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Generalized boundary value problems for nonlinear elliptic equations *

Laurent Véron

Abstract

We give here an overview of some recent developments in the study of the description of all the positive solutions of

$$-\Delta u + |u|^{q-1}u = 0 \tag{1}$$

in a domain Ω where $q > 1$.

1 Introduction

Given a partial differential equation in a domain Ω of $\mathbb{R}^N$, a natural question is to find a way to describe all its solutions by means of their possible boundary value. For example, if Ω is smooth, any nonnegative harmonic function u in Ω admits a boundary trace which is a Radon measure μ on $\partial\Omega$ and the Riesz-Herglotz representation formula holds,

$$u(x) = \int_{\partial\Omega} P(x,y)d\mu(y) \tag{2}$$

for any $x \in \Omega$, where $P(x,y)$ is the Poisson kernel in Ω. If Ω is not smooth, the Poisson kernel is replaced by the Martin kernel and a representation formula holds.

This article is concerned with is the description of the positive solutions of (1) and this study is known as the nonlinear trace theory. A description as in the linear case is still far out of reach, but thank to the works of Le Gall [24, 25, 26], Marcus and Véron [28, 29, 30, 31, 32], Dynkin and Kuzneztsov [11, 12, 13, 14, 15] this program is now well advanced. In this survey I want to present the historical progression which led to the nonlinear trace problem, and in particular present the preliminary works of Gmira and Véron (1989-1991) [17] dealing with the measure boundary data and the isolated boundary singularities, the question of existence and uniqueness of the large solutions in particular in non-smooth domains and the role of the Borel measures and the

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boundary capacities framework for representing all the positive solutions. We shall also give applications to conformal deformations of hyperbolic space.

The associated boundary value problem (or generalised Dirichlet problem) is the following,

$$\begin{array}{c} -\Delta u + |u|^{q-1}u = 0, \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega, \end{array} \tag{3}$$

where g may be a function (regular or not, a Radon measure, a generalized Borel measure).

2 The regular non-linear Dirichlet problem in regular domains

By using convex analysis, L^p and Schauder's regularity theory of elliptic equations, it is classical that for any g in $C^{2,\alpha}(\partial\Omega)$ there exists a unique u in $C^{2,\alpha}(\bar{\Omega})$ solution of (3). The extension to merely integrable boundary data is due to Brezis [4]

Theorem 1 *Assume $g \in L^1(\partial\Omega)$, then there exists a unique function $u \in L^1(\Omega) \cap L^q(\Omega, \rho_\Omega dx)$, where $\rho_\Omega(x) = \operatorname{dist}(x, \partial\Omega$ such that*

$$\int_\Omega \left(-u\Delta\zeta + |u|^{q-1}u\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g dH_{N-1} \tag{4}$$

$\forall\zeta \in C_0^{1,1}(\bar{\Omega})$. Moreover the mapping $g \mapsto u = P_\Omega^q(g)$ is increasing and

$$\|u_1 - u_2\|_{L^1(\Omega)} + \|\rho_\Omega\left(h(u_1) - h(u_2)\right)\|_{L^1(\Omega)} \leq C\|g_1 - g_2\|_{L^1(\partial\Omega)} \tag{5}$$

where $u_j = P_\Omega^q(g_j)$, $j = 1, 2$ and $h(r) = |r|^{q-1}r$.

In this statement, dH_{N-1} denotes the $(N-1)$-dimensional Hausdorff measure. Notice also that $\zeta \in C_0^{1,1}(\bar{\Omega}) \Rightarrow |\zeta(x)| \leq C\rho_\Omega(x)$, which gives its meaning to the condition $u \in L^q(\Omega, \rho_\Omega dx)$. The key point in Brezis proof is the following result

Lemma 1 *For any $(f, g) \in L^1(\Omega, \rho_\Omega dx) \times L^1(\partial\Omega)$ there exists a unique $v \in L^1(\Omega)$ such that*

$$-\int_\Omega v\Delta\zeta dx = \int_\Omega f\zeta dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g dH_{N-1} \tag{6}$$

$\forall\zeta \in C_0^{1,1}(\bar{\Omega})$. Moreover, if $\zeta \geq 0$,

$$-\int_\Omega |v|\Delta\zeta dx + \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu}|g| dH_{N-1} \leq \int_\Omega f\zeta sign(v) dx \tag{7}$$

and the same estimate holds if one replaces $|v|$ by v_+.

3 The a priori estimate of Keller and Osserman

One of the most striking aspects of the equation (1) is the fact that all the solutions are locally uniformly bounded. More precisely, by using suitable local radial supersolutions of (1), Keller [20] and Osserman [34] proved independently that there always holds

$$|u(x)| \leq C(N,q)\rho_\Omega(x)^{2/(1-q)}. \tag{8}$$

One of the important consequence of this inequality consists in the construction of positive solution of (3) with a boundary value g belonging to $C(\partial\Omega;[0,\infty])$.

Theorem 2 *Let Ω be a Lipschitz bounded domain. Then for each g in $C(\partial\Omega;[0,\infty])$, there exists at least one solution of (3).*

Proof. For any positive integer k let u_k be the solution of

$$\begin{gathered} -\Delta u_k + |u|_k^{q-1} u_k = 0, \text{ in } \Omega, \\ u_k = g_k = \min(k,g) \text{ on } \partial\Omega, \end{gathered}$$

By the maximum principle, the sequence $\{u_k\}$ is positive and increasing. Moreover it is locally bounded thanks to (8). Therefore it converges to some solution u of (1). If $x_0 \in \partial\Omega$ is such that $g(x_0) < \infty$, then the continuity of g and the elliptic equations boundary regularity theory imply that the set of functions $\{u_k\}$ remains equicontinuous in some neighborhood of x_0. Consequently the limit function u achieves the value of g in a relative neighborhood of x_0 on $\partial\Omega$. If $g(x_0) = \infty$, then $\{u_k(x_0)\}$ is not bounded from above.

The Lipschitz regularity assumption can be relaxed and replaced by the Wiener regularity criterion. The problem of uniqueness remains open up to 1993 when Kondratiev and Nikishkin [21] discovered that there may exist infinitely many solutions in the framework of boundary data in the class of continuous functions with possibly infinite value. This will be completely understood in the framework of the boundary trace theory.

4 Measure boundary data

In this section Ω is a smooth bounded domain. The Poisson formula which expresses the Poisson potential of a boundary Radon measure is the following

$$P_\mu(x) = \int_\Omega P(x,y)d\mu(y) \tag{9}$$

The first extension of Theorem 1 to measure boundary data, is due to Gmira and Véron [17] for the existence part (1989-1991) and Marcus and Véron [31] for the stability part (1996).

Theorem 3 *Suppose that $1 < q < (N+1)/(N-1)$, then for any $\mu \in \mathcal{M}(\partial\Omega)$ there exists a unique $u \in L^1(\Omega) \cap L^q(\Omega, \rho_\Omega dx)$ such that*

$$\int_\Omega \left(-u\Delta\zeta + |u|^{q-1}u\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu \tag{10}$$

$\forall\zeta \in C_0^{1,1}(\bar{\Omega})$. Moreover the mapping $g \mapsto u = P_\Omega^q(\mu)$ is increasing. If μ_n converges weakly to μ when n goes to infinity, $u_n = P_\Omega^q(\mu_n)$ converges to $u = P_\Omega^q(\mu)$, locally uniformly in Ω.

Proof. Uniqueness follows from estimate (5). The proof of existence relies on the following well-known estimates on the Poisson kernel: there exists $C = C(\Omega) > 0$ such that

$$C^{-1}\rho(x)|x-y|^{-N} \leq P(x,y) \leq C\rho(x)|x-y|^{-N}$$

for all $(x,y) \in \Omega \times \partial\Omega$. Consequently

$$\|P(.,y)\|_{L^p(\Omega)} \leq K_{p,\Omega} \quad \forall 1 < p < N/(N-1) \quad \forall y \in \partial\Omega,$$
$$\|P(.,y)\|_{L^p(\Omega;\rho dx)} \leq K^*_{p,\Omega} \quad \forall 1 < p < (N+1)/(N-1) \quad \forall y \in \partial\Omega.$$

Let $\{g_n\} \subset L^1(\partial\Omega)$ such that $g_n \to \mu$ in the sense of measures, and set $u_n = P_\Omega^q(g_n)$. From the maximum principle,

$$-P_{g_n^-} \leq u_n \leq P_{g_n^+}.$$

Since

$$P_{g_n^\pm} = \int_{\partial\Omega} P(x,y) g_n^\pm(y) dH_{N-1},$$

we take $f \in L^{p'}(\Omega)$ with $p' = p/(p-1)$ and $1 \leq p < N/(N-1)$, and we have

$$\begin{aligned}
\int_\Omega P_{g_n^\pm}(x) f(x) dx &= \int_\Omega \int_{\partial\Omega} P(x,y) g_n^\pm(y) dH_{N-1}(y) f(x) dx \\
&= \int_{\partial\Omega} \left(\int_\Omega P(x,y) f(x) dx\right) g_n^\pm(y) dH_{N-1}(y) \\
&\leq K_{p,\Omega} \|f\|_{L^p(\Omega)} \int_{\partial\Omega} g_n^\pm(y) dH_{N-1}(y)
\end{aligned}$$

This implies

$$\|P_{g_n^\pm}\|_{L^p(\Omega)} \leq K_{p,\Omega} \|g_n^\pm\|_{L^1(\partial\Omega)} \leq K. \tag{11}$$

Similarly

$$\|P_{g_n^\pm}\|_{L^p(\Omega;\rho dx)} \leq K^*_{p,\Omega} \|g_n^\pm\|_{L^1(\partial\Omega)} \leq K^*, \tag{12}$$

for $1 \leq p < (N+1)/(N-1)$. If we take $q < p$ in (11), and $1 < p$ in (12), we deduce that $\{u_n\}$ and $\{|u_n|^{q-1}u_n\}$ are uniformly integrable and therefore weakly compact in $L^1(\Omega)$ and $L^1(\Omega;\rho dx)$ respectively. From the Osserman-Keller estimate $\{u_n\}$ remains also locally uniformly bounded in Ω and by Ascoli's theorem there exist a sequence $\{u_{n_k}\}$ and a $C^2(\Omega)$-function u such that $u_{n_k}\to u$ locally uniformly and weakly in $L^1(\Omega)$. Moreover $|u_{n_k}|^{q-1}u_{n_k}\to|u|^{q-1}u$ weakly in $L^1(\Omega;\rho dx)$. Letting $n\to\infty$ in the next exprssion

$$\int_\Omega\left(-u_{n_k}\Delta\zeta+|u|_{n_k}{}^{q-1}u_{n_k}\zeta\right)dx=-\int_{\partial\Omega}\frac{\partial\zeta}{\partial\nu}g_{n_k}dH_{N-1},$$

yields to (10).

