2002-Fez conference on Partial Differential Equations,
Electronic Journal of Differential Equations, Conference 09, 2002, pp 139–147.
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Nonlinear elliptic systems with exponential nonlinearities *

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Abstract

In this paper we investigate the existence of solutions for

$$\begin{gathered} -\operatorname{div}(a(|\nabla u|^N)|\nabla u|^{N-2}u) = f(x,u,v) \quad \text{in } \Omega \\ -\operatorname{div}(a(|\nabla v|^N)|\nabla v|^{N-2}v) = g(x,u,v) \quad \text{in } \Omega \\ u(x) = v(x) = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, f and g are nonlinearities having an exponential growth on Ω and a is a continuous function satisfying some conditions which ensure the existence of solutions.

1 Introduction

Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary $\partial\Omega$.
In this paper we shall be concerned with existence of solutions for the problem

$$\begin{gathered} -\operatorname{div}(a(|\nabla u|^N)|\nabla u|^{N-2}u) = f(x,u,v) \quad \text{in } \Omega \\ -\operatorname{div}(a(|\nabla v|^N)|\nabla v|^{N-2}v) = g(x,u,v) \quad \text{in } \Omega \\ u(x) = v(x) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Where the nonlinearities $f, g : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions having an exponential growth on Ω: i.e.,

(H1) For all $\delta > 0$

$$\lim_{|(u,v)|\to\infty} \frac{|f(x,u,v)| + |g(x,u,v)|}{e^{\delta(|u|^N+|v|^N)^{1/(N-1)}}} = 0 \quad \text{Uniformly in } \Omega.$$

Let us mention that there are many results in the scalar case for problem involving exponential growth in bounded domains; see for example [4], [6]. The

* *Mathematics Subject Classifications:* 35J70, 35B45, 35B65.
Key words: Nonlinear elliptic system, exponential growth, Palais-Smale condition.

Published December 28, 2002.

objective of this paper is to extend these results to a more general class of elliptic systems using variational method. Here we will make use the approach stated by Rabinowitz [8].

Note that for nonlinearities having polynomial growth, several results of such problem have been established. We can cite, among others, the articles: [9] and [10]. In order to prove the compactness condition of the functional associated to a problem (1.1) we assume the following hypothesis

(H2) $u\frac{\partial F}{\partial u} \geq \frac{\mu}{2}F$ and $v\frac{\partial F}{\partial v} \geq \frac{\mu}{2}F$, where $F = F(x,u,v)$ and such that $\frac{\partial F}{\partial u} = f(x,u,v)$, $\frac{\partial F}{\partial v} = g(x,u,v)$ with $F(x,u,v) > 0$ for $u > 0$ and $v > 0$, $F(x,u,v) = 0$ for $u \leq 0$ or $v \leq 0$ with $\mu > N$ and $U = (u,v) \in \mathbb{R}^2$.

We shall find weak-solution of (1.1) in the space $W = W_0^{1,N}(\Omega) \times W_0^{1,N}(\Omega)$ endowed with the norm

$$\|U\|_W^N = \int_\Omega |\nabla U|^N \, dx = \int_\Omega (|\nabla u|^N + |\nabla v|^N) \, dx$$

where $U = (u,v) \in W$. Motivated by the following result due to Trudinger and Moser (cf. [7].[11]), we remark that the space W is embeded in the class of Orlicz-Lebesgue space

$$L_\phi = \{U : \Omega \to \mathbb{R}^2, \text{ measurable } : \int_\Omega \phi(U) < \infty\},$$

where $\phi(s,t) = \exp\left(s^{\frac{N}{N-1}} + t^{\frac{N}{N-1}}\right)$. Moreover,

$$\sup_{\|(u,v)\|_W \leq 1} \int_\Omega \exp\left(\gamma(|u|^{\frac{N}{N-1}} + |v|^{\frac{N}{N-1}}\right) dx \leq C \quad \text{if } \gamma \leq \omega_{N-1},$$

where C is a real number and ω_{N-1} is the dimensional surface of the unit sphere.

On this paper, we make the following assumptions on the function a.

