16th Conference on Applied Mathematics, Univ. of Central Oklahoma,
Electronic Journal of Differential Equations, Conf. 07, 2001, pp. 99–102.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Almost periodic solutions of nonlinear hyperbolic equations with time delay *

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Abstract

The almost periodicity of bounded solutions is established for a nonlinear hyperbolic equation with piecewise continuous time delay. The equation represents a mathematical model for the dynamics of gas absorption.

1 Introduction

In this paper we are interested in determining almost periodicity for a unique bounded solution of nonlinear hyperbolic equations with time delay. The initial value problem under investigation is the following:

$$u_{xt}(x,t) + a(x,t)u_x(x,t) = C(x,t,u(x,[t])) \tag{1}$$

$$u(0,t) = u_0(t), \tag{2}$$

where a and C are defined in the domain $D : (0,l) \times \mathbb{R} \to \mathbb{R}$, and $[t]$ denotes the greatest integer function: $[t] = n$ when $n \le t < n+1$, for an integer n. In this case the delay function is piecewise constant. The existence of a unique bounded solution of problem (1)–(2) has been discussed earlier [1].

Equation (1) with condition (2) under assumption

$$a(x,t) \ge m > 0 \quad \text{in } D$$

has a unique bounded solution via Volterra integral equation

$$u(x,t) = u_0(t) + \int_0^x \int_{-\infty}^t e^{-\int_\tau^t a(\xi,\theta)\,d\theta} C(\xi,\tau,u(\xi,n))\,d\tau\,d\xi \tag{3}$$

Let us notice that, in case of periodicity, the period has to be the same for all the functions involved [2]. This result is based on the equivalence of (1)–(2) with integral equation (3), and it can be stated as the following assertion.

Theorem 1 *If $u_0(t)$, $a(x,t)$, and $C(x,t,u(x,[t])$ are periodic in t with period T, then the unique bounded solution of (3) is also periodic in t, with the same period T.*

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* *Mathematics Subject Classifications:* 35B10, 35B15, 35J60, 35L70.
Key words: Nonlinear hyperbolic equation, time delay, almost periodic solution.

Published July 20, 2001.

Proof From (3), one obtains

$$u(x,t+T) = u_0(t+T) + \int_0^x \int_{-\infty}^{t+T} e^{-\int_\tau^{t+T} a(\xi,\theta)\,d\theta} C(\xi,\tau,u(\xi,n))\,d\tau\,d\xi.$$

Making the substitution $\tau = \eta + T$ and taking into account

$$\int_{\eta+T}^{t+T} a(\xi,\theta)\,d\theta = \int_{\eta+T}^{\eta} a(\xi,\theta)\,d\theta + \int_{\eta}^{t} a(\xi,\theta)\,d\theta + \int_{t}^{t+T} a(\xi,\theta)\,d\theta$$

we have

$$u(x,t+T) = u_0(t) + \int_0^x \int_{-\infty}^{t} e^{-\int_\eta^{t} a(\xi,\theta)\,d\theta} C(\xi,\eta,u(\xi,n))\,d\eta d\xi = u(x,t)$$

which proves the periodicity of u in t with period T.

Definition 2 (Bohr's Definition of ϵ–almost periodicity) For any $\epsilon > 0$, there exists a number $l(\epsilon) > 0$ with property that any interval of length $l(\epsilon)$ of the real line contains at least one point with abscissa δ, such that

$$|u(x,t+\delta) - u(x,t)| < \epsilon, \ (x,t) \in D\,,$$

the number δ is called translation number of $u(x,t)$ corresponding to ϵ, or an ϵ-almost period of $u(x,t)$.

The following lemma will be used to prove that the unique bounded solution (in D) of equation (3) is almost periodic in t.

Lemma 3 *Assume the following conditions hold true in regard to the equation*

$$V_t(x,t) + a(x,t)V(x,t) = f(x,t), \ \textit{in } D : (0,l)\times\mathbb{R} \to \mathbb{R} \tag{4}$$

1. *$a(x,t)$, $f(x,t)$ are almost periodic in t, uniformly with respect to x;*
2. *$a(x,t) \geq m > 0$ in D.*