The stability result is proved by the same device. If $\mu_n\to\mu$ weakly in $\mathcal{M}(\partial\Omega)$, it remains bounded in the total variation norm and therefore $\{P_{|\mu|_n}\}$ remains bounded in $L^{p_1}(\Omega)\cap L^{p_2}(\Omega;\rho dx)$ for any $1\le p_1<N/(N-1)$ and $1\le p_2<(N+1)/(N-1)$. Since $|P^q_\Omega(\mu_n)|\le P_{|\mu_n|}$, the needed compactness in order to let $\{n_k\}$ in

$$\int_\Omega\left(-u_{n_k}\Delta\zeta+|u_{n_k}|^{q-1}u_{n_k}\zeta\right)dx=-\int_{\partial\Omega}\frac{\partial\zeta}{\partial\nu}d\mu_{n_k}.$$

Therefore the full convergence result follows from uniqueness.

Remark. The problem (3) may not have a solution for any measure when $q\ge(N+1)/(N-1)$. For example there exists no solution if $\mu=\delta_a$ for some $a\in\partial\Omega$. The full treatment of the solvability of (3) is now well understood thanks to the works of Dynkin and Kuznetsov ($(N+1)/(N-1)\le q<2$), Le Gall ($q=2$) and Marcus and Véron ($q>2$). This treatment involves the notion of removable sets which are associated to Bessel capacities. It will be presented in Section 9.

Remark. A more elaborated analytic tool (weighted Marcinkiewicz spaces) allowed Gmira and Véron [17] to prove an existence and uniqueness result for the general problem

$$\begin{aligned}-\Delta u+g(u)=0,\ \text{in}\ \Omega,\\ u=\mu\ \text{on}\ \partial\Omega,\end{aligned}\tag{13}$$

where g is continuous and non-decreasing, $\mu\in\mathcal{M}(\partial\Omega)$, and

$$\int_\Omega\left|g(P_{|\mu|})\right|\rho(x)dx<\infty.$$

By a solution we mean an integrable function u defined in Ω such that $g(u)\in L^1(\Omega;\rho dx)$ which satisfies, for every $\zeta\in C_0^{1,1}(\bar\Omega)$,

$$\int_\Omega\left(-u\Delta\zeta+g(u)\zeta\right)dx=-\int_{\partial\Omega}\frac{\partial\zeta}{\partial\nu}\zeta d\mu.\tag{14}$$

The above integrability condition appears to be more general, however in the particular case where $g(r)=r|r|^{q-1}$, it may not be satisfied even with $\mu=$

$hdx \in L^1(\partial\Omega)$ when $q \geq (N+1)/(N-1)$. It is then natural to isolate the singular part of the measure by writing the Lebesgue decomposition

$$\mu = \mu_S + \mu_R$$

where μ_R is the regular part (with respect to the (N-1)-Hausdorff measure), and μ_S, the singular one. If g satisfies a Δ_2-condition,

$$|g|(r + r') \leq \theta(|g|(r) + |g|(r'))$$

for some $\theta > 0$, whenever $rr' \geq 0$, Marcus and Véron [41](1996) proved the existence of a solution (always unique) under the mere condition

$$\int_\Omega |g|(P_{|\mu|_S})\rho(x)dx < \infty.$$

The case of the exponential is different. For example if $g(r) = e^{2r}$ the existence of a solution of (13) is insured if, for some $p \in (1, \infty]$, there holds

$$\begin{gathered} \exp(2P_{\mu_S}) \in L^{p/(p-1)}(\partial\Omega), \\ \exp(2\mu_R) \in L^p(\partial\Omega). \end{gathered} \tag{15}$$

This was obtained by Grillot and Véron in 1997 [18].

5 Isolated singularities

As we have seen it above, if $1 < q < (N+1)/(N-1)$ and $a \in \partial\Omega$, for any $k > 0$ the function $u_{k,a} = P^q_\Omega(k\delta_a$ is a solution of (1) which vanishes on $\partial\Omega \setminus \{a\}$. Moreover, when k increases, it is the same with $u_{k,a}$. From the Osserman-Keller estimate, this sequence is locally uniformly bounded in Ω , therefore it converges to some positive solution $u_{\infty,a}$ of (1). By using some local estimate on the boundary it can be checked that $\{u_{k,a}\}$ is equicontinuous on any compact subset of Ω. Therefore $u_{\infty,a}$ vanishes on $\partial\Omega \setminus \{a\}$. This function $u_{\infty,a}$ is a solution of (1) has the maximal blow-up rate at a among the solutions vanishing on $\partial\Omega \setminus \{a\}$. As for the behaviour of $u_{k,a}$ near a it can be obtained from perturbation theory, since the blow-up estimate on Poisson's kernel yields to

$$0 \leq P^q_\Omega(k\delta_a) \leq P_{k\delta_a} = kP_{\delta_a} = kP(x,a),$$

that is

$$0 \leq u_{k,a} \leq Ck|x-a|^{-N}\rho_\Omega$$

Finally it is possible to prove that the non-linear term is negligible near a in some sense and

$$\lim_{x\to a} \frac{u_{k,a}}{P(x,a)} = k. \tag{16}$$

Always in the range $1 < q < (N+1)/(N-1)$, the function $u_{\infty,a}$ has a much stronger blow-up than $u_{k,a}$. The precise expression of this blow-up is made clearer by introducing spherical coordinates centered at a. In fact there exists a unique function ω defined on the half unit sphere S_+^{N-1}, the equator of which belongs to the plane $T_a\partial\Omega$ tangent to $\partial\Omega$ at the point a, such that

$$\lim_{x\to a} |x-a|^{2/(q-1)} u_{\infty,a}(x) = \omega\left((x-a)/|x-a|\right). \tag{17}$$

Moreover $u_{\infty,a}$ is the unique solution of (1) vanishing on $\partial\Omega\setminus\{a\}$ which satisfies (16) with $k=\infty$.

The following result due to Gmira and Véron [17]asserts that the behaviours (16) and (17) characterise all the isolated singularities of the solutions of (1) which coincide on $\partial\Omega\setminus\{a\}$ with a continuous function $h\in C(\partial\Omega)$.

Theorem 4 *Suppose $\partial\Omega$ is smooth, $a\in\partial\Omega$, $1<q<(N+1)/(N-1)$, and $u\in C(\bar\Omega\setminus\{a\})\cap C^2(\Omega)$ is a positive solution of (1) in Ω which coincides with h on $\partial\Omega\setminus\{a\}$. Then*

(i) either $u=P_\Omega^q(h)$, and u is regular in $\bar\Omega$;

(ii) or there exists $k>0$ such that $u=P_\Omega^q(h+k\delta_a)$ and $u\approx u_{k,a}$ in the sense of (16);

(iii) or $u=P_\Omega^q(h+\infty\delta_a)=\lim_{k\to\infty}u=P_\Omega^q(h+k\delta_a)$ and $u\approx u_{\infty,a}$ in the sense of (17).

Remark. It is always possible to assume that u vanishes on the boundary except the point a. Actually, if it is not the case, it follows by the maximum principle that there exists a solution $\tilde u$ of (1) vanishing on $\partial\Omega\setminus\{a\}$ and such that

$$|u(x)-\tilde u(x)|\le\sup_{y\in\partial\Omega}|h(y)|,\qquad \forall x\in\Omega.$$

Remark. The above classification can be extended to changing sign solutions when $(N+2)/N\le q<(N+1)/(N-1)$: in case (ii) there is no sign restriction on k, and in case (iii), we have either $u\approx u_{\infty,a}$ or $u\approx -u_{\infty,a}$.

Sketch of the proof of Theorem 4. The full proof is highly technical, therefore we shall restrict ourselves to the case where $\partial\Omega$ is flat near a. By scaling and translating it can be supposed that $a=0$ and $\partial\Omega\supset T_0\partial\Omega\cap\{x:\,|x|\le 1\}$ We shall not impose the positivity of u in order to see at what level this condition versus $(N+2)/N\le q<(N+1)/(N-1)$ appears. Let $(r,\sigma)\in\mathbb{R}_+\times S^{N-1}$ be the spherical coordinates, then u solution of (1) satisfies

$$\partial_{rr}u+\frac{N-1}{r}\partial_r u+\frac{1}{r^2}\Delta_{S^{N-1}}u=|u|^{q-1}u \tag{18}$$

in $(0,1]\times S_+^{N-1}$, and vanishes on $(0,1]\times\partial S_+^{N-1}$. Setting $t=\ln(1/r)$ and $w=r^{2/(q-1)}u$, then w is bounded and it satisfies

$$\partial_{tt}w+\beta_{q,N}\partial_t w+\Delta_{S^{N-1}}w+\alpha_{q,N}w=|w|^{q-1}w \tag{19}$$

in $[0, \infty) \times S_+^{N-1}$, where

$$\beta_{q,N} = 2\frac{q+1}{q-1} - N \text{ and } \alpha_{q,N} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right).$$

Since w is bounded in $[0, \infty) \times S_+^{N-1}$ and vanishes on $[0, \infty) \times \partial S_+^{N-1}$, it follows from the regularity theory of linear elliptic equations that $\nabla_S^\alpha \partial_t^\beta w$ remains uniformly bounded for any $|\alpha| + |\beta| \leq 3$ (here ∇_S is the covariant derivative symbol). Multiplying (19) by $\partial_t w$ gives

$$\beta_{q,N} \int_{S_+^{N-1}} (\partial_t w)^2 d\sigma = \frac{dE}{dt} \tag{20}$$

with

$$E = \int_{S_+^{N-1}} \left(\frac{1}{2}|\nabla_S w|^2 + \frac{1}{q+1}|w|^{q+1} - \frac{\alpha_{q,N}}{2}|w|^2 - \frac{1}{2}(\partial_t w)^2\right) d\sigma$$

But $\beta_{q,N} \neq 0$ since $q \neq (N+2)/(N-2)$. Since E is bounded,

$$\int_0^\infty \int_{S_+^{N-1}} (\partial_t w)^2 d\sigma dt < \infty. \tag{21}$$

By differentiating (19) with respect to t and multiplying by $\partial_{tt} w$, there also holds

$$\int_0^\infty \int_{S_+^{N-1}} (\partial_{tt} w)^2 d\sigma dt < \infty, \tag{22}$$

and by using the previous regularity estimates and (21), (22), it follows

$$\lim_{t\to\infty} \partial_t w = \lim_{t\to\infty} \partial_{tt} w = 0, \tag{23}$$

in $L^2(S_+^{N-1})$ and actually uniformly on $\bar{S}_+^{N-1}$. Consequently, the ω-limit set of the trajectory $\mathcal{T} = \bigcup_{t\geq 0}\{w(t,.)\}$ is included into a non-empty, compact and connected component of the set

$$\mathcal{S} = \left\{\varphi \in C^2(S_+^{N-1}) : \ \Delta_{S^{N-1}}\varphi + \alpha_{q,N}\varphi = |\varphi|^{q-1}\varphi \text{ in } S_+^{N-1}, \ \varphi = 0 \text{ on } \partial S_+^{N-1}\right\}$$

1- If $\alpha_{q,N} \leq N - 1 = \lambda_1(S_+^{N-1})$, which is equivalent to $q \geq (N+1)/(N-1)$, $\mathcal{S}$ is reduced to the zero function. This is easily done by multiplying by φ, and integrating over S_+^{N-1}.
2- If $2N = \lambda_2(S_+^{N-1}) \leq \alpha_{q,N} < \lambda_1$, which is equivalent to $(N+2)/N \leq q < (N+1)/(N-1)$, the set $\mathcal{S}$, besides the zero function, contains two nonzero elements, ω and $-\omega$, which keep constant sign (($\omega > 0$ for example), and are rotationally invariant.
3- If $1 < q < (N+1)/(N-1)$ $\mathcal{S} \cap C_+^2(\bar{(}S_+^{N-1}) = \{0, \omega\}$

It follows that either if $u \geq 0$ or if $(N+2)/N \leq q < (N+1)/(N-1)$, the function w admits a limit ℓ at infinity, and $\ell \in \{0, \omega, -\omega\}$.

