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Heteroclinic connections for a class of non-autonomous systems *

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Abstract

We prove the existence of heteroclinic connections for a system of ordinary differential equations, with time-dependent coefficients, which is reminiscent of the ODE arising in connection with traveling waves for the Fisher equation. The approach is elementary and it allows in particular the study of the existence of positive solutions for the same system that vanish on the boundary of an interval $(t_0, +\infty)$.

1 Introduction

When one looks for one-dimensional traveling waves $u(x - ct)$ for the Fisher equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u)$$

(that models a diffusion phenomenon in biomathematics), one finds the ordinary differential equation

$$u'' + cu' + f(u) = 0\,. \tag{1}$$

Here $c > 0$ represents the admissible wave speed; the function f takes positive values between two zeros, say 0 and a ($a > 0$): see for example [7, 2].

In this paper we consider the following system, which is a non-autonomous multi-dimensional analogue of (1):

$$u_i'' + p_i(t)u_i' + f_i(u) = 0, \quad i = 1, \dots, n\,, \tag{2}$$

where $u = (u_1, \cdots u_n)$. The vector field $f = (f_1, \cdots, f_n)$ is assumed to be defined in some n-dimensional box $[0, a_1] \times \cdots \times [0, a_n]$ ($a_i > 0, \forall i = 1, \dots, n$), the vertices $(0, \cdots, 0)$ and $(a_1, \cdots, a_n)$ being its only zeros. More precisely, we state the following basic assumptions:

(H1) For each $i \in 1, \cdots, n$, $f_i : [0, a_1] \times \cdots \times [0, a_n] \to \mathbb{R}_+$ is a Lipschitz continuous function such that $f_i(0, \cdots, 0) = 0 = f_i(a_1, \cdots a_n)$ and $f_i(u) > 0$ if $u_i > 0$ and $u \neq a := (a_1, \cdots a_n)$.

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(H2) The functions $p_i : \mathbb{R} \to \mathbb{R}_+$ will be assumed throughout to be continuous and

$$c_i := \inf_{t \in \mathbb{R}} p_i(t) > 0.$$

We look for *positive solutions* $u(t) = (u_1(t); \cdots, u_n(t))$, i.e., solutions that have positive components. Accordingly, the word *solution* will be used to mean *positive solution* throughout.

The relevant problem is to find "monotonic" heteroclinics (in the sense that their components are decreasing functions) that connect the equilibria $(a_1, \cdots a_n)$ and 0. This problem has been very much studied for the autonomous scalar equation, various approaches being available in a vast literature: we refer the reader to [2, 3, 6] and the bibliography in those papers. The autonomous system has been dealt with in [1]; we owe a lot to the ideas there, and we would like to stress that our approach, which is also elementary, works in a slightly more general setting in the sense that it allows not only time dependence but also consideration of models where the vector field f may vanish to a higher order at $u = 0$. In addition we could equally consider a more general form of (2) where nonlinear terms $b_i(t)f_i(u)$ replace $f_i(u)$ and the functions b_i are bounded above and below by positive numbers (see [8]).

An important role is played by the functions $g_i(u)$, defined (for those u such that $0 < u_i \leq a_i$) by

$$f_i(u) = g_i(u)u_i.$$

Study of the autonomous case (1) has shown that a sufficient condition for the existence of the mentioned heteroclinic is a bound of the form $\sup_{0<u<a} \frac{f(u)}{u} := M \leq \frac{c^2}{4}$. In the case of (2) with $p_i(t) \equiv c_i$ it has been shown in [1] that $\sup g_i(u) \leq \frac{c_i^2}{4}$ remains a sufficient condition for the existence of the heteroclinic. We shall show that this still holds for our system (2) as long as the meaning of c_i is that given in assumption (H2); in addition we remark that we do not need an assumption used in [1] according to which (the continuous extension of) g_i takes its maximum value at the origin.

When (2) reduces to a single equation, boundary value problems for (2) in a finite interval appear also in connection with nonlinear elliptic problems in an annulus (see [5], [4]). By analogy we consider a second problem: that of finding (nontrivial) positive solutions of (2) defined in an unbounded interval of the form $[t_0, +\infty)$ and satisfying $u(t_0) = 0 = u(+\infty)$. We shall apply the arguments used in the construction of heteroclinics to obtain as a by-product the existence of a continuum of solutions to this problem when f, defined in $\mathbb{R}^n_+$, is (componentwise) non-negative and has its only zero at the origin.