(a1) $a : \mathbb{R}^+ \to \mathbb{R}$ is continuous

(a2) There exist positive constants $p \in]1,N]$, b_1, b_2, c_1, c_2 such that

$$c_1 + b_1 u^{N-p} \leq u^{N-p} a(u^N) \leq c_2 + b_2 u^{N-p} \quad \forall u \in \mathbb{R}^+;$$

(a3) The function $k : \mathbb{R} \to \mathbb{R}$, $k(u) = a(|u|^N)|u|^{N-2}u$ is strictly increasing and $k(u) \to 0$ as $u \to 0^+$.

Remark Note that operator considered here has been studied by Hirano [5] and by Ubilla [11] with nonlinearities having polynomial growth.

We shall denote by λ_1 the smallest eigenvalue [9] for the problem

$$-\Delta_N u = \lambda |u|^{\alpha-1} u |v|^{\beta+1} \quad \text{in } \Omega \subset \mathbb{R}^N$$
$$-\Delta_N v = \lambda |u|^{\alpha+1} |v|^{\beta-1} v \quad \text{in } \Omega \subset \mathbb{R}^N$$
$$u(x) = v(x) = 0 \quad \text{on } \partial\Omega;$$

i.e.,

$$\lambda_1 = \inf\Big\{ \frac{\alpha+1}{N}\int_\Omega |\nabla u|^N\,dx + \frac{\beta+1}{N}\int_\Omega |\nabla v|^N\,dx : \\ (u,v)\in W,\ \int_\Omega |u|^{\alpha+1}|v|^{\beta+1}\,dx = 1\Big\}$$

where $\alpha+\beta = N-2$ and $\alpha,\beta > -1$.

Definition We say that a pair $(u,v)\in W$ is a weak solution of (1.1) if for all $(\varphi,\psi)\in W$,

$$\begin{aligned} \int_\Omega a(|\nabla u|^N)|\nabla u|^{N-2}\nabla u\nabla\varphi\,dx &= \int_\Omega f(x,u,v)\varphi\,dx \\ \int_\Omega a(|\nabla v|^N)|\nabla v|^{N-2}\nabla v\nabla\psi\,dx &= \int_\Omega g(x,u,v)\psi\,dx \end{aligned} \tag{1.2}$$

Now state our main results.

Theorem 1.1 *Suppose that f and g are continuous functions satisfying (H1), (H2) and that a satisfies (a1), (a2) and (a3), with $Nb_2 < \mu b_1$. Furthermore, assume that*

$$\lim_{|U|\to 0}\sup \frac{pF(x,U)}{|u|^{\alpha+1}|v|^{\beta+1}} < (c_1 + b_1\delta_p(N))\lambda_1 \tag{1.3}$$

uniformly on $x\in\Omega$, where $\delta_p(N) = 1$ if $N = p$ and $\delta_p(N) = 0$ if $N \neq p$. Then problem (1.1) *has a nontrivial weak solution in W.*

Remarks

1) Here we note that in case that $(a2)$ holds for $p = N$, the condition $(a2)$ can be rewritten as follows:
   $(a2')$ There exist c_1, c_2 such that

   $$c_1 \le a(u^N) \le c_2 \quad \text{for all} \quad u\in\mathbb{R}^+.$$

   If $a(t) = 1, (a2')$ holds with $c_1 = c_2 = 1$ and therefore, we obtain the result given in [3].

2) If $a(u) = 1 + u^{\frac{p-N}{N}}$, conditions $(a2)$ and $(a3)$ hold, then the problem (1.1) can be formulated as follows

   $$\begin{aligned} -\Delta_N u - \Delta_p u &= f(x,u,v) \\ -\Delta_N v - \Delta_p v &= g(x,u,v); \end{aligned}$$

   where $\Delta_p \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian operator.

2 Preliminaries

The maximal growth of $f(x,u,v)$ and $g(x,u,v)$ will allow us to treat variationally system (1.1) in the product Sobolev space W. This exponential growth is relatively motivated by Trudinger-Moser inequality ([4], [11]).