If $\ell = 0$ a fine analysis similar to the one performed by Chen, Matano and Véron [7] yields to the existence of some $\varepsilon > 0$ such that

$$|w(t,\sigma)| \leq Ce^{-\varepsilon t}$$

for $t \geq 0$. This type of estimate is not easy to obtain. It is obtained by contradiction and the fact that either $\alpha_{q,N}$ is not an eigenvalue of $-\Delta_S$ in $W_0^{1,2}(S_+^{N-1})$, or, if it is, the equivariance properties of w are not compatible with the ones of the element of $\ker(-\Delta_S - \alpha_{q,N} I)$. From this follows an improved estimate of the following type

$$|w(t,\sigma)| \leq C \min(e^{-\theta\varepsilon t}, e^{(N-1-2/(q-1))t})$$

where $\theta = \theta(N,q) > 1$. This estimate is of linear type and easy to obtained by a representation formula. Since θ, a final estimate of the type

$$|w(t,\sigma)| \leq Ce^{(N-1-2/(q-1))t}$$

is derived in a finite number of iterations. The projection of w onto $\ker(-\Delta_S - (N-1)I)$ yields to the existence of some $\varphi \in \ker(-\Delta_S - (N-1)I)$ such that

$$\lim_{t\to\infty} e^{(2/(q-1)-N+1)t} w(t,.) = \varphi(.)$$

in the $C^2(S_+^{N-1}$-topology. Finally, if $\varphi = 0$, comparison of u with $\pm\varepsilon P(.,0)$ implies that u is actually the zero-function.

Remark. If the assumptions on q and the sign of u are relaxed, the only thing which can be proved is that the limit set of the trajectory $\mathcal{T}$ is included into a connected component of the set $\mathcal{S}$. However, in such cases exept for the constant-sign solutions, and the zero function, the connected components of $\mathcal{S}$ are continuum. Therefore it is not possible to assert that w converges precisely to a particular element.

6 Large solutions

Let Ω be a domain with a compact boundary. By a large solution of (1) we mean a positive solution u such that

$$\lim_{\rho(x)\to 0} u(x) = \infty. \tag{24}$$

Since it is classical to approximate Ω by an increasing sequence Ω_n of smooth subdomains, on each of them we can construct a positive solution u_n of (1) in Ω_n with infinite value on the boundary , we obtain a decreasing sequence of solutions of (1), each of them dominating in Ω_n any solution u of (1). Therefore the sequence $\{u_n\}$ converges to the maximal solution $\bar{u}_\Omega$ of (1) in Ω (this is essentially the results of Keller and Osserman). The questions which are raised are therefore

1- Existence of large solution or, equivalently, is the maximal solution a large solution? This question is linked to the regularity of $\partial\Omega$.

2- Uniqueness of the large solution.
The first question which is a problem linked to the regularity of $\partial\Omega$, and it has been solved in the case $q = 2$ by Dhersin and Le Gall [8] , by probabilistic methods. If $a \in \partial\Omega$, they proved that a Wiener-type criterion at the point a, associated to some Bessel capacity is a necessary and sufficient condition for the existence of a solution u of (1) such that

$$\lim_{x \to a} u(x) = \infty.$$

Their methods heavily relies on probability theory and the question remains open to cover the full range of exponent $q > 1$. The following easy to prove result is essentially due to Marcus and Véron [28] (although not explicitely written in this reference, it clearly corresponds to their cone condition with dimension 0).

Theorem 5 *Suppose* $1 < q < N/(N-2)$*, then there exists a large solution of (1) in* Ω*.*

Proof. At every point $a \in \partial\Omega$ the maximal solution $\bar{u}_\Omega$ constructed above is bounded from below by the explicit positive singular solution U_a of (1)

$$x \mapsto U_a(x) = \alpha_{q,N}^{1/(q-1)} |x-a|^{2/(1-q)}. \tag{25}$$

Notice that the existence of U_a is insured by the positivity of $\alpha_{q,N}$, which is equivalent to $1 < q < N/(N-2)$.

When $q \geq N/(N-2)$ large solutions may exist provided $\partial\Omega$ satisfies some weak geometric constraint. For example Marcus and Véron [28] proved

Theorem 6 *Suppose* $1 < q < (N-1)/(N-3)$*, and* Ω *satisfies the exterior segment property, then there exists a large solution of (1) in* Ω*.*

The exterior segment property asserts that there exists $\varepsilon > 0$ such that, for any $a \in \partial\Omega$, there exists a segment $I_{\varepsilon,a} \subset \Omega^c$ with length ε and a as one of its end-points. It is also proved in [28] that if $q \geq N/(N-2)$, the blow-up rate of the maximal solution may be much weaker than the one given by (25).

The first result of uniqueness of the large solution are due to Iscoe [19], Bandle and Marcus [1, 2] and Véron [42]. The technique used by Bandle and Marcus or Véron needs a C^2-regularity of $\partial\Omega$ and relies on a precise blow-up estimate of any large solution u at the boundary, namely

$$\lim_{\rho(x) \to 0} \rho^{2/(q-1)}(x) u(x) = C(q) > 0 \tag{26}$$

From this estimate and the maximum principle it follows

$$u \leq (1+\varepsilon)\tilde{u} \quad \text{in } \Omega,$$

if u and $\tilde{u}$ are two maximal solutions, and this holds for any $\varepsilon > 0$. Letting $\varepsilon \to 0$ and exchanging the role of u and $\tilde{u}$ implies $u = \tilde{u}$. The result of Iscoe, only stated in the case $q = 2$ gives uniqueness without the possible existence in the case where Ω is star-shapped with respect to some point, say 0. It use the equi-variant properties of equation (1)under the transformation

$$u \mapsto u_k, \quad \text{where } u_k(x) = k^{2/(q-1)}u(kx), \quad \forall k > 0.$$

If u and $\tilde{u}$ are two large solutions in Ω, then u_k is a large solution in $\Omega_k = k^{-1}\Omega$. If $k > 1$, $\Omega_k \subset \Omega$ and therefore $u_k \geq \tilde{u}$. Letting $k \to 1$ and exchanging again the role of u and $\tilde{u}$ implies uniqueness.

In 1995 Marcus and Véron [28] proved a uniqueness result extending Iscoe's one to a wide class of domains

Theorem 7 *Suppose $\partial\Omega$ is locally the graph of a continuous function, then there exists at most one large solution of (1) in Ω.*

The proof of this result is not easy. The key point is that the assumption on $\partial\Omega$ implies that any two large solutions u and $\tilde{u}$ satisfy

$$\lim_{\rho(x)\to 0} \frac{u(x)}{\tilde{u}(x)} = 1. \tag{27}$$

Finally, in 1997 Marcus and Véron [29] introduced a new technique for proving uniqueness of a wide class of solutions of (1) with positively homogeneous boundary data. In the case of uniqueness of large solutions, their results is the following

Theorem 8 *Suppose $1 < q < N/(N-2)$ and $\partial\Omega = \partial\bar{\Omega}^c$, then there exists exactely one large solution of (1) in Ω.*

Proof. The meaning of positively homogeneous boundary data is that they are unchanged by any multiplication by positive real numbers: they take only two possible values, 0 and ∞. Existence of the maximal solution $\bar{u}_\Omega$ follows from Theorem 5. Let u be another large solution and assume $u \neq \bar{u}_\Omega$. Then $u < \bar{u}_\Omega$.

Step 1 There exists a constant $K = K(N,q) > 1$ such that

$$K^{-1} \leq \frac{u(x)}{\bar{u}_\Omega(x)} \leq K, \quad \forall x \in \Omega. \tag{28}$$

Since $\partial\Omega = \partial\bar{\Omega}^c$; for any $a \in \partial\Omega$, there exists a sequence $\{a_n\} \subset \bar{\Omega}^c$ such that $a_n \to a$ when n goes to infinity. Because u blows-up on $\partial\Omega$ and the explicit solution U_{a_n} (defined in (25)), is finite, $U_{a_n} \leq u$, and letting n go to infinity, $U_a \leq u$ in Ω. If x is any point in Ω, let $a \in \partial\Omega$ such that $\rho_\Omega(x) = |x-a|$. Then

$$u(x) \geq U_a(x) = \alpha_{q,N}^{1/(q-1)}|x-a|^{2/(1-q)} = \alpha_{q,N}\rho_\Omega^{2/(1-q)}(x).$$

But, by the Keller-Osserman estimate (8),

$$\bar{u}_\Omega(x) \leq C(q,N)\rho_\Omega^{2/(1-q)}(x).$$

Consequently (28) holds with $K(N,q) = C(q,N)/\alpha_{q,N}^{1/(q-1)})$.

