2002-Fez conference on Partial Differential Equations,
Electronic Journal of Differential Equations, Conference 09, 2002, pp 161–170.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

On the spectrum of the p-biharmonic operator *

Abdelouahed El Khalil, Siham Kellati & Abdelfattah Touzani

Abstract

This work is devoted to the study of the spectrum for p-biharmonic operator with an indefinite weight in a bounded domain.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, not necessary regular; $1 < p < \infty$ and $\rho \in L^r(\Omega)$, $\rho \neq 0$, an unbounded weight function which can change its sign, with $r = r(N,p)$ satisfying the conditions

$$r \begin{cases} > \frac{N}{2p} & \text{for } \frac{N}{p} \geq 2 \\ = 1 & \text{for } \frac{N}{p} < 2. \end{cases}$$

We assume that $|\Omega_\rho^+| \neq 0$, where $\Omega_\rho^+ = \{x \in \Omega; \rho(x) > 0\}$ and $\lambda \in \mathbb{R}$. We consider the eigenvalue problem

$$\begin{gathered} \Delta_p^2 u = \lambda \rho(x) |u|^{p-2} u \quad \text{in } \Omega \\ u \in W_0^{2,p}(\Omega). \end{gathered} \tag{1.1}$$

Here $\Delta_p^2 := \Delta(|\Delta u|^{p-2}\Delta u)$, the operator of fourth order called the p-biharmonic operator. For $p = 2$, the linear operator $\Delta_2^2 = \Delta^2 = \Delta.\Delta$ is the iterated Laplacian that multiplied with positive constant appears often in Navier-Stokes equations as being a viscosity coefficient. Its reciprocal operator denoted $(\Delta^2)^{-1}$ is the celebrated Green's operator [5].

It is important to indicate that here we don't suppose any boundary conditions on the high order partial derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}$ on the boundary set $\partial\Omega$ of the domain Ω. The particular case $\rho \equiv 1$ and $u = \Delta u = 0$ on $\partial\Omega$ was considered recently by Drábek and Ôtani [2]. There the authors proved the existence, the simplicity, and the isolation of the first eigenvalue of (1.1) by using a transformation of a problem to a known Poisson's problem, and using the well-known advanced regularity of Agmon-Douglis-Niremberg [3]. Note that this transformation processus is not applicable to our situation because the quantity Δu

* *Mathematics Subject Classifications:* 35P30, 34C23.
Key words: p-biharmonic operator, Duality mapping, Palais-Smale condition, unbounded weight.

Published December 28, 2002.

does not necessary vanished on $\partial\Omega$ and the eventual regularity is not required in any bounded domain.

The main objective of this work is to show that problem (1.1) has at least one non-decreasing sequence of positive eigenvalues $(\lambda_k)_{k\geq 1}$, by using the Ljusternich-schnirelmann theory on C^1 manifolds, see e.g. [6]. Our approach is based only on some properties of the considered operator. So that we give a direct characterization of λ_k involving a minimax argument over sets of genus greater than k.

We set

$$\lambda_1 = \inf\left\{\|\Delta v\|_p^p, v \in W_0^{2,p}(\Omega); \int_\Omega \rho(x)|v|^p dx = 1\right\},$$

where $\|.\|_p$ denotes the $L^p(\Omega)$-norm. It is not difficult to show that $\|\Delta u\|_p$ defines a norm in $W_0^{2,p}(\Omega)$ and $W_0^{2,p}(\Omega)$ equipped with this norm is a uniformly convex Banach space for $1 < p < +\infty$. The norm $\|\Delta .\|_p$ is uniformly equivalent on $W_0^{2,p}(\Omega)$ to the usual norm of $W_0^{2,p}(\Omega)$ [3].

This paper is organized as follows: In section 2, we establish some definitions and show certain basic lemmas. In section 3, we use a variational technique to prove the existence of a sequence of the positive eigenvalues of p-biharmonic operator with any unbounded weight.

2 Preliminaries

Throughout this paper, all solutions are weak, i.e, $u \in W_0^{2,p}(\Omega)$ is a solution of (1.1), if for all $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi \, dx = \lambda \int_\Omega \rho(x)|u|^{p-2}u\varphi \, dx.$$

If $u \in W_0^{2,p}(\Omega)\backslash\{0\}$, then u shall be called an eigenfunction of the p-biharmonic operator (or of (1.1)) associated to the eigenvalue λ. The following proposition states some fundamental properties of the p-biharmonic operator.

Proposition 2.1 *For any bounded domain Ω and $1 < p < +\infty$, Δ_p^2 satisfies the following:*

(i) Δ_p^2 *is an hemicontinuous operator from* $W_0^{2,p}(\Omega)$ *into* $W^{-2,p'}(\Omega)$.

(ii) Δ_p^2 *is a bounded monotonous, and coercive operator.*

(iii) $\Delta_p^2 : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$ *is a bicontinuous operator. Here* $p' = \frac{p}{p-1}$.

