, and this does not contradict the fact that $v\ne 0$ in $B_1\cap\omega$. To overcome this difficulty, the following simple lemma will be useful.

Lemma 2.4. *Let $\Omega\subset\mathbb{R}^N$ be bounded and suppose that some sequence $(u_j)\subset C^{4,\alpha}(\Omega)$ $(0<\alpha<1)$ satisfies*

$$||u_j||_{H^2(\Omega)}\le C \quad \text{and} \quad ||\Delta^2 u_j||_{L^1(\Omega)}\to 0. \tag{2.25}$$

Then, up to a subsequence, $u_j\to u$ weakly in $H^2(\Omega)$, $u\in C^\infty(\Omega)$ and $\Delta^2u=0$.

Proof. 1) We may already assume that $u_j\to u$ weakly in $H^2(\Omega)$ and $u_j\to u$ a.e. in Ω. Fix any balls B_1, B_2 with $\overline{B_1}\subset B_2\subset\overline{B_2}\subset\Omega$. To prove the lemma it is enough to show that $u\in C^\infty(B_1)$ and $\Delta^2u=0$ in B_1. We denote

$f_j = \Delta^2 u_j \in C^{0,\alpha}(\Omega)$.
2) By minimization, there exists a unique $v_j \in H_0^2(B_2)$ such that

$$\int_{B_2} \Delta v_j \Delta\varphi = \int_{B_2} f_j\varphi, \quad \forall\varphi \in H_0^2(B_2). \tag{2.26}$$

Using [11, Th 1], we have that $v_j \in C^4(B_2)$ and $\Delta^2 v_j = f_j$. Moreover, if $\varphi \in \mathcal{D}(B_2)$, by (2.25) and (2.26), we see that

$$\int_{B_2} \Delta v_j \Delta\varphi = \int_{B_2} \Delta^2 u_j\varphi = \int_{B_2} \Delta u_j \Delta\varphi \leq C(\int_{B_2} (\Delta\varphi)^2)^{1/2}.$$

Since $\mathcal{D}(B_2)$ is a dense subset of $H_0^2(B_2)$, we conclude that (v_j) is bounded in $H_0^2(B_2)$. Hence, up to a subsequence, $v_j \to v$ weakly in $H_0^2(B_2)$. The assumption $f_j \to 0$ in $L^1(B_1)$ and (2.26) then imply

$$v \in H_0^2(B_2) \quad \text{and} \quad \int_{B_2} \Delta v \Delta\varphi = 0,\ \forall\varphi \in \mathcal{D}(B_2).$$

In particular, $v \in H_0^1(B_2)$ and $\Delta v = 0$, so that $v = 0$.
3) Denote $w_j := u_j - v_j \in C^4(B_2)$ and $g_j := \Delta w_j$. Then

$$\Delta g_j = 0 \quad in \quad B_2 \quad \text{and} \quad ||g_j||_{L^2(B_2)} \leq C. \tag{2.27}$$

Fix any $\varphi \in \mathcal{D}(B_2)$ such that $\varphi = 1$ in B_1 and multiply the equation in (2.27) by $g_j\varphi$. This yields $\int_{B_1} |\nabla g_j|^2 \leq C\int_{B_2} g_j^2 \leq C'$. Thus, up to a subsequence, $g_j \to g$ weakly in $H^1(B_1)$. Again from (2.27), it follows that

$$\int_{B_1} \langle\nabla g, \nabla\varphi\rangle = 0, \quad \forall\varphi \in \mathcal{D}(B_2),$$

and so $\Delta g = 0$. In particular, $g \in C^\infty(B_1)$. Now, $w_j \to u$ weakly in $H^2(B_1)$, and so $\Delta w_j \to \Delta u$ weakly in $L^2(B_1)$. As a consequence,

$$\Delta u = g \in C^\infty(B_1).$$

This implies $u \in C^\infty(B_1)$ and $\Delta^2 u = \Delta g = 0$. □

Now we can state our main result of this section. We recall that, in particular, this will complete the proof of Theorem 1.1.

Proposition 2.5. *Let ω and β be given by* (2.2) *and* (2.4) *and assume (in case $d < \infty$) that $\langle\ell, \overline{x}\rangle = 0$. Given $2 < p < 2N/(N-4)$, suppose that $u \in C^4(\omega) \cap C^3(\overline{\omega})$ satisfies*

$$\Delta^2 u = (\beta^+(x) - L\beta^-(x))|u|^{p-2}u \ \ in\ \omega, \quad u = \Delta u = 0 \ \ on\ \partial\omega.$$

If u is bounded and has finite index then $u = 0$.