Step 2 End of the proof. Set

$$w = u - \frac{1}{2K}(\bar{u} - u).$$

Because of Step 1,

$$\left(\frac{1}{2} + \frac{1}{2K}\right) u < w < u,$$

which implies in particular that w blows-up on $\partial\Omega$. Moreover

$$\Delta w = \left(1 + \frac{1}{2K}\right) u^q - \frac{1}{2K}\bar{u}^q \leq \left(\left(1 + \frac{1}{2K}\right) u - \frac{1}{2K}\bar{u}\right)^q = w^q$$

which means that w is a super solution for (1). Since $(\frac{1}{2} + \frac{1}{2K})u$ is a subsolution, there exists a large solution u_1 of (1) which satisfies

$$\left(\frac{1}{2} + \frac{1}{2K}\right) u < u_1 < u \quad \text{and} \quad \bar{u}_\Omega - u_1 \geq \left(1 + \frac{1}{2K}\right)(\bar{u}_\Omega - u)$$

in Ω (see [35] for an a proof of this classical result under these conditions). Iterating this process, we consruct a sequence $\{u_n\}$ of large solutions of (1), which always satisfy $u_0 = u$,

$$\bar{u}_\Omega \leq K u_n,$$

and

$$\left(\frac{1}{2} + \frac{1}{2K}\right) u_{n-1} < u_n < u_{n-1} \quad \text{and} \quad \bar{u}_\Omega - u_n \geq \left(1 + \frac{1}{2K}\right)(\bar{u}_\Omega - u_{n-1}).$$

Consequently

$$\bar{u}_\Omega - u_n \geq \left(1 + \frac{1}{2K}\right)^n (\bar{u}_\Omega - u).$$

Because $\bar{u}_\Omega - u$ is locally bounded in Ω, we derive a contradiction by letting n go to infinity.

Remark. It is clear that uniqueness cannot hold if $\Omega = \tilde{\Omega} \cup F$, where $F = \{a_j\}$ is finite. In that case, $\partial\Omega = \partial\tilde{\Omega} \cup F$ and a large solution u of (1) in Ω may have a weak blow-up such as

$$u(x) = c|x - a_j|^{2-N}(1 + o(1)$$

for any $c > 0$ near a_j (we assume $N > 2$, the cases $N = 2, 1$ needing only very minor modifications). However, it has been noticed by Chasseigne and Vazquez [6] that a large solution u is unique if one imposes the strongest blow-up on $\partial\Omega$, that is

$$\lim_{\rho(x)\to 0} \rho(x)^{N-2} u(x) = \infty.$$

7 The boundary trace

For simplicity of exposition, in the next sections, we shall restrict ourself to the case where $\Omega = B = \{x \in \mathbb{R}^N : |x| < 1\}$, although all the results we shall propose extend to a C^2 bounded domain. We recall that (r, σ) are the spherical coordinates in R^N. It is classical, and easy to check, that every positive harmonic function u in B possesses a boundary trace given by a positive Radon measure $\mu \in \mathcal{M}(\partial B)$ which is attained in the sense of weak convergence of measures:

$$\lim_{r\to 1} \int_{S^{N-1}} u(r,\sigma)\zeta(\sigma)d\sigma = \int_{S^{N-1}} \zeta(\sigma)d\mu \tag{29}$$

for every $\zeta \in C(S^{N-1})$ and $u = P_\mu$. In this writing, we identify ∂B and S^{N-1}. The trace result is still valid if harmonic is replaced by super-harmonic [9] and a representation formula holds. Moreover, the positivity assumption on u can be replaced by an integrability condition such as $\Delta u \in L^1(B; (1 - r)dx)$ and in that case the boundary trace is a general Radon measure on ∂B.

The existence of a boundary trace for positive solutions of (1) is due to Le Gall in 1993 [24, 25] when $q = 2$. Actually, in this pioneering work, Le Gall gives a probabilistic representation of any positive solution of (1) in this case. The notion of trace that we present is due to Marcus and Véron [29](1995); it is a purely analytic presentation and is extendible to a wider class of nonlinearities.

Theorem 9 *Suppose $q > 1$ and u is a positive solution of (1) in B. Then for any $\omega \in \partial B$ we have the following alternative.*
(i) Either for every relatively open neighbourhood $U \subset \partial B$ of ω

$$\lim_{r\to 1} \int_U u(r,\sigma)d\sigma = \infty, \tag{30}$$

(ii) or there exists a relatively open neighbourhood $U \subset \partial B$ of ω such that for every $\zeta \in C^\infty(U)$

$$\lim_{r\to 1} \int_U u(r,\sigma)\zeta(\sigma)d\sigma = \ell(\zeta), \tag{31}$$

where ℓ is a positive linear functional on $C^\infty(U)$.

If V is an open domain of S^{N-1}, we denote by φ_V the first eigenfunction of $-\Delta$ in $W_0^{1,2}(V)$ with the normalization condition

$$\max_V \varphi_V = 1.$$

The corresponding eigenvalue is quoted λ_V , and if the relative boundary of V is C^2, the Hopf lemma applies with ν_S the normal outward unit vector to V (tangent to S^{N-1}),

$$\frac{\partial \varphi_V}{\partial \nu_S} < 0 \quad \text{on } \partial V.$$

Lemma 2 *Let V be an open domain of S^{N-1} with a C^2 boundary, u a positive solution of (1) in B and $\alpha > (N+1)/(N-1)$. Then we have the following dichotomy.*
(i) Either

$$\int_0^1 \int_V u^q \varphi_V^\alpha (1-r) r^N dr d\sigma = \infty,$$

and in that case

$$\lim_{r\to 1} \int_V (u\varphi_V^\alpha)(r,\sigma) d\sigma = \infty,$$

(ii) or

$$\int_0^1 \int_V u^q \varphi_V^\alpha (1-r) r^N dr d\sigma < \infty,$$

and in that case, for any C^2 function on V which satisfies

$$|\zeta| \le k\varphi_V^\alpha \quad \text{and } |\Delta\zeta| \le k\varphi_V^{\alpha-2} \tag{32}$$

for some $k > 0$, the following limit exists

$$\lim_{r\to 1} \int_V u(r,\sigma)\zeta d\sigma = \ell(\zeta),$$

and the mapping $\zeta \mapsto \ell(\zeta)$ is a positive linear functional defined on the subset of C^2 functions which satisfy (32).

Proof. **Step 1**. There holds

$$I_\alpha = \int_V \left|\Delta_S^{N-1}\varphi_V^\alpha\right|^{q/(q-1)} \varphi_V^{-q\alpha/(q-1)} d\sigma < \infty.$$

From Hopf boundary Lemma

$$\varphi_V(\sigma) \ge C_1 \operatorname{dist}_{S^{N-1}}(\sigma, \partial V)$$

for any $\sigma \in V$, where $\operatorname{dist}_{S^{N-1}}$ is the geodesic distance on S^{N-1} and $C_1 > 0$. Since

$$\Delta_S^{N-1}\varphi_V^\alpha = -\alpha\lambda_V\varphi_V^\alpha + \alpha(\alpha-1)\varphi_V^{\alpha-2}|\nabla\varphi_V|^2,$$

then

$$\left|\Delta_S^{N-1}\varphi_V^\alpha\right|^{q/(q-1)} \leq C_2 \left(\operatorname{dist}_{S^{N-1}}(.,\partial V)\right)^{q(\alpha-2)/(q-1)},$$

and finally

$$\left|\Delta_S^{N-1}\varphi_V^\alpha\right|^{q/(q-1)}\varphi_V^{-q\alpha/(q-1)} \leq C_3 \left(\operatorname{dist}_{S^{N-1}}(.,\partial V)\right)^{q((\alpha-2)-\alpha)/(q-1)}.$$

But $\alpha > (N+1)/(N-1)$, and the claim follows follows.

Step 2 Reduction of the equation. We shall assume $N > 2$, the case $N = 2$ requiring only minor technical modifications. We transform the equation (18) satisfied by u in polar coordinates by setting

$$s = \frac{r^{N-2}}{N-2} \quad \text{and } u(r,\sigma) = r^{2-N}v(s,\sigma),$$

and we get

$$s^2\frac{\partial^2 v}{\partial s^2} + \frac{1}{(N-2)^2}\Delta_{S^{N-1}}v - Ks^{N/(N-2)-q}v^q = 0, \tag{33}$$

in $(0,a)\times S^{N-1}$, where $K = K(N) > 0$ and $a = (N-2)^{-1}$, from which follows

$$s^2\frac{d^2}{ds^2}\int_V v\varphi_V^\alpha d\sigma + \frac{1}{(N-2)^2}\int_V v\Delta_{S^{N-1}}\varphi_V^\alpha d\sigma - Ks^{N/(N-2)-q}\int_V v^q\varphi_V^\alpha d\sigma = 0.$$

We set

$$X(s) = \int_V v\varphi_V^\alpha d\sigma \quad \text{and} \quad Y(s) = \left(\int_V v^q\varphi_V^\alpha d\sigma\right).$$

Since, by Hölder's inequality and Step 1,

$$Y(s) \leq I_\alpha^{1-1/q}Y(s),$$

it infers

$$\begin{array}{ll} (i) & s^2X'' + JY - Ks^{N/(N-2)-q}Y^q \geq 0 \\ (ii) & s^2X'' - JY - Ks^{N/(N-2)-q}Y^q \leq 0 \end{array} \tag{34}$$

where $J = I^{1-1/q}(N-2)^{-2}$.

Step 3 End of the proof. *Case 1*: suppose $\int_0^1\int_V u^q\varphi_V^\alpha(1-r)r^N dr d\sigma = \infty$. Since (34-i) implies

$$X'' \geq AY^q - B$$

on $[a/2,a)$, where $A > 0$ and $B > 0$ do not depend on s, a double integration gives

$$X(s) \geq X(a/2) + (s-a/2)X'(a/2) + A\int_{a/2}^s (s-\tau)Y^q(\tau)d\tau - B(s-a/2)^2/2,$$

and $\lim_{s\to a} X(s) = \infty$, which is assertion (i).

Case 2: suppose $\int_0^1 \int_V u^q \varphi_V^\alpha (1-r) r^N dr d\sigma < \infty$. Then $\int_{a/2}^a (a-s) Y^q ds < \infty$. Moreover $X'' \leq AY^q + B$, and

$$\frac{d^2}{ds^2}\left(X(s) + A\int_{a/2}^{s}(s-\tau)Y^q(\tau)d\tau - \frac{B}{2}(s-a/2)^2\right) \leq 0,$$

which infers that $s \mapsto X(s)$ admits a finite limit at infinity. If ζ is a C^2-function satisfying (32), we set $X_\zeta(s) = \int_V v\zeta d\sigma$ and derive

$$s^2\frac{d^2X_\zeta}{ds^2} + \frac{1}{(N-2)^2}\int_V v\Delta_{S^{N-1}}\zeta d\sigma - Ks^{N/(N-2)-q}\int_V v^q\zeta d\sigma = 0 \tag{35}$$

from (33). But

$$\left|\int_V v\Delta_{S^{N-1}}\zeta d\sigma\right| \leq k\left(\int_V v^q\varphi_V^\alpha d\sigma\right)^{1/q}\left(\int_V \varphi_V^{\alpha-2q/(q-1)}d\sigma\right)^{1-1/q}$$

and

$$\left|\int_V v^q\zeta d\sigma\right| \leq k\int_V v^q\varphi_V^\alpha d\sigma.$$

Consequently

$$\int_{a/2}^{a}\left|\int_V v\Delta_{S^{N-1}}\zeta d\sigma\right|(a-s)ds < \infty\,, \quad \int_{a/2}^{a}\left|\int_V v^q\zeta d\sigma\right|(a-s)ds < \infty.$$

Integrating twice the equality (35) implies that $s \mapsto X_\zeta(s)$ admits a finite limit at infinity and this limit depends linearly of ζ. moreover it is nonnegative if such is the case of ζ. We call this limit $\ell(\zeta)$.