In section 2 we collect some auxiliary results. In section 3 we deal with the existence of heteroclinics. In section 4 we briefly consider the existence of positive solutions vanishing in the boundary of an infinite interval.

2 Some auxiliary results for scalar equations

We start with three very simple observations about the scalar equation

$$u'' + p(t)u' + f(u) = 0, \tag{2_1}$$

where p is positive and f satisfies the one-dimensional analogue of (H1). Recall that *solution* means *positive solution*.

Remarks. 1) The solution for the initial value problem for (2_1) with $u(t_0) = u_0 \geq 0$ and $u'(t_0) = u_1$ exists in the maximal interval $[t_0, s)$ with $s < +\infty$ only if $\lim_{t\to s} u(t) = 0$.

2) A nonconstant solution of (2_1) has at most a critical point, which must be an absolute maximum.

3) Let $c > 0$, $\mu > 0$, $c^2 \geq 4M$ and $0 \leq \epsilon \leq \mu(\frac{c}{2} + \frac{\sqrt{c^2-4M}}{2})$. Then the solution $u(t)$ of the initial value problem

$$u'' + cu' + Mu = 0 \tag{3}$$

$$u(0) = \mu, \quad u'(0) = -\epsilon \tag{4}$$

is positive in $[0, +\infty)$ and tends to zero as $t \to +\infty$. (See [8].)

Lemma 2.1 *Let continuous functions p, q, l, m be given such that $p(t) \geq q(t) > 0$, $0 \leq l(t) \leq m(t)$ in the interval $[t_0, t_1]$. Let u and v be the respective solutions of*

$$u'' + p(t)u' + l(t)u = 0, \tag{5}$$

$$v'' + q(t)v' + m(t)v = 0 \tag{6}$$

such that $u(t_0) = v(t_0) \geq 0$ and $u'(t_0) = v'(t_0)$. Assume in addition that $p(t) \equiv q(t)$ in case $u'(t_0) = v'(t_0) > 0$. Then if $v(t) \geq 0$ in $[t_0, t_1]$ we have $u(t) \geq v(t)$ in $[t_0, t_1]$.

Proof. If $u(t_0) = v(t_0) = u'(t_0) = v'(t_0) = 0$ or $p \equiv q$ and $l \equiv m$ there is nothing to prove. Otherwise, starting with $u(t_0) = v(t_0) \geq 0$ and $u'(t_0) = v'(t_0) + \epsilon$ ($\epsilon > 0$) it follows that $u > v$ in some interval $(t_0, t_0 + \delta)$. Suppose that there exists $\bar{t} \leq t_1$ such that $u(t) > v(t)$ $\forall t \in (t_0, \bar{t})$ and $u(\bar{t}) = v(\bar{t})$. Set $P(t) := \int_{t_0}^{t} p(s)\, ds$. Multiplying (5) and (6) by $e^{P(t)}$, then (5) by v, (6) by u, integrating in $[t_0, \bar{t}]$ and subtracting we obtain, since $v'(t) < 0$ in case $v'(t_0) \leq 0$ (according to Remark 2)

$$\begin{aligned} 0 \quad > \quad & [e^{P(t)}(u'(t)v(t) - u(t)v'(t)]_{t_0}^{\bar{t}} + \int_{t_0}^{\bar{t}} e^{P(t)}(p(t) - q(t))v'(t)u(t)\, dt \\ & + \int_{t_0}^{\bar{t}} e^{P(t)}(l(t) - m(t))u(t)v(t)\, dt = 0, \end{aligned}$$

a contradiction. Hence $u \geq v$ in $[t_0, t_1]$. In the limit as $\epsilon \to 0^+$ the statement follows.