Proof. 1) In case $\beta(x) = c \neq 0$, this was proved in Proposition 2.2 (see also the remark following it). So, we may assume $\ell \neq 0$. Moreover, by Proposition 2.3 and previous remarks, we may already assume that $L = 0$, $\beta(x) = \langle \ell, x\rangle$ and $d = +\infty$ or else $d = 0$ in (2.2).
2) We borrow the argument and the notation from the proof of Proposition 2.3. As before, the sequence (u_j) converges weakly in H^2 to some function v satisfying

$$\int_{B_1\cap\omega} |v|^p = 1 \quad \text{and} \quad v = 0 \ \text{ in } B_1^+ \cap \omega,$$

where $B_1^+ := B_1 \cap \{\beta > 0\}$. Observe that $B_1^+ \cap \omega \neq \emptyset$ (for instance, the vector $(\ell - \overline{x})/(2|\ell - \overline{x}|)$ is in $B_1^+ \cap \omega$). *We claim that*

$$f_j := \mu_j \beta^+ |u_j|^{p-2} u_j \to 0 \quad \text{in } L^1(B_1 \cap \omega). \tag{2.28}$$

Assume the claim for a moment. Observing that $u_j \in C^{4,\alpha}_{\rm loc}$ (for any $0 < \alpha < 1$), the previous lemma implies that $\Delta^2 v = 0$. Since $v = 0$ in $B_1^+ \cap \omega$ and $B_1 \cap \omega$ is connected, we deduce by unique continuation that $v = 0$ in $B_1 \cap \omega$, which contradicts the fact that $\int_{B_1\cap\omega} |v|^p = 1$ and proves the proposition.
3) In order to establish (2.28), we follow the argument in [13], which consists in showing that

$$\mu_j \int_{B_1\cap\omega} \beta^+ |u_j|^{p-2} \le C \quad \text{and} \quad \mu_j \int_{B_1\cap\omega} \beta^+ |u_j|^p \to 0. \tag{2.29}$$

Now, the first estimate follows immediately from the assumption that u has finite index. Indeed, (2.6) implies that

$$\int_\omega \beta^+ |u|^{p-2} \varphi^2 \le C(1 + \int_\omega |\nabla\varphi|^2) \le C R^{N-4}$$

for every large R, and this proves the first inequality in (2.29). In order to prove the second estimate, it is of course enough to show that

$$\mu_j \int_{B_1\cap\omega} (\beta^+)^3 |u_j|^p \to 0 \tag{2.30}$$

and

$$\mu_j \int_{B_1^+\cap\omega} |u_j|^p \le C. \tag{2.31}$$

4) Similarly to (2.16) (with φ replaced with $\beta^+\varphi$ in (2.6)), we have

$$\int_{B_R\cap\omega} (\beta^+)^3 |u|^p \le C(1 + R^{-2} \int_{B_{2R}^+\cap\omega} u^2 + \int_{B_{4R}^+\cap\omega} u\Delta u). \tag{2.32}$$

Here we have denoted $B_R^+ = B_R \cap \{\beta > 0\}$ and have used the fact that β is a linear function. We also recall that (2.23) holds and that we are assuming (by contradiction) the inequality in (2.17). Hence, (2.32) implies

$$\int_{B_R\cap\omega} (\beta^+)^3 |u|^p \le C R^{-2} \int_{B_{8R}^+\cap\omega} u^2.$$

In particular, this yields for the sequence (u_j):

$$\mu_j \int_{B_1\cap\omega} (\beta^+)^3 |u_j|^p \leq C \int_{B_8^+\cap\omega} u_j^2 \tag{2.33}$$

It is clear that we may assume that $v = 0$ in $B_8^+ \cap \omega$ and not only in $B_1^+ \cap \omega$ (see (2.20) and (2.21)). Thus (2.33) implies (2.30).
5) As a final step, we integrate $\mathrm{div}(\beta\varphi|u|^p\ell/p)$ in $\omega^+ := \omega \cap \{\beta > 0\}$ (where, as usual, $\varphi \in \mathcal{D}(B_{2R})$ and $\varphi = 1$ in B_R). This yields

$$|\ell|^2 \int_{\omega^+} \varphi \frac{|u|^p}{p} = -\int_\omega \beta^+ \frac{|u|^p}{p} \langle \nabla\varphi, \ell\rangle - \int_\omega (\Delta^2 u)\varphi\langle\nabla u, \ell\rangle.$$

The last integral can be estimated exactly as in Lemma 2.1 – just replace, in its proof, $|x|^2/2$ by the linear map $\langle \ell, x\rangle$ (the assumption $\langle \ell, \overline{x}\rangle = 0$ insures that the boundary terms in step 3 of the quoted proof do vanish). Denoting by ℓ_i the i-th component of ℓ, it then follows that

$$-\int_\omega \varphi\Delta^2 u\langle\nabla u, \ell\rangle = \int_\omega [\frac{(\Delta u)^2}{2}\langle\varphi, \ell\rangle - \Delta u\langle\nabla u, \ell\rangle\Delta\varphi - 2\Delta u(\sum_{i,k} u_{ik}\varphi_k\ell_i)].$$

We combine the last two identities and conclude, for the sequence (u_j), that

$$\mu_j \int_{B_1^+\cap\omega} |u_j|^p \leq C \int_{B_2\cap\omega} [\beta^+|u_j|^p + (\Delta u_j)^2 + |\Delta u_j|\,|\nabla u_j| + |\Delta u_j| \sum_{i,k} |\frac{\partial^2 u_j}{\partial x_i \partial x_k}|].$$

Since (u_j) is bounded in $H^2(B_2 \cap \omega)$, this implies (2.31) and ends the proof of the proposition. □

Acknowledgments. The present work is part of the Master thesis [14] and is the content of a talk given at the USA-Chile meeting. The first author acknowledges the organizers for the invitation and the warm hospitality. We thank Djairo De Figueiredo for pointing us reference [2]. The first author is supported by FCT, praxis xxi, FEDER and project praxis/2/2.1/mat/125/94.

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