Proof of Theorem 9. If $\omega \in S^{N-1}$ and there exists an open neighbourhood V such that Lemma 1-i holds, there existence of a nonegative Radon measure μ_V such that

$$\lim_{r\to 1}\int_V u(r,\sigma)\zeta(\sigma)d\sigma = \int_V \zeta d\mu \quad \forall \zeta \in C_c(V) \tag{36}$$

If such a neighbourhood does not exist, we are in case (i).

Let $\mathcal{R}$ be the set of the $\omega \in S^{N-1}$ such that (ii) holds; $\mathcal{R}$ is relatively open and there exists a unique (non-negative) Radon measure μ such that $\mu_{\llcorner V} = \mu_V$. The set $\mathcal{S} = S^{N-1} \setminus \mathcal{R}$ is closed.

Definition. The couple $(\mathcal{S}, \mu)$ is called the boundary trace of u. The set $\mathcal{S}$ is the singular part of this trace and the measure μ on $\mathcal{R}$, the regular part of the trace. We denote

$$\mathrm{tr}_{\partial B}(u) = \mathrm{tr}(u) = (\mathcal{S}, \mu).$$

For convenience it is often useful to introduce the Borel measure framework. Actually there is a one to one correspondence between the family CM of couples $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a compact subset of $\S^{N-1}$ and μ a positive Radon measures on $\mathcal{R}=S^{N-1}\setminus\mathcal{S}$ and the set $\mathcal{B}^{+}_{reg}(S^{N-1})$ of outer regular, positive Borel measures on S^{N-1}. If $\beta\in\mathcal{B}^{+}_{reg}$, the singular set $\mathcal{R}_\beta$ and the singular part $\mathcal{S}_\beta$ are defined as follows:

$$\mathcal{R}_\beta=\{\omega\in S^{N-1}:\ \exists U\text{ rel. open neighborhood of }\omega\text{ s.t. }\beta(U)<\infty\},$$

$$\mathcal{S}_\beta=\{\omega\in S^{N-1}:\ \forall U\text{ rel. open neighborhood of }\omega\text{ s.t. }\beta(U)=\infty\}.$$

The correspondence $CM\leftrightarrow\mathcal{B}^{+}_{reg}(S^{N-1})$ is given by

$$\mathcal{M}((\mathcal{S},\mu))=\bar\mu\text{ where }\bar\mu(A)=\begin{cases}\mu(A)\text{ if }A\subseteq\mathcal{R},\\ \infty\text{ if }A\cap\mathcal{S}\neq\emptyset,\end{cases}\tag{37}$$

for any Borel subset A of S^{N-1}, and

$$\mathcal{M}^{-1}(\beta)=(\mathcal{S}_\beta,\beta\rceil_{\mathcal{R}_\beta}).\tag{38}$$

With this correspondence, we denote

$$\mathrm{Tr}(u)=\mathcal{M}(\mathrm{tr}(u))\in\mathcal{B}^{+}_{reg}(S^{N-1}),$$

and a more general formulation for (3) is therefore what we call the generalized Dirichlet problem, namely

$$\begin{array}{c}-\Delta u+u^q=0,\ \text{ in }B,\\ \mathrm{Tr}(u)=\bar\mu\in\mathcal{B}^{+}_{reg}(S^{N-1}).\end{array}\tag{39}$$

The problem is completely different according $1<q<(N+1)/(N-1)$ (the subcritical case) or $q\geq(N+1)/(N-1)$ (the supercritical case).

8 Generalized Dirichlet problem: the subcritical case

The following estimate links the local integral blow-up estimate corresponding to singular boundary points and pointwise blow-up estimate.

Theorem 10 *Suppose $1<q<(N+1)/(N-1)$, u is a non negative solution of (1) in B with $\mathrm{tr}(u)=(\mathcal{S},\mu)$ and $\omega\in\mathcal{S}$. Then*

$$\lim_{r\to1}u(r,\omega)=\infty,$$

and more precisely

$$u(x)\geq U_\omega(x),\quad\forall x\in B,\tag{40}$$

where U_ω is defined in (25).

Proof. A scaling-concentration argument is used. Since $\omega \in \mathcal{S}$, for any $\eta > 0$

$$\lim_{r\to 1}\int_{D_\eta(\omega)} u(r,\sigma)d\sigma = \infty,$$

where $D_\eta(\omega) = \{\sigma \in S^{N-1} : \operatorname{dist}_{S^{N-1}}(\sigma,\omega) < \eta\}$ We recall that $u_k(x) = k^{2/(q-1)}u(kx)$, and u_k is a soluton of (1) in $B_{1/k} = k^{-1}B$. For $0 < \varepsilon < 1$ we denote

$$M_{\varepsilon,\eta} = \int_{D_\eta(\omega)} u(1-\varepsilon,\sigma)d\sigma.$$

For $m > 0$ large enough and η small enough there exists $\varepsilon = \varepsilon(\eta, m) > 0$ such that $m = M_{\varepsilon,\eta}$. Let $\chi_{D_\eta(\omega)}$ be the characteristic function of the set $D_\eta(\omega)$ and w_η be the solution of (1) in B with boundary value $u_{1-\varepsilon}(1,.)\chi_{D_\eta(\omega)}$. Clearly $w_\eta \leq u_{1-\varepsilon}$. Since $\lim_{\eta\to 0}\varepsilon = 0$, then $\lim_{\eta\to 0} w_\eta = P_B^q(m\delta_\omega) = u_{m,\omega}$. Moreover $\lim_{\varepsilon\to 0} u_{1-\varepsilon} = u$. Therefore $u_{m,\omega} \leq u$, for any $m > 0$, which implies (40).

Remark. The proof below cannot work in the supercritical case, and actually the result does not hold as we shall see it below.

It is proved by Le Gall [24, 25] in the case $N = 2 = q$, and by Marcus and Véron [29] in the general case, that the boundary data problem (39) is well posed. The method of Le gall is mainly a probabilistic one while Marcus and Véron use only analytic tools.

Theorem 11 *Suppose* $1 < q < (N+1)/(N-1)$. *Then the correspondence* $u \mapsto \mathrm{Tr}(u)$ *which assigns to each nonnegative solution* u *of (1) in* B*, its boundary trace in* $\mathcal{B}^+_{reg}(S^{N-1})$ *is one to one.*

Since the proof is long and very technical, we shall only give a sketch of it, pointing out the main steps.

Lemma 3 *There exits a minimal solution* $\underline{u}_{\mathcal{S},\mu}$.

Sketch of the proof. Let $\{y_k\}_{k\in\mathbb{N}}$ be a dense sequence in $\mathcal{S}$ and

$$\mathcal{S}_\varepsilon = \{\sigma \in S^{N-1} : \operatorname{dist}_{S^{N-1}}(\sigma,\mathcal{S}) \leq \varepsilon\}.$$

We set

$$\underline{\mu}_n = n\sum_{k=0}^{n}\delta_{y_k} + \chi_{\mathcal{S}^c_{1/n}}\mu.$$

Since the closure of $\mathcal{S}^c_{1/n}$ is a compact subset of of $\mathcal{S}^c$, $\underline{\mu}_n$ is a bounded Radon measure. By comparison principle and Theorem 3, the sequence $\{\underline{u}_n\} = \{P_B^q(\underline{\mu}_n)\}$ is increasing. Moreover, it follows from Lemma 1 and Theorem 10 that if u is any solution of (39), then $\underline{u}_n \leq u$. When $n \to \infty$, $\underline{u}_n$ increases and converges to the minimal solution $\underline{u}_{\mathcal{S},\mu}$.

Lemma 4 *There exists a maximal solution* $\overline{u}_{\mathcal{S},\mu}$.

Sketch of the proof. We denote by $\overline{\mu}_n$ the Borel measure with singular set $\mathcal{S}_{1/n}$ and regular part $\mu_n = \chi_{\overline{\mathcal{S}}^c_{1/n}}\mu$ Since $\overline{\mu}_n$ is the increasing limit of the sequence of Radon measures $\{g_{n,k}\}$ defined by

$$g_{n,k} = k\chi_{\overline{\mathcal{S}}_{1/n}} + \chi_{\overline{\mathcal{S}}^c_{1/n}}\mu,$$

we first construct the solution $\overline{u}_{n,k} = P^q_B(g_{n,k})$ of (1) and after (as $k \to \infty$) a solution $\overline{u}_n$ of (39) with $\mathrm{Tr}(\overline{u}_n) = \overline{\mu}_n$. Moreover the sequences $\{\overline{u}_n\}$ is non-increasing, and, if u is any solution of (39), then $\overline{u}_n \geq u$. When $n \to \infty$, $\overline{u}_n$ decreases and converges to the maximal solution $\overline{u}_{\mathcal{S},\mu}$.

Lemma 5 *There always holds*

$$\overline{u}_{\mathcal{S},\mu} - \underline{u}_{\mathcal{S},\mu} \leq \overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0}.$$

Sketch of the proof. For $r, s > 0$ set

$$\Phi(r,s) = \begin{cases} \frac{r^q - s^q}{r-s} & \text{if } r \neq s, \\ qs^{q-1} & \text{if } r = s. \end{cases}$$

By convexity,

$$\left\{ \begin{array}{l} r_0 \geq s_0,\, r_1 \geq s_1 \\ r_1 \geq r_0,\, s_1 \geq s_0, \end{array} \right\} \Longrightarrow \Phi(r_1, s_1) \geq \Phi(r_0, s_0).$$

We set (with the notations of Lemmas 3-4)

$$\underline{u}_n = \underline{u}_{n,\mu} \quad \text{and} \quad \overline{u}_n = \overline{u}_{n,\mu},$$

and

$$Z_{n,\mu} = \overline{u}_{n,\mu} - \underline{u}_{n,\mu} \quad \text{and} \quad \lambda_{n,\mu} = \Phi(\overline{u}_{n,\mu}, \underline{u}_{n,\mu}).$$

Then

$$\Delta Z_{n,\mu} = \lambda_{n,\mu} Z_{n,\mu}$$

and

$$\Delta(Z_{n,\mu} - Z_{n,0}) - \lambda_{n,\mu}(Z_{n,\mu} - Z_{n,0}) = (\lambda_{n,\mu} - \lambda_{n,0})Z_{n,0}.$$

Since

$$\overline{u}_{n,\mu} \geq \underline{u}_{n,\mu},\ \overline{u}_{n,\mu} \geq \overline{u}_{n,0} \quad \text{and} \quad \overline{u}_{n,0} \geq \underline{u}_{n,0},\ \underline{u}_{n,\mu} \geq \underline{u}_{n,0},$$

$\lambda_{n,\mu} \geq \lambda_{n,0}$. Therefore

$$\Delta(Z_{n,\mu} - Z_{n,0}) - \lambda_{n,\mu}(Z_{n,\mu} - Z_{n,0}) \geq 0.$$

Moreover $Z_{n,\mu} = Z_{n,0}$ on ∂B. Therefore

$$Z_{n,\mu} \leq Z_{n,0} \quad \text{in} \quad B.$$

Actually an approximation argument is needed to take into account the fact that the solutions have infinite values in $\overline{\mathcal{S}}_{1/n}$. Conclusion follows by letting $n \to \infty$.