Let us write, in accordance with previous notation for the n-dimensional case,

$$f(u) = ug(u), \quad 0 \leq u \leq a.$$

Lemma 2.2 *Assume p is continuous in $\mathbb{R}$, f is a Lipschitz continuous function in $[0,\mu]$ such that $f(0) = 0$ and $f(u) > 0$ if $0 < u < \mu$; and $p(t) \geq c \geq 2\sqrt{M}$ where $M := \sup_{0<u<\mu} g(u)$. Given $t_0 \in \mathbb{R}$ assume in addition that $p(t)$ is bounded for $t > t_0$. Then the solution of the Cauchy problem*

$$u'' + p(t)u' + f(u) = 0, \quad u(t_0) = \mu, \;\; u'(t_0) = -\epsilon$$

where we assume that μ and $\epsilon > 0$ are as in Remark 3, is positive and strictly decreasing in $[t_0, \infty)$ and vanishes at $+\infty$.

Proof. Apply Lemma 2.1 with $q(t) = c$, $l(t) = g(u(t))$, $m(t) = M$. Take Remarks 2 and 3 into account. The fact that $u(+\infty) = 0$ is an easy consequence of the boundedness of p, u and u' since for $t > t_0$ and some $t^* \in (t_0, t)$

$$u'(t) + \epsilon + p(t^*)(u(t) - u_0) + \int_{t_0}^{t} f(u(s))\, ds = 0, \quad t > t_0$$

and we infer that $\int_{t_0}^{+\infty} f(u(s))\, ds$ converges.

Remark. It is immediately recognized that the above result still holds if $\epsilon = 0$ provided $f(u) > 0 \; \forall u \in (0,\mu]$.

3 Heteroclinics

In this section we give a simple analytic argument to prove the existence of heteroclinics under hypotheses (H1)-(H2). For the sake of clarity we start with the case of the scalar equation

$$u'' + p(t)u' + f(u) = 0\,, \tag{2_1}$$

where $f : [0,a] \to \mathbb{R}_+$ has the property (H1) for $n = 1$ and we write accordingly

$$c := \inf_{t\in\mathbb{R}} p(t) > 0; \quad f(u) = ug(u).$$

A basic assumption, which cannot be improved when $p \equiv c$ is a constant and $g(u)$ is decreasing, is $c \geq 2\sqrt{\sup_{0<u<a} g(u)}$. When one deals with other models, namely when $g(0) = 0$, that lower bound can be improved. Condition (ii) in the following proposition is motivated by this setting.

Proposition 3.1 *Assume p is continuous, f is Lipschitz continuous in some interval $[0,\mu]$, $f(0)=0$, $f(u)>0$ $0<u<\mu$, and that either*

(i) $c \geq 2\sqrt{\sup_{0<u<\mu} g(u)}$ *or*

(ii) there exists $\nu \in (0,\mu)$ such that $N := \sup_{0<u<\mu} f(u)$ and $M := \sup_{0<u<\nu} g(u)$ satisfy $c^2 \geq 4M$ and

$$\frac{N}{c} \leq \nu(\frac{c}{2} + \frac{\sqrt{c^2-4M}}{2}) \tag{7}$$

Suppose in addition that $p(t)$ is bounded for $t > t_0$. Then for each sufficiently small $\epsilon > 0$ the solution $u(t,t_0,\epsilon)$ of (2) such that $u(t_0,t_0,\epsilon)=\mu$ and $u'(t_0,t_0,\epsilon)=-\epsilon$ is positive in $[t_0,+\infty)$ and

$$\lim_{t\to+\infty} u(t,t_0,\epsilon) = 0.$$

Proof. In case (i) holds, this is only Lemma 2.2. Otherwise note that the solution $u(t,t_0,\epsilon)$ has no critical points and therefore is strictly decreasing. It cannot remain above a positive constant by the argument used at the end of the proof of Lemma 2.2. Let t_1 be such that $u(t_1,t_0,\epsilon)=\nu$. The equation itself shows that $u'(t_1,t_0,\epsilon) \geq -N/c$ (consider separately the cases where t_1 lies in an interval of convexity or of concavity of the solution) and therefore Lemma 2.2 can be applied.

Remark. According to the remark after Lemma 2.1 it is obvious, via the same arguments, that the Proposition holds even if $\epsilon = 0$ except in case $f(\mu)=0$.