Then the unique bounded solution of (4), given by

$$V(x,t) = \int_{-\infty}^{t} e^{-\int_\tau^{t} a(x,\theta)\,d\theta} f(x,\tau)\,d\tau, \tag{5}$$

is almost periodic in t, uniformly with respect to x, and

$$|V(x,t)| \leq \frac{1}{m} \sup |f(x,t)|\,, \quad (x,t) \in D. \tag{6}$$

Proof We obtain from (4), changing t to $t+\delta$:

$$V_t(x,t+\delta)+a(x,t+\delta)V(x,t+\delta)=f(x,t+\delta),$$

and subtracting (4) from it,

$$\begin{aligned}&[V(x,t+\delta)-V(x,t)]_t+a(x,t+\delta)\left[V(x,t+\delta)-V(x,t)\right]\\ =\;& f(x,t+\delta)-f(x,t)-\left[a(x,t+\delta)-a(x,t)\right]V(x,t).\end{aligned}$$

Taking into account the almost periodicity of $a(x,t)$, $f(x,t)$ and boundedness of $V(x,t)$ in D, one obtains in D, according to (6):

$$\begin{aligned}\sup|V(x,t+\delta)-V(x,t)| \;&\le\; \frac{1}{m}\sup|f(x,t+\delta)-f(x,t)|\\ &\quad+\frac{M}{m}\sup|a(x,t+\delta)-a(x,t)|\,,\end{aligned}$$

where $M=\sup|V(x,t)|$, $(x,t)\in D$. We choose δ such that,

$$|f(x,t+\delta)-f(x,t)|<\frac{m\epsilon}{2},\quad\text{and}\quad |a(x,t+\delta)-a(x,t)|<\frac{m\epsilon}{2M}$$

for sufficiently large t, i.e., $f(x,t)$ must be an $(m\epsilon)/2$-almost periodic and $a(x,t)$ is $(m\epsilon)/2M$-almost periodic. Then

$$\sup|V(x,t+\delta)-V(x,t)|\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\quad\text{for all such }\delta\in\mathbb{R}. \tag{7}$$

In other words, for any $\epsilon>0$, there exists a number $l(\epsilon)>0$ with the property that any interval $(a,a+l)\in\mathbb{R}$ contains an ϵ-almost period of $V(x,t)$. This means that $V(x,t)$ is an almost periodic function in t, uniformly with respect to $x\in[0,l]$ by Bohr's definition of almost periodicity. Let us conclude now with the result on almost periodicity of the unique bounded solution of (3) in D.

Theorem 4 *Consider equation (1) in D, and assume* $u_0(t)$, $a(x,t)$, *and* $C(x,t,u(x,[t]))$ *are almost periodic in t, uniformly with respect to* $x\in[0,l]$, *and* $a(x,t)\ge m>0$. *Also assume that* $C(x,t,u(x,[t]))$ *is continuous on* $D\times\mathbb{R}$, *with* $C(x,t,0)$ *bounded on D, and satisfies the Lipschitz condition*

$$|C(x,t,u(x,[t]))-C(x,t,V(x,[t]))|\le L\,|u(x,[t])-V(x,[t])|$$

where L is a positive constant. Then the unique bounded solution of (1)–(2) in D is almost periodic in t, uniformly with respect to $x\in[0,l]$.

Proof Let the first approximation be $u_0(x,t)\equiv 0$. Next approximation is then

$$u_1(x,t)=u_0(t)+\int_0^x\int_{-\infty}^t e^{-\int_\tau^t a(\xi,\theta)\,d\theta}C(\xi,\tau,0)\,d\tau\,d\xi\,.$$

Since $V(x,t)=\frac{\partial}{\partial x}u_1(x,t)$, then from the equation

$$V_t(x,t)+a(x,t)V(x,t)=C(x,t,0)$$

by Lemma 2 we obtain the almost periodicity of $V(x,t)$. But

$$u_1(x,t) = u_0(t) + \int_0^x V(\xi,t)\,d\xi.$$

This shows that $u_1(x,t)$ is almost periodic in t, uniformly with respect to $x \in [0,l]$, and

$$u_2(x,t) = u_0(t) + \int_0^x \int_{-\infty}^t e^{-\int_\tau^t a(\xi,\theta)\,d\theta} C(\xi,\tau,u_1(\xi,\tau))\,d\tau\,d\xi.$$

The relation $\overline{V}(x,t) = \frac{\partial}{\partial x} u_2(x,t)$ and equation

$$\overline{V_t}(x,t) + a(x,t)\overline{V}(x,t) = C(x,t,u_1(x,t))$$

implies almost periodicity of

$$u_2(x,t) = u_0(t) + \int_0^x \overline{V}(\xi,t)\,d\xi,$$

by Lemma 2. Then $u_3(x,t)$ is almost periodic by a similar argument. Consequently, all successive approximations $u_n(x,t), \quad n = 1,2,\ldots$ are almost periodic functions in t, uniformly with respect to $x \in [0,l]$. Hence the solution

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

is also almost periodic in t, uniformly with respect to $x \in [0,l]$.

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