Lemma 6 *There exits* $L = L(N,q) > 1$ *such that*

$$\overline{u}_{\mathcal{S},0} \leq L\underline{u}_{\mathcal{S},0}.$$

Sketch of the proof. *Case 1* Suppose $\omega \in \mathcal{S}$. Since

$$U_\omega(r,\omega) \leq \underline{u}_{\mathcal{S},0}(r,\omega) \leq \overline{u}_{\mathcal{S},0}(r,\omega) \leq \bar{u}_B(r,\omega)$$

where $\bar{u}_B$ is the maximal solution of (1) in B, and

$$U_\omega(r,\omega) \geq K(N,q)\bar{u}_B(r,\omega),$$

the estimate holds.

Case 2 Suppose $\omega \in \mathcal{R}$. The way for obtaining this estimate is a bit more technical, however, if we assume that B is replaced $\mathbb{R}^N_+$ and ∂B by the hyperplane $H = \mathbb{R}^{N-1}$, the argument goes as follows. By a change of coordinates, it can be assumed that $0 = \omega$ is the origin. Let $D_\rho \subset H$ be the largest open (N-1)-ball with center 0 such that $D_\rho \subseteq \mathcal{R}$ (ρ is finite since $\mathcal{S} \neq \emptyset$), and let $a \in \partial D_\rho \cap \mathcal{S}$. If we denote $x = (x_1, x')$ the current point in $\mathbb{R}_+ \times \mathbb{R}^{N-1}$, then

$$u_{\infty,\omega}(x_1,0) \leq \underline{u}_{\mathcal{S},0}(x_1,0) \leq \overline{u}_{\mathcal{S},0}(x_1,0) \leq \overline{u}_{D^c_\rho,0}(x_1,0).$$

In order to estimate the bounds of the quotient

$$Q_\rho(x_1) = \frac{u_{\infty,\omega}(x_1,0)}{\overline{u}_{D^c_\rho,0}(x_1,0)},$$

when x_1 runs from 0 to ∞, a scaling argument shows that it can be assumed that $\rho = 1$. For small values of x_1 the lower bounds is positive and given by Hopf's lemma, while for large values of x_1, the lower bound follows from the fact that

$$\lim_{x_1\to\infty} x_1^{2/(q-1)} u_{\infty,\omega}(x_1,0) = \omega(x_1/|x_1|),$$

(in this formula $\omega(.)$ is the spherical function defined in (17)) while

$$\lim_{x_1\to\infty} x_1^{2/(q-1)} \overline{u}_{D^c_\rho,0}(x_1,0) = \left(\frac{2(q+1)}{(q-1)^2}\right)^{1/(q-1)}.$$

Consequently there exists $L_1 > 0$ such that

$$\overline{u}_{\mathcal{S},0}(x_1,0) \leq L\underline{u}_{\mathcal{S},0}(x_1,0).$$

In the real situation where we are dealing with B and not with $\mathbb{R}^N_+$, we proceed by contradiction and scaling, reducing the situation to the half space case.

Proof of Theorem 10. It follows the proof of Theorem 8 (in a simplified way). By Lemma 5 it is sufficient to prove uniqueness in the case where $\mathrm{tr}(u) = (\mathcal{S}, 0)$. Assuming that

$$\underline{u}_{\mathcal{S},0} \neq \overline{u}_{\mathcal{S},0} \Longrightarrow \underline{u}_{\mathcal{S},0} < \overline{u}_{\mathcal{S},0},$$

then

$$w = \underline{u}_{\mathcal{S},0} - \frac{1}{2L}(\overline{u}_{\mathcal{S},0} - \underline{u}_{\mathcal{S},0})$$

is a supersolution of (1) dominated by $\underline{u}_{\mathcal{S},0}$, but dominating the subsolution $(1/2 + 1/(2L))\underline{u}_{\mathcal{S},0}$. Therefore there exists a solution $\tilde{u}$ of (1) such that

$$(\frac{1}{2} + \frac{1}{2L})\underline{u}_{\mathcal{S},0} \leq \tilde{u} \leq w < \underline{u}_{\mathcal{S},0}.$$

It follows that $tr(\tilde{u}) = (\mathcal{S}, 0)$, which contradicts the minimality of $\underline{u}_{\mathcal{S},0}$.

9 Generalized Dirichlet problem: the supercritical case

As we have seen it in Sections 3-4, neither any Radon measure, nor any closed subset of ∂B are eligible for being the regular or the singular part of the boundary trace $(\mathcal{S}, \mu)$ of a positive solution of (1) in$B$. Roughly speaking, a Radon measure is admissible if it does not charge too thin sets, while a closed subset is admissible for being a singular set if it is not too ramified. Moreover there is a compatibility requirement between $\mathcal{S}$ and μ. Those different notions will be made rigourous with the help of boundary Bessel capacities. The results that we present are due to Le Gall [25] in the case q =2, Dynkin and Kuznetsov in the case $1 < q < 2$ [11, 12, 13]and Marcus and Véron [32]in the case $q > 2$. Because of the technical difficulties of the various aspects of the theory of supercritical boundary trace we shall essentially restrict ourself to present the main results in the case $q > 2$, with some ideas of how to obtain them. *Up to now a unified proof covering all the cases $q \geq (N+1)/(N-1)$ is still missing.*

Removable sets

Defintion. (i) A subset $E \subset \partial B$ is said **q-removable** if any non-negativefunction $u \in C^2(B) \cap C(\bar{B} \setminus E)$ which satisfies (1) in B and vanishes on $\partial B \setminus E$ is identically zero.
(ii) A subset $E \subset \partial B$ is said **conditionally q-removable** if any non-negative function $u \in C^2(B) \cap C(\bar{B} \setminus E)$ which satisfies (1) in B and vanishes on $\partial B \setminus E$ is such that $u \in L^q(B; (1-r)dx)$.

The main result concerning the removability of set is the following

Theorem 12 *Assume $q \geq (N+1)/(N-1)$ and $E \subset \partial B$ is a Borel set. Then the following assertions are equivalent.*
(i) E is q-removable
(ii) E is conditionally q-removable
(iii) $C_{2/q,q'}(E) = 0$.

Moreover an arbitrary set $A \subset \partial B$ is q-removable if and only if every closed subset of A is q-removable.

In the statement of the Theorem, $C_{2/q,q'}(E)$ denote the Bessel capacities of order $2/q$ and exponent q. In the space $\mathbb{R}^d$ the Bessel capacities $C_{\alpha,p}(E)$ ($\alpha > 0$,

$1 < p < \infty$) is defined by

$$C_{\alpha,p}(E) = \inf\left\{\int_{\mathbb{R}^d} f^p dx : \ f \geq 0, \ G_\alpha * f \geq 1 \text{ on } E\right\}$$

where G_α denotes the Bessel potential of order α. Another way to define capacities is to use Besov spaces. For $h \in L^p(\mathbb{R}^d)$ $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$, put

$$T^{p,q}_\alpha(h)(t) = |t|^{-d/q-\alpha} \|h(t+.) - h(.)\|_{L^p}$$

for $t \in \mathbb{R}^d$ and denote

$$B^{p,q}_\alpha(\mathbb{R}^d) = \left\{h \in L^p(\mathbb{R}^d) : \ T^{p,q}_\alpha(h) \in L^q(\mathbb{R}^d)\right\},$$

with norm

$$\|h\|_{B^{p,q}_\alpha} = \|h\|_{L^p} + \|T\|^{p,q}_\alpha(h)_{L^q}.$$

When $\alpha = 1$, $T^{p,q}_\alpha(h)$ is replaced by

$$T^{p,q}_1(h) = |t|^{-d/q-1} \|h(t+.) + h(-t+.) - 2h(.)\|_{L^p}$$

When $\alpha > 1$ the spaces are defined by induction. From this spaces (which coincide with the classical Sobolev spaces when α is not an integer and $p = q$), we define the capacity of a compact subset $E \in \mathbb{R}^d$ by $C_{B^{p,q}_\alpha}(E)$ by

$$C_{B^{p,q}_\alpha}(E) = \inf\left\{\|f\|_{B^{p,q}_\alpha} : \ f \in \mathcal{S}(\mathbb{R}^d), \ f \geq 1 \text{ on } E\right\}.$$

The $C_{B^{p,q}_\alpha}$-capacity of an open set is defined by the supremum of the capacities of its compact subsets, and the capacity of a general set by the infimum of the capacities of the open sets in which it is contained.

When $\alpha > 0$, $1 < p < \infty$, and $p \leq d/\alpha$ the following equivalence holds: there exists $K = K(d, p, \alpha) > 0$ such that for any subset of $\mathbb{R}^d$,

$$K^{-1} C_{\alpha,p}(E) \leq C_{B^{p,p}_\alpha}(E) \leq K C_{\alpha,p}(E).$$

On a submanifold of $\mathbb{R}^N$, the capacity is defined by using local charts.

In the forthcoming lemmas, we give the strategy for proving Theorem 12.

Lemma 7 *Suppose that $q \geq (N+1)/(N-1)$ and let $E \subset \partial B$ be compact and such that $C_{2/q,q'}(K) = 0$. Then K is conditionally q-removable.*

Sketch of the proof. In the case $q > 2$ Marcus and Véron [32] proceed as follows: If $\eta \in C^2(\partial B)$ is such that $0 \leq \eta \leq 1$ is identically 1 is a neighborhood of K set $\zeta = (1 - P_\eta)^{2q'} \varphi_1$, where φ_1 is the positive first eigenfunction of $-\Delta$ in $W^{1,2}_0(B)$. Then it is proved by approximation that

$$\int_B (-u\Delta\zeta + u^q \zeta) dx = 0. \tag{41}$$

From (41) and Hölder's inequality,

$$\int_B u^q \zeta dx \le \left(\int_B u^q \zeta dx\right)^{1/q} \left(\int_B \zeta^{-q'/q} |\Delta\zeta|^{q'} dx\right)^{1/q'}.$$

Using sharp regularizing estimates of the Poisson potential,

$$\int_B \zeta^{-q'/q} |\Delta\zeta|^{q'} dx \le C_1 \|\eta\|_{W^{2/q,q'}} + C_2.$$

From the assumption on K, there exists a sequence of functions $\eta_k \in C^2(\partial B)$ such that $0 \le \eta_k \le 1$ and $\|\eta_k\|_{W^{2/q,q'}} \to 0$ as $k \to \infty$. Since this implies in particular that $1 - P_{\eta_k} \to 1$, it follows

$$\int_B u^q \varphi_1 dx \le C_2,$$

which implies the claim.