Theorem 3.2 *Assume (H1)-(H2) with $n=1$ and, in addition to the hypotheses of proposition 3.1 with $\mu=a$, that $p(t)$ is bounded. Then (2_1) has a strictly decreasing heteroclinic solution connecting a and 0.*

Proof. With respect to $\mu = a$ in Proposition 3.1 take a sequence t_m decreasing to $-\infty$ and consider the solution $u(.,t_1,\epsilon_1)$ where ϵ_1 is a small positive number. According to proposition 3.1, $0<u(t,t_1,\epsilon_1)\leq a$ for $t \geq t_1$ and there exists $\bar{t}$ such that $u(\bar{t},t_1,\epsilon_1)=a/2$.
Claim: There exists $m_2 > 1$ such that

$$u(\bar{t},t_{m_2},\epsilon_1) < a/2.$$

Proof of the Claim: Otherwise we would have $u(\bar{t},t_m,\epsilon_1) \geq a/2$ for all $m > 1$ and, by a t_m translation, this can be written

$$u_m(\bar{t}-t_m) \geq a/2 \tag{8}$$

in terms of the solution of

$$u_m'' + p_m(t)u_m' + f(u_m) = 0, \quad u_m(0) = a, \ \ u_m'(0) = -\epsilon_1 \,,$$

where $p_m(t) = p(t + t_m)$. The boundedness of p_m, u_m and u_m' and Ascoli's theorem enable us, by extracting subsequences and a diagonal procedure, to suppose that (where we set $d := \sup_{t\in\mathbb{R}} p(t)$)

$$p_m \to p_\infty \ \text{ in } \ L^\infty\text{weak-*}, \ \ c \le p_\infty(t) \le d$$

$$u_m \to u \ \text{ in } \ C^1(K), \ \ K \ \text{ any compact interval in } [0, +\infty).$$

Since

$$u'' + p_\infty(t)u' + f(u) = 0, \quad u(0) = 1, \ \ u'(0) = -\epsilon_1$$

(and it is easy to see that proposition 3.1 still applies to solutions in the Carathéodory sense) there exists $\tilde{t}$ such that $u(\tilde{t}) = a/4$. Since $u_m \to u$ uniformly in $[0, \tilde{t}]$ and $\bar{t} - t_m \to +\infty$ this contradicts (8) and so the Claim holds.

To go on with the proof we observe that if $\delta > 0$ is sufficiently small we have $u(\bar{t}, t_{m_2}, \delta) > a/2$, since $u(., t_{m_2}, \delta) \to a$ as $\delta \to 0^+$ in $[t_{m_2}, \bar{t}]$. By the intermediate value theorem we can pick up $0 < \epsilon_2 < \epsilon_1$ such that $u(\bar{t}, t_{m_2}, \epsilon_2) = a/2$. This argument can be iterated so as to construct decreasing sequences $\tau_k = t_{m_k}$ and ϵ_k with the property that $u(\bar{t}, \tau_k, \epsilon_k) = a/2$.

Using again the boundedness of $u(., \tau_k, \epsilon_k)$ and $u'(., \tau_k, \epsilon_k)$ and the diagonal procedure we can pass to a subsequence (which for convenience is denoted by the same symbol) so that for any compact interval $K \subset \mathbb{R}$,

$$u(., \tau_k, \epsilon_k) \to u \ \text{ in } \ C^1(K).$$

The limit function u thus obtained is, of course, a decreasing solution to (1), such that $u(\bar{t}) = a/2$ and $0 < u(t) < a \ \forall t \in \mathbb{R}$ (by the uniqueness theorem for the initial-value problem u cannot take the values 0 or a). Finally we can repeat the argument used in the proof of Lemma 2.2 to conclude that $\lim_{t\to-\infty} u(t) = a$, $\lim_{t\to+\infty} u(t) = 0$ and $\lim_{t\to\pm\infty} u'(t) = 0$.

Remarks. 1) It can be shown, using an argument similar to the proof of the Claim, that $\epsilon_k \to 0$.
2) See [8] to see how in some instances (with $g(0) = 0$) assumption (ii) is an improvement over (i).

We now turn to the study of system (2).

Theorem 3.3 *Assume (H1)-(H2), the functions p_i are bounded and*

$$c_i \ge 2\sqrt{M}_i, \quad M_i := \sup_{(0,a_1)\times\cdots\times(0,a_n)} g_i(u); \quad i = 1, \ldots, n.$$

Then (2) has a heteroclinic solution, whose components are strictly decreasing, connecting $a = (a_1, \cdots a_n)$ and 0.