Note that if the functions f and g are continuous and have an exponential growth, then there exist positive constants C and γ such that

$$|f(x,u,v)| + |g(x,u,v)| \le C \exp\left(\gamma(|u|^{\frac{N}{N-1}} + |v|^{\frac{N}{N-1}})\right), \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2. \tag{2.1}$$

Consequently the functional $\Psi : W \to \mathbb{R}$ defined as

$$\Psi(u,v) = \int_\Omega F(x,u,v)\, dx$$

is well defined, belongs to $C^1(W,\mathbb{R})$, and has

$$\Psi'(u,v)(\varphi,\psi) = \int_\Omega f(x,u,v)\varphi + g(x,u,v)\psi\, dx.$$

To prove this statements, we deduce from (2.1) that there exists $C_1 > 0$ such that

$$|F(x,u,v)| \le C_1 \exp(\gamma(|u|^{\frac{N}{N-1}} + |v|^{\frac{N}{N-1}})), \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^2.$$

Thus, since

$$\exp\left(\gamma(|u|^{\frac{N}{N-1}} + |v|^{\frac{N}{N-1}})\right) \in L^1(\Omega), \quad \forall (u,v) \in W,$$

we have the result.

It follows from the assumptions on the function a that for all $t \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{N} A(|t|^N) &\ge \frac{b_1}{N}|t|^N + \frac{c_1}{p}|t|^p \\ \frac{1}{N} A(|t|^N) &\le \frac{b_2}{N}|t|^N + \frac{c_2}{p}|t|^p, \end{aligned}$$

where $A(t) = \int_0^t a(s)\, ds$. Furthermore the function $g(t) = A(|t|^N)$ is strictly convex. Consequently, the functional $\Phi : W \to \mathbb{R}$ defined as

$$\Phi(u,v) = \frac{1}{N}\int_\Omega A(|\nabla u|^N) + A(|\nabla v|^N)\, dx$$

is well defined, weakly lower semicontinuous, Frechet differentiable and belongs to $C^1(W,\mathbb{R})$.

Therefore, if the function a satisfies conditions (a1), (a2) and (a3) and the nonlinearities f and g are continuous and satisfy (2.1), we conclude that the functional $J : W \to \mathbb{R}$, given by

$$J(u,v) = \frac{1}{N}\int_\Omega A(|\nabla u|^N) + A(|\nabla v|^N)\, dx - \int_\Omega F(x,u,v)\, dx$$

is well defined and belongs to $C^1(W,\mathbb{R})$. Also for all $(u,v)\in W$,

$$J'(u,v)(\varphi,\psi)=\int_\Omega a(|\nabla u|^N)|\nabla u|^{N-2}\nabla u\nabla\varphi+a(|\nabla v|^N)|\nabla v|^{N-2}\nabla u\nabla\psi\,dx \\ -\int_\Omega f(x,u,v)\varphi+g(x,u,v)\psi\,dx\,.$$

Consequently, we are interested in using Critical Point theory to obtain weak solutions of (1.1).

Lemma 2.1 *Assume that f and g are continuous and have an exponential growth. Let (u_n,v_n) be a sequence in W such that (u_n,v_n) converge weakly on $(u,v)\in X$, then*

$$\int_\Omega f(x,u_n,v_n)(u_n-u)\,dx\to 0, \\ \int_\Omega g(x,u_n,v_n)(v_n-v)\,dx\to 0,$$

as $n\to\infty$.

Proof. Let (u_n,v_n) be a sequence converging weakly to some (u,v) in W. Thus, there exist a subsequence, denoted again by (u_n,v_n) such that

$$u_n\to u \quad \text{in } L^p(\Omega), \\ v_n\to v \quad \text{in } L^q(\Omega),$$

as $n\to\infty$ and for all $p,q>1$. On the other hand, we have

$$\begin{aligned}\int_\Omega |f(x,u_n,v_n)|^p\,dx &\le C\int_\Omega \exp(p\gamma(|u_n|^{\frac{N}{N-1}}+|v_n|^{\frac{N}{N-1}}))\,dx \\ &\le C(\int_\Omega \exp(sp\gamma|u_n|^{\frac{N}{N-1}}))^{\frac{1}{s}}(\int_\Omega \exp(s'p\gamma|v_n|^{\frac{N}{N-1}}))^{\frac{1}{s'}} \\ &\le C\Big(\int_\Omega \exp(sp\gamma\|u_n\|^{\frac{N}{N-1}}_{W_0^{1,N}(\Omega)}(\frac{|u_n|^{\frac{N}{N-1}}}{\|u_n\|_{W_0^{1,N}(\Omega)}}))\Big)^{1/s} \\ &\quad\times\Big(\int_\Omega \exp(s'p\gamma\|v_n\|^{\frac{N}{N-1}}_{W_0^{1,N}(\Omega)}(\frac{|v_n|^{\frac{N}{N-1}}}{\|v_n\|_{W_0^{1,N}(\Omega)}}))\Big)^{1/s'}.\end{aligned}$$