Lemma 8 *Any conditionally q-removable closed subset $E \subset \partial B$ is q-removable.*

Proof. If E is not q-removable, there exists a nonnegative nonzero solution u of (1) vanishing on $\partial B \setminus E$. Therefore $Tr(u)$ is a Borel measure which is zero outside E. Since E is conditionally q-removable, $u^q \in L^1(B; (1-r)dx)$. Therefore the singular part of the boundary trace of u is empty and

$$Tr(u) = \mu \in \mathcal{M}_+(\partial B).$$

For $n \ge 1$, $\tilde{u}_n = nu$ is a supersolution of (1) with boundary trace $n\mu$. Therefore there exists a nonnegative solution u_n of (1) such that $Tr(u_n) = n\mu$, and in particular

$$\int_B (\lambda_1 u_n + u_n^q)\varphi_1 dx = -n \int_{\partial B} \frac{\partial \varphi_1}{\partial \nu} d\mu$$

Letting $n \to \infty$ implies that the increasing sequence $\{u_n\}$ converges to some solution u_∞ in B with $Tr(u_\infty) = 0$ outside E. Moreover,

$$\int_B (\lambda_1 u_\infty + u_\infty^q)\varphi_1 dx = \infty \Longrightarrow \int_B u_\infty^q \varphi_1 dx = \infty.$$

This contradicts the fact that $u_\infty^q \in L^1(B; (1-r)dx)$ because E is conditionally q-removable.

Remark. It is clear that if a subset $E \in \partial B$ is q-removable, it is conditionnaly q-removable.

By sharp linear estimates on the Poisson potential, the following result holds

Lemma 9 *Suppose $q \ge 2$, then $\mu \mapsto P_B(\mu)$ maps $\mathcal{M}_+(\partial B) \cap W^{-2/q,q}(\partial B)$ into $L^q(B; (1-r)dx)$, and there holds*

$$\|P_\mu\|_{L^q(B;(1-r)dx)} \le C(q) \|\mu\|_{W^{-2/q,q}(\partial B)}.$$

As an important consequence we have

Corollary 1 *Suppose $q \geq 2$ and $\mu \in \mathcal{M}_+(\partial B) \cap W^{-2/q,q}(\partial B)$. Then there exists $u = P^q_B(\mu)$.*

Proof. By Lemma 9, $P_\mu \in L^q(B;(1-r)dx)$. For $k > 0$ we denote $\{u_k\}$ the solution of

$$\begin{aligned} -\Delta u_k + (\min u_k, k)^q = 0, \text{ in } B, \\ u_k = \mu \text{ on } \partial B, \end{aligned}$$

Clearly $u_k \leq P_\mu$ which implies that the sequence $\{(\min u_k, k)^q\}$ is uniformly integrable in $L^1(B;(1-r)dx)$. When k increases, u_k decreases and converges to some u which belongs to $L^1(B) \cap L^q(B;(1-r)dx)$. Letting $k \to \infty$ in the weak formulation (which is valid for every $\zeta \in C_0^{1,1}(\bar{B})$)

$$\int_B (-u_k \Delta\zeta + \zeta(\min u_k, k)^q) dx = -\int_B \frac{\partial \zeta}{\partial \nu} d\mu,$$

infers that $u = P^q_B(\mu)$.

From a dual definition of the $C_{2/q,q'}$-capacity of a closed subset $E \subseteq \partial B$, follows the implication

$$C_{2/q,q'}(E) > 0 \Longrightarrow \exists \mu \in \mathcal{M}_+ \cap W^{-2/q,q}(\partial B) \; s.t. \; \mu(E) = \mu(\partial B) > 0. \quad (42)$$

Consequently there holds

Lemma 10 *Suppose $q \geq \max(2, (q+1)/(q-1)$. If $E \subseteq \partial B$ is closed and such that $C_{2/q,q'}(E) > 0$, then E is not conditionally q-removable.*

Proof. By Lemma 9 there exists $\mu \in \mathcal{M}_+ \cap W^{-2/q,q}(\partial B)$ such that $\mu(E) = \mu(\partial B) > 0$. By Corollary 1, $u = P^q_B(\mu)$ exists and $\mathrm{tr}(u) = (\emptyset, \mu)$, which implies in particular that u vanishes on $\partial B \setminus E$.

The proof of Theorem 12 is completed by the next result whose proof is too technical to be presented here.

Lemma 11 *A set $A \subset \partial B$ is q-removable if and only if every closed subset of A is q-removable.*

q-traces

Definition. A Radon measure μ on ∂B is called a q-trace if there exists a solution u of (1) such that $u = P^q_B(\mu)$. The set of q-traces is denoted by $\mathcal{M}^q(\partial B)$.

The main result concerning q-traces is

Theorem 13 *Assume $q \geq (N+1)/(N-1)$. A measure μ on ∂B is a q-trace if and only if for any Borel subset $A \subseteq \partial B$*

$$\mu(A) = 0 \quad \textit{whenever} \quad C_{2/q,q'}(A) = 0. \quad (43)$$

Besides what has been proved in the preceding subsection, the two next lemmas are needed, which both are mere adaptations of results of Baras and Pierre [3], and Meyers [33].

Lemma 12 *Let $\mu \in \mathcal{M} \cap W^{-2/q,q}(\partial B)$. Then μ does not charge the sets with $C_{2/q,q'}$-capacity zero.*

Lemma 13 *Suppose $\mu \in \mathcal{M}^+(\partial B)$ does not charge the sets with $C_{2/q,q'}$-capacity zero. Then there exists a sequence $\{\mu_n\} \subset W^{2/q,q'}(\partial B) \cap \mathcal{M}^+(\partial B)$, such that $\{\mu_n\}$ converges in increasing to μ.*

Sketch of the proof of Theorem 13. We shall restrict ourselves to the cases $q \geq 2$ and the measures are nonnegative, the general case needing some approximation argument. Let $\mu \in \mathcal{M}^q_+(\partial B)$ and $E \subset \partial B$ be a Borel subset such that $C_{1/q,q'}(E) = 0$. Set $\mu_E = \chi_E \mu$. Since $0 \leq \mu_E \leq \mu$, the construction given in Corollary 1 yields $\mu_E \in \mathcal{M}^q_+(\partial B)$ and $P^q_B(\mu_E) \leq P^q_B(\mu)$. But $C_{1/q,q'}(E) = 0$ and Theorem 12 implies that E is q-removable. therefore $P^q_B(\mu_E) = 0$ and consequently $\mu_E = 0$.

Conversely, let $\mu \in \mathcal{M}^+(\partial B)$ which does not charge the sets with $C_{2/q,q'}$-capacity zero. Then there exists an increasing sequence of positive measures $\{\rho_n\}$ belonging to $W^{-2/q,q}(\partial B)$ and converging to μ. Then $\rho_n \in \mathcal{M}^q_+(\partial B)$, and the sequence $\{u_n\} = \{P^q_B(\rho_n)\}$ is increasing and converges to some u. Moreover

$$\int_B (\lambda_1 u_n + u_n^q)\varphi_1 dx = -\int_{\partial B} \frac{\partial \varphi_1}{\partial \nu} d\rho_n. \tag{44}$$

Because the right-hand side of (44) is convergent, the same holds for the left-hand side (by the Beppo-Levi theorem). Therefore $u \in L^q(B, (1-r)dx) \cap L^1(B)$, which is sufficient to derive that $u = P^q_B(\mu)$ from the weak formulation of the fact that $u_n = P^q_B(\rho_n)$.

The generalized Dirichlet problem

Given a Borel measure $\bar{\mu} \in \mathcal{B}^+_{reg}(\partial B)$ with $\mathcal{M}^{-1}(\bar{\mu}) = (\mathcal{S}, \mu) \in CM$, we recall that

$$\mathcal{S}_\varepsilon = \{\sigma \in S^{N-1} : \mathrm{dist}_{_{S^{N-1}}}(S^{N-1}, \mathcal{S}) \leq \varepsilon\}.$$

and

$$\mu_\varepsilon = \chi_{\mathcal{S}^c_\varepsilon} \mu.$$

Let $\overline{u}_{\mathcal{S}_\varepsilon}$ be the maximal solution of (1) in B with $\mathrm{tr}(\overline{u}_{\mathcal{S}_\varepsilon}) = (\mathcal{S}_\varepsilon, 0)$. Because of the construction it is easy to see that

$$0 < \varepsilon < \delta \Longrightarrow \overline{u}_{\mathcal{S}_\varepsilon} \leq \overline{u}_{\mathcal{S}_\delta}.$$

When $\varepsilon \to 0$, $\overline{u}_{\mathcal{S}_\varepsilon}$ converges locally uniformly in B to a nonnegative solution u^* of (1). Let $\mathcal{S}^*_q$ be the singular part of the boundary trace of u^*. Clearly $\mathcal{S}^*_q \subseteq \mathcal{S}$.

If $\mathcal{R} = \partial B \setminus \mathcal{R}$ and $\mu \in \mathcal{M}_+(\mathcal{R})$ is such that $\mu_K = \chi_K \mu$ is a q-trace for any compact subset $K \subset \mathcal{R}$, we denote $u_K = P_B^q(\mu_K)$. If K_n is an increasing sequence of compact subset of $\mathcal{R}$ such that $\bigcup K_n = \mathcal{R}$, the sequence $\{u_{K_n}\}$ is increasing and converges to a solution $\check{u}$ of (1). Let $\partial_\nu \mathcal{S}$ be the singular part of the boundary trace of $\check{u}$. Again it is easy to verify that $\partial_\nu \mathcal{S} \subseteq \mathcal{S}$.

The following result is the main result concerning the solvability of (39) in the supercritical case.

Theorem 14 *Assume $q \geq (N+1)/(N-1)$ and let $\bar{\mu} \in \mathcal{B}^+_{reg}(\partial B)$, with regular set $\mathcal{R} = \mathcal{R}_{\bar{\mu}}$, regular part $\mu = \bar{\mu}_{|\mathcal{R}_{\bar{\mu}}}$ and singular part $\mathcal{S} = \mathcal{S}_{\bar{\mu}} = \partial B \setminus \mathcal{R}_{\bar{\mu}}$. Then problem (39) possesses a maximal solution $\bar{u}_{\bar{\mu}}$ if and only if the following two conditions are satisfied:*
(i) For every Borel subset $A \subset \mathcal{R}$, $C_{2/q,q'}(A) = 0 \Longrightarrow \mu(A) = 0$.
(ii) $\mathcal{S} = \mathcal{S}_q^* \cup \partial_\nu \mathcal{S}$.