Proof. As in the previous proof, we start by considering solutions $u(.,t_0,\epsilon)$ to (2) such that $u(t_0,t_0,\epsilon) = a$ and $u_i'(t_0,t_0,\epsilon) = -\epsilon$, $i = 1,\cdots n$. Using Proposition 3.1, one easily sees that for $\epsilon > 0$ sufficiently small such solutions are defined in $[t_0,\infty)$, their components being strictly decreasing and vanishing at $+\infty$. Now take a sequence $t_m \to -\infty$ and $u(.,t_1,\epsilon_1)$ where ϵ_1 is small. Selecting the first component, u_1, we easily establish, as in the proof of theorem 3.2, that there exists $\bar{t}$ and subsequences $\tau_k = t_{m_k} \to -\infty$, $\epsilon_k \to 0^+)$ so that

$$u_1(\bar{t},\tau_k,\epsilon_k) = a_1/2 \tag{9}$$

(it is sufficient to argue as in the proof of theorem 3.2 with respect to the equation for the first component).

Next consider the sequence $u_2(.,\tau_k,\epsilon_k)$ and let s_k be numbers such that

$$u_2(s_k,\tau_k,\epsilon_k) = a_2/2.$$

We claim that the sequence $s_k - \bar{t}$ is bounded: for suppose for instance that along a subsequence $s_k - \bar{t} \to +\infty$ (the case $s_k - \bar{t} \to -\infty$ is analogous); integrating the second equation of the system (2) in $[\bar{t}, s_k]$ we obtain

$$\begin{aligned} u_2'(s_k,\tau_k,\epsilon_k) - u_2'(\bar{t},\tau_k,\epsilon_k) + p_2(t_k^*)(u_2(s_k,\tau_k,\epsilon_k) - u_2(\bar{t},\tau_k,\epsilon_k)) \\ + \int_{\bar{t}}^{s_k} u_2(t,\tau_k,\epsilon_k) g_2(u(t,\tau_k,\epsilon_k))\,dt \quad = \quad 0\,, \end{aligned}$$

where $t_k^* \in [\bar{t}, s_k]$. Now the first factor in the integrand is greater than $a_2/2$; using (H1) we see that the second is bounded away from zero (because u_1 takes values $< a_1/2$ while u_2 takes values $> a_2/2$); therefore we have reached a contradiction.

Since this argument can be repeated with respect to the remaining components, along with (9) we construct sequences $s_k^{(j)}$, $j = 2,\cdots,n$ such that

$$u_j(s_k^{(j)},\tau_k,\epsilon_k) = a_j/2 \tag{10}$$

and $s_k^{(j)} - \bar{t}$ is bounded. Now, as in theorem 3.2 we go to the limit through a diagonal subsequence: $u(.,\tau_k,\epsilon_k) \to v$ uniformly in compact intervals, and v is a solution of (2) with decreasing components. Moreover we may assume that $s_k^{(j)} \to t_j$, $j = 2,\cdots,n$ and therefore on account of (9)-(10) we obtain

$$v_1(\bar{t}) = a_1/2, \quad v_j(t_j) = a_j/2, \;\; j = 2,\cdots n. \tag{11}$$

We assert that $0 < v_i(t) < a_i \quad \forall t \in \mathbb{R}$, and in particular $v_i'(t) < 0 \quad \forall t \in \mathbb{R}$, $i = 1,\ldots,n$. Indeed suppose for instance that $v_1(t) = a_1$ for $t \le t^*$. Then the first equation of (2) implies that $g_1(a_1, v_2(t),\cdots,v_n(t)) = 0$ if $t \le t^*$, so that by (H1) we have $v_j(t) = a_j$ for $t \le t^*$ and $j = 2,\cdots,n$. By uniqueness of solutions of the Cauchy problem, it follows that $v \equiv a$, contradicting (11). An easier argument shows that v_j cannot take the value 0. Then, arguing as in the proof of lemma 2.2 one concludes that $v(-\infty) = a$ and $v(+\infty) = 0$. Finally we illustrate the proof of the fact that $v'(\pm\infty) = 0$ by showing that $v_1'(-\infty) = 0$. If this were not the case we could select sequences $t_k < s_k \to -\infty$ and a number $\delta > 0$ with $v_1'(t_k) \to 0$ and $|v_1'(s_k)| \geq \delta$. Multiplying the first equation in (2) by v_1' and integrating yields

$$\frac{1}{2}[v_1'^2(s_k) - v_1'^2(t_k)] + \int_{t_k}^{s_k} p_1 v_1'^2 + \int_{t_k}^{s_k} f_1(v) v_1' = 0,$$

where, by the mean value theorem and what has been already proved, the last summand tends to 0 as $k \to \infty$; this is a contradiction and the proof is complete.