Since (u_n,v_n) is a bounded sequence, we may choose γ sufficiently small such that

$$sp\gamma\|u_n\|_{W_0^{1,N}(\Omega)}{}^{\frac{N}{N-1}}<\alpha_N \quad\text{and}\quad s'p\gamma\|v_n\|_{W_0^{1,N}(\Omega)}{}^{\frac{N}{N-1}}<\alpha_N.$$

Then

$$\int_\Omega |f(x,u_n,v_n)|^p\,dx\le C_1$$

for n large and some constant $C_1 > 0$. By the same argument, we have also

$$\int_\Omega |g(x, u_n, v_n)|^q \, dx \le C_2$$

for n large and some constant $C_2 > 0$. Using Hölder inequality, we obtain

$$\begin{aligned}\int_\Omega f(x, u_n, v_n)(u_n - u)\, dx &\le \left[\int_\Omega |f(x, u_n, v_n)|^{p'}\right]^{1/p'} [|u_n - u|^p]^{1/p} \\ &\le C\,[|u_n - u|^p]^{1/p}\end{aligned}$$

and

$$\begin{aligned}\int_\Omega g(x, u_n, v_n)(v_n - v)\, dx &\le \left[\int_\Omega |g(x, u_n, v_n)|^{q'}\right]^{1/q'} [|v_n - v|^q]^{1/q} \\ &\le C'\,[|v_n - v|^q]^{1/q}.\end{aligned}$$

Thus the proof is completed since $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^q(\Omega)$. □

Lemma 2.2 *Assume that f and g are continuous satisfying (H1). Then the functional J satisfies Palais-Smale condition (PS) provided that every sequence (u_n, v_n) in W is bounded.*

Proof. Note that

$$\begin{aligned}J'(u_n, v_n)(\varphi, \psi) =& \Phi'(u_n, v_n)(\varphi, \psi) - \int_\Omega f(x, u_n, v_n)\varphi + g(x, u_n, v_n)\psi\, dx \\ \le& \varepsilon_n \|(\varphi, \psi)\|_W,\end{aligned} \tag{2.2}$$

for all $(\varphi, \psi) \in W$, where $\varepsilon_n \to 0$ as $n \to \infty$. Since $\|(u_n, v_n)\|_W$ is bounded, we can take a subsequence, denoted again by (u_n, v_n) such that

$$\begin{aligned} u_n \to u \quad &\text{in } L^p(\Omega), \\ v_n \to u \quad &\text{in } L^q(\Omega),\end{aligned}$$

as n approaches ∞ and $\forall p, q > 1$. Then considering in one hand $\varphi = u_n - u$ and $\psi = 0$ in (2.2) and with the help of Lemma 2.1 , we obtain

$$\Phi'(u_n, v_n)(u_n - u, 0) \to 0,$$

as n approaches ∞. Since $u_n \rightharpoonup u$ weakly, as $n \to \infty$ and $\Phi' \in (S_+)$, the result is proved. We have the same result for v_n by considering $\psi = v_n - v$ and $\varphi = 0$ in (2.2). Finally, we conclude that $(u_n, v_n) \to (u, v)$ as $n \to \infty$. □

Lemma 2.3 *Assume that the function a satisfies (a1), (a2) and (a3) with $Nb_2 < \mu b_1$, and that the nonlinearities f and g are continuous and satisfy (H1). Then the functional J satisfies the Palais-Smale condition (PS).*

Proof. Using (a1), (a2) and (a3) with $Nb_2 < \mu b_1$, we obtain positive constants c, d such that

$$\frac{\mu}{N}A(t) - a(t)t \geq ct - d \quad \forall t \in \mathbb{R}^+. \tag{2.3}$$

Now, let (u_n, v_n) be a sequence in W satisfying condition (PS). Thus

$$\frac{1}{N}\int_\Omega A(|\nabla u_n|^N) + \frac{1}{N}\int_\Omega A(|\nabla v_n|^N)\,dx - \int_\Omega F(x, u_n, v_n)\,dx \to c \tag{2.4}$$

as n goes to ∞.