We shall not give the proof, although it is interesting to note that a key inequality which give the condition (ii) for solving (39) is the following

$$\max(\check{u}, u^*) \leq \bar{u}_{\bar{\mu}} \leq \check{u} + u^*, \tag{45}$$

in which formula u^* and $\check{u}$ have been defined above.

A striking difference between the subcritical case and the supercritical case is the loo of uniqueness. It was first proved by Le Gall [26] in the case $q = 2$ and then by Marcus and Véron [32] in the general case $q \geq (N+1)/(N-1)$ that there may exists infinitely many solutions of (39) with a given singular trace. In particular, for every $\gamma > 0$, there exists a solution u_γ with $\operatorname{tr}(u_\gamma) = (\partial B, 0)$ and such that $u_\gamma(0) < \gamma$. Moreover, by sharpening the construction of the u_γ it is proved in [32] that for any $\varepsilon > 0$ there exists a Borel subset $A \subset \partial B$ with $\operatorname{meas}(\partial B \setminus A) < \varepsilon$ and a solution u of (39) such that $Tr(u) = (\partial B, 0)$ and $\lim_{r \to 1} u(r, \sigma) = 0$ for every $\sigma \in A$. This clearly indicates that the formulation of boundary traces in terms of the usual topology on ∂B is not sufficient to describe the variety of phenomena which may occur at the boundary and that a sharper notion is needed. A first and important step towards this direction has been made by Dynkin and Kuznetsov [15][22, 23] in a series of recent papers. In these works they introduce a topology thiner than the usual one in $\mathbb{R}^d$ and they introduce a new class of problems in which some uniqueness is proved. *However the problem of finding The notion of boundary trace which gives rise to a one to one and onto correspondence between the set of all positive solutions of (39) and their boundary trace is still open.*

Conformal deformations of hyperbolic space

The problem of conformal deformation of Riemannian metrics is one of the most interesting field of applications of semilinear elliptic equations. We perform the identification of the hyperbolic N-space $\mathbb{H}^N$ with (B, g_H), where

$$g_H(x) = \left(\frac{2}{1 - |x|^2}\right)^2 \eta(x)$$

with $\eta_{ij} = \delta_{ij}$. Given $K \in C^{\infty}(B)$ with $N > 2$, three classical problems arising from conformal geometry are the following
I- Does there exists a positive function v on such that the metric $g_v = v^{4/(N-2)}$ has scalar curvature K ?
II- Is g_v a complete Riemannian metric ?
III- If so is v unique in the class of complete Riemannian metrics conformal to g_H with scalar curvature K ?

It is classical that the function v satisfies

$$C_N \Delta v + K(x) v^{(N+2)/(N-2)} = 0 \quad \text{in} \quad B \tag{46}$$

with $C_N = 4(N-1)/(N-2)$. This problem has been thoroughly studied by Loewner and Nirenberg [27](1974), Ni (1982), Aviles and McOwen (1985-1988) and more recently (1993-94) by Ratto, Rigoli and Véron [35]. An interesting case which is associated to supercritical trace problems occurs when K is nonpositive (at least near ∂B). The completeness assumption means that the geodesic distance from inside up to the boundary is always infinite, which in this case is equivalent to

$$\int_0^1 v^{(N-2)/2}(\gamma(t))dt = \infty \tag{47}$$

for any $\gamma \in C^{0,1}([0,1]; \bar{B})$, with $\gamma([0,1)) \subset B$, and $\gamma(1) \in \partial B$. This means that v has some kind of blow-up near $\partial\Omega$. Many existence results concerning this equation have been proven. Some uniqueness results also hold under a strong blow-up assumption, namely

$$\lim_{|x| \to 1} v(x) = \infty,$$

and a positivity assumption on K (see [35] for details), but the problem of uniqueness of v under the mere completeness assumption (47) remains completely open, even when $K = 1$.

References

[1] Bandle C. & Marcus M., *Asymptotic behavior of solutions and their derivative for semilinear elliptic problems with blow-up on the boundary*, Ann. Inst. H. Poincaré 12, 155-171 (1995).

[2] Bandle C. & Marcus M., *Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behavior*, J. Anal. Math. 58, 9-24 (1992).

[3] Baras P. & Pierre M.; *Singularités éliminables pour des équations semilineaires*, Ann. Inst. Fourier Grenoble 34, 185-206 (1984).

[4] Brezis H., *Une équation semi-linéaire avec conditions aux limites dans* L^1, unpublished paper. See also [41]-Chap. 4.

[5] Brezis H. & Strauss W., *Semilinear second order ellitic equations in* L^1 , J. Math. Soc. Japan 25, 565-590 (1973).

[6] Chasseigne E. & Vazquez J.L., *Well-posedness of fast- diffusion equations for optimal classes of data*, Preprint Univ. Autônoma de Madrid (2000).

[7] Chen X.Y., Matano H. & Veron L.; *Anisotropic singularities of nonlinear elliptic equations*, J. Funct. Anal. 83, 50-97 (1989).

[8] Dhersin J.S. & Le Gall E., *Wiener test for super Brownian motion and the Brownian snake*, Prob. Theory relat. Fields 108, 103-129 (1997).

[9] Doob J., *Classical Potential Theory and its Probabilistic Counterpart*, Springer-Verlag, Berlin/New-York (1984).

[10] Dynkin E.B., *A probabilistic approach to one class of nonlinear differential equations*, Prob. Theory Relat. Fields 89, 89-115 (1991).

[11] Dynkin E.B. & Kuznetsov S.E., *Superdiffusions and removable singularities for quasilinear P.D.E.*, Comm. Pure, Appl. Math. 49, 125-176 (1996).

[12] Dynkin E.B. & Kuznetsov S.E., *Linear additive functionals of superdiffusion and related nonlinear P.D.E.*, Trans. A.M.S. 348, 1959-1988 (1996).

[13] Dynkin E.B. & Kuznetsov S.E., *Trace on the boundary for solutions of nonlinear differential equations*, Trans. A.M.S. 350, 4499-4519 (1998).

[14] Dynkin E.B. & Kuznetsov S.E., *Solutions of nonlinear differential equtions on a Riemannian manifold and their trace on the Martin boundary* , Trans. A.M.S. 350, 4521-4552 (1998).

[15] Dynkin E.B. & Kuznetsov S.E., *Fine topology and fine trace on the boundary associated with a class of quasilinear differential equations*, Comm. Pure Appl. Math. 51, 897-936 (1998).

[16] Gilbarg D. & Trudinger N.S., *Partial Differential Equations of Second Order*, 2nd Ed. Springer-Verlag, Berlin/New-York (1983)

[17] Gmira A. & Véron L., *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J 64, 271-324 (1991).

[18] Grillot M. & Véron L., *Boundary trace of solutions of the Prescribed Gaussian curvature equation*, Proc. Roy. Soc. Edinburgh 130 A, 1-34 (2000).

[19] Iscoe I.,*On the support of measure-valued critical branching Brownian motion*, Ann. Prob. 16, 200-221 (1988).

[20] Keller J.B., *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. 10, 503-510 (1957).

[21] Kondratiev V. & Nikishkin A.,*On positive solutions of singular value problemsfor the equation* $\Delta u = u^k$, Russian J. Math. Phys. 1, 123-138 (1993).

[22] Kuznetsov S.E., *σ-moderate solutions of $Lu = u^{\alpha}$ and fine trace on the boundary* C.R. Acad. Sci. Paris 326 Ser. I, 1189-1194 (1998).

[23] Kuznetsov S.E., *On uniqueness of a solution of $Lu = u^{\alpha}$ with given trace*, Electr. Comm. in Probability (to appear).

[24] Le Gall J.F., *Les solutions positives de $\Delta u = u^2$ dans le disque unité*, C.R. Acad. Sci. Paris 317 Ser. I, 873-878 (1993).

[25] Le Gall J.F., *The brownian snake and solutions of $\Delta u = u^2$ in a domain*, Prob. Theory Rel. Fields 102, 393-432 (1995).

[26] Le Gall J.F., *A probabilistic approach to the trace at the boundary for solutions of a semilinear parabolic partial differential equation*, J. Appl. Math. Stoch. Anal. 9, 399-414 (1996).

[27] Loewner C. & Nirenberg L., *Partial differential equations invariant under conformal or projective transformations*, in Contributions to Analysis, 245-272, Academic Press, Orlando Flo (1974).

[28] Marcus M. & Véron L., *Uniqueness and asymptotic behaviour of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré 14, 237-274 (1997).

[29] Marcus M. & Véron L., *Traces au bord des solutions positives d'équations elliptiques non-linéaires*, C.R. Acad. Sci. Paris 321 Ser. I, 179-184 (1995).

[30] Marcus M. & Véron L., *Traces au bord des solutions positives d'équations elliptiques et paraboliques non-linéaires: résultats d'existence et d'unicité*, C.R. Acad. Sci. Paris 323 Ser. I, 603-608 (1996).

[31] Marcus M. & Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rat. Mech. Anal. 144, 201-231 (1998).

[32] Marcus M. & Véron L., *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. 77, 481-524 (1998).

[33] Meyers N.G., *A theory of capacities for potential of functions in Lebesgue classes*, Math. Scan. 26, 255-292 (1970).

[34] Osserman R., *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. 7, 1641-1647 (1957).

[35] Ratto A., Rigoli M. & Véron L., *Scalar curvature and conformal deformation of hyperbolic space* J. Funct. Anal. 121, 15-77 (1994).

[36] Sheu Y.C., *Removable singularities for solutions of some nonlinear differerential equations*, Duke Math. J. 74, 701-711 (1994).

[37] Stein E., *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press 30 (1970).

[38] Treibel H., *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Publ. Co (1978).

[39] Véron L., *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. T.,M. & A 5, 225-242 (1981).

[40] Véron L., *Comportement asymptotique des solutions d'équations elliptiques semi-linéaires dans* $\mathbb{R}^N$, Ann. Mat. Pura Appl. 127, 25-50 (1981).

[41] Véron L., *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Research Notes in Math. 353, Addison Wesley Longman Inc. (1996).

[42] Véron L., *Semilinear elliptic equations with uniform blow-up on the boundary*, J. Anal. Math. 59, 231-250 (1992).

Laurent Véron
Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 6083
Faculté des Sciences et Techniques
Parc de Grandmont
F 37200 Tours France
e-mail: veronl@univ-tours.fr