Remark. As in theorem 3.2, we could use a set of conditions like (7) to improve the lower bounds on $c_i, \quad i = 1, \ldots, n$ in case the functions g_i approach 0 as $u_i \to 0$.

4 Positive solutions vanishing at the endpoints of an unbounded interval

In this section we consider f satisfying

(H3) For each $i \in 1, \cdots n$, $f_i : {R_+}^n \to \mathbb{R}_+$ is a locally Lipschitz continuous function such that $f_i(0) = 0$ and $f_i(u) > 0$ if $u_i > 0$.

We consider the problem of finding *nontrivial* positive solutions to

$$u_i'' + p_i(t)u_i' + f_i(u) = 0, \quad i = 1, \ldots, n \tag{12}$$

$$u(0) = 0 = u(+\infty) \tag{13}$$

where by *nontrivial* we mean that each component u_i of such solution is positive in $(0, +\infty)$. For definiteness the initial endpoint is taken to be $t_0 = 0$, but our results can obviously be restated with an arbitrary left endpoint.

Before stating the result we note the following fact: denote by $u(\cdot, A)$ the solution of the Cauchy problem

$$u_i'' + p_i(t)u_i' + f_i(u) = 0, \quad u_i(0) = 0, \;\; u_i'(0) = A;$$

then $u_i(\cdot, A)$ has, for every $A > 0$, a maximum $\mu_i(A)$ depending continuously on A and $\mu_i(0^+) = 0$, $\mu_i(+\infty) = +\infty$. To see this, take $t^* > 0$ in a neighborhood of 0, $A_i^* = u_i'(t^*, A) > 0$, $u_i^* = u_i(t^*, A) > 0$. Let K_i be the least upper bound of the (scalar) solution of

$$z'' + p_i(t)z' = 0, \quad z(t^*) = u_i^*, \;\; z'(t^*) = A_i^*,$$

and define $\delta_i := \inf\{g_i(x) : \;\; u_i^* \leq x_i \leq K_i;\; x_j \leq K_j \;\text{ if } j \neq i\} > 0$. Comparing $u_i(\cdot, A)$ with the solution to

$$v'' + p_i(t)v' + \delta_i v = 0, \quad v(t^*) = u_i^*, \;\; v'(t^*) = A_i^*,$$

it is easy to see, using Lemma 2.1, that (since $v \leq z$) $u_i(t, A) \leq v(t)$ as long as $u_i(t, A) > u_i^*$. But the behavior of $v(t)$ implies that $v(t)$ returns to the value u_i^* and therefore $u_i(t, A)$ attains a maximum. The assertion about $\mu_i(0^+)$ comes from the fact that $K_i \to 0$ as $A \to 0^+$. The other assertion is straightforward.

Proposition 4.1 *Assume (H2)-(H3). Assume p is bounded in $[0, \infty)$ and let $c_i := \inf_{t \geq 0} p_i(t)$. Given positive numbers α_i, $i = 1, \cdots, n$ such that $M_i := \sup\{g_i(u) : \ 0 < u_i < \alpha_i\} \leq c_i^2/4$, then there exists a number $A_0 > 0$ such that whenever $0 < A \leq A_0$ $u(\cdot, A)$ is a nontrivial solution of (12)-(13).*

Proof. It suffices to define $A_0 := \sup\{A > 0 : \ (\mu_1(A), \cdots, \mu_n(A)) \in (0, \alpha_1] \times \cdots \times (0, \alpha_n]\}$. Then if $0 < A \leq A_0$ and $u_i(t_i^*, A) = \mu_i$ it is easy to conclude, using lemma 2.2 and the remark after it, that each component $u_i(\cdot, A)$ remains positive in $[t_i^*, +\infty)$ and vanishes at $+\infty$.

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