$$\begin{aligned} &\Big|\int_\Omega a(|\nabla u_n|^N)|\nabla u_n|^N + a(|\nabla v_n|^N)|\nabla v_n|^N \\ &\quad -(\int_\Omega f(x, u_n, v_n)u_n + g(x, u_n, v_n)v_n)\,dx\Big| \leq \varepsilon_n\|(u_n, v_n)\|_W, \end{aligned} \tag{2.5}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Multiplying (2.4) by μ, subtracting (2.5) from the expression obtained and using (2.3), we have

$$\begin{aligned} \Big|\int_\Omega |\nabla u_n|^N + |\nabla v_n|^N - \int_\Omega (\mu F(x, u_n, v_n) - (f(x, u_n, v_n)u_n + g(x, u_n, v_n)v_n)\,dx\Big| \\ \leq c + \varepsilon_n\|(u_n, v_n)\|_W. \end{aligned}$$

From this inequality and using hypothesis (H1), we deduce that (u_n, v_n) is bounded sequence in W. Now, with the help of Lemma 2.2, we conclude the proof. □

3 Proofs of the existence results

Lemma 3.1 *Assume that the hypotheses of Theorem 1.1 hold. Then, there exist* $\eta, \rho > 0$ *such that* $J(u, v) \geq \eta$ *if* $\|(u, v)\|_X = \rho$. *Moreover,* $J(t(u, v)) \to -\infty$ *as* $t \to +\infty$ *for all* $(u, v) \in W$.

Proof. By (1.3) and (2.1), we can choose $\eta_1 < c_1 + b_1\delta_p(N)$ such that for $r > N$,

$$F(x, u, v) \leq \frac{1}{p}\eta_1\lambda_1|u|^{\alpha+1}|v|^{\beta+1} + C|u|^r e^{\gamma|u|^{\frac{N}{N-1}}} e^{\gamma|v|^{\frac{N}{N-1}}},$$

for all $(x, u, v) \in \Omega \times W$. For $\|u\|_{W_0^{1,N}}$ and $\|v\|_{W_0^{1,N}}$ small, from Hölder's and Trudinger-Moser's inequalities, we obtain

$$\begin{aligned} J(u, v) \quad \geq \quad & \frac{b_1}{N}\|u\|^N_{W_0^{1,N}} + \frac{c_1}{p}\|u\|^p_{W_0^{1,N}} - \frac{\eta_1}{p}\|u\|^p_{W_0^{1,N}} - C_1\|u\|^r_{W_0^{1,N}} \\ & + \frac{b_1}{N}\|v\|^N_{W_0^{1,N}} + \frac{c_1}{p}\|v\|^p_{W_0^{1,N}} - \frac{\eta_1}{p}\|v\|^p_{W_0^{1,N}} - C_1\|v\|^r_{W_0^{1,N}}. \end{aligned}$$

Since $\eta_1 < c_1 + b_1\delta_p(N)$ and $p \leq N < r$, we can choose $\rho > 0$ such that $J(u,v) \geq \eta$ if $\|(u,v)\|_W = \rho$ for some $\eta > 0$. On the other hand, we can prove easily that

$$J(t(u,v)) \rightarrow -\infty \quad \text{as} \quad t \rightarrow +\infty$$

So, by the Mountain-Pass Lemma [2], problem (1.1) has nontrivial solution $(u,v) \in W$ which is a critical point of J. This completes the proof of Theorem 1.1.
At the end, we give an example which illustrates conditions given on the nonlinearities f and g.

Example Let

$$F(x,u,v) = (1+\delta_p(N))\frac{\lambda}{p}|u|^{\alpha+1}|v|^{\beta+1} + (1-\chi(u,v))exp\left(\frac{\sigma(|u|^N+|v|^N)^{\frac{1}{N-1}}}{Log(|u|+|v|+2)}\right)$$

where $\chi \in C^1(\mathbb{R}^2,[0,1]), \chi \equiv 1$ on some ball $B(0,r) \subset \mathbb{R}^2$ with $r > 0$, and $\chi \equiv 0$ on $\mathbb{R}^2 \backslash B(0,r+1)$.
Thus, it follows immediately that $(H_1),(H_2)$ and (1.3) are satisfied. Then problem (1.1) has a nontrivial weak solution provided that $\lambda < \lambda_1$.

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