Proof (i) Define on $W_0^{2,p}(\Omega)$ the potential functional

$$A(u) = \frac{1}{p}\|\Delta u\|_p^p.$$

This functional is convex and of class C^1 on $W_0^{2,p}(\Omega)$. Further its derivative operator is $A' = \Delta_p^2$. So this yields the hemicontinuity.
(ii) By a simple calculation we can show that $\|\Delta_p^2 u\|_* = \|\Delta u\|_p^{p-1}$, where $\|.\|_*$ is the dual norm associated to $\|\Delta.\|_p$. This implies that Δ_p^2 is bounded and is a monotonous operator. The continuity and coercivity are obvious .
(iii) The fact that, for any $u, v \in W_0^{2,p}(\Omega)$, $\|\Delta u\|_p = \|\Delta v\|_p$ if $\Delta_p^2 u = \Delta_p^2(v)$ and $(W_0^{2,p}(\Omega), \|\Delta.\|_p)$ is a uniformly convex space completes the proof.

Definition Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle .,. \rangle$. T a mapping acting from X into X^*. T is said to belong to the class (S^+), if for any sequence $\{u_n\}$ in X with u_n converges weakly to $u \in X$ and $\limsup_{n\to+\infty}\langle Tu_n, u_n - u\rangle \le 0$. It follows that u_n converges strongly to u in X. We write $T \in (S^+)$.

3 Main results

We will use Ljusternick-Schnirelmann theory on C^1-manifolds [6]. Consider the following two functionals defined on $W_0^{2,p}(\Omega)$:

$$A(u) = \frac{1}{p}\|\Delta u\|_p^p, \quad B(u) = \frac{1}{p}\int_\Omega \rho(x)|u|^p dx.$$

We set $\mathcal{M} = \{u \in W_0^{2,p}(\Omega); pB(u) = 1\}$.

Lemma 3.1 *(i) A and B are even, and of class C^1 on $W_0^{2,p}(\Omega)$. (ii) $\mathcal{M}$ is a closed C^1-manifold.*

Proof (i) It is clear that B is of class C^1 on $W_0^{2,p}(\Omega)$. $\mathcal{M} = B^{-1}\{\frac{1}{p}\}$ so B is closed. Its derivative operator B' satisfies $B'(u) \neq 0\ \forall u \in \mathcal{M}$ (i.e., $B'(u)$ is onto $\forall u \in \mathcal{M}$), so B is a submersion, then $\mathcal{M}$ is a C^1-manifold. □

Remark 3.2 Observe that $J : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$,

$$J(u) = \begin{cases} \|\Delta u\|_p^{2-p}\Delta_p^2 u & \text{if } u \neq 0 \\ 0 & \text{if } u = 0 \end{cases}$$

is the duality mapping of $(W_0^{2,p}(\Omega), \|\Delta.\|_p)$.

The following lemma is the key to show existence.

Lemma 3.3 *(i) $B' : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$ is completely continuous.*
(ii) The functional A satisfies the Palais-Smale condition on $\mathcal{M}$, i.e., for $\{u_n\} \subset \mathcal{M}$, if $A(u_n)$ is bounded and

$$\epsilon_n := A'(u_n) - g_n B'(u_n) \to 0 \quad \text{as } n \to +\infty, \tag{3.1}$$

where $g_n = \langle A'(u_n), u_n\rangle / \langle B'(u_n), u_n\rangle$. Then $\{u_n\}_{n\ge1}$ has a convergent subsequence in $W_0^{2,p}(\Omega)$.

Proof (i) Step 1: Definition of B'.
First case: $\frac{N}{p} > 2$ and $r > \frac{N}{2p}$. Let $u, v \in W_0^{2,p}(\Omega)$. By Hölder's inequality, we have

$$\Big|\int_\Omega \rho(x)|u(x)|^{p-2}u(x)v(x)dx\Big| \leq \|\rho\|_r \|u\|_s^{p-1}\|v\|_{p_2}$$

where $\frac{1}{p_2} = \frac{1}{p} - \frac{2}{N}$ and s is given by

$$\frac{p-1}{s} + \frac{1}{p_2} + \frac{1}{r} = 1. \tag{3.2}$$

Therefore,

$$\frac{p-1}{s} = 1 - \frac{1}{r} - \frac{1}{p_2} > 1 - \frac{2p}{N} - \frac{1}{p_2} = \frac{p-1}{p_2}.$$

Then it suffices to take

$$\max(1, p-1) < s < p_2 \tag{3.3}$$

so that B' is well defined.
Second case: $\frac{N}{p} = 2$ and $r > \frac{N}{2p}$. In this case $W_0^{2,p}(\Omega) \hookrightarrow L^q(\Omega)$, for any $q \in [p, +\infty[$. There is $q \geq p$ such that $\frac{1}{q} + \frac{1}{r} + \frac{p-1}{p} = \frac{1}{q} + \frac{1}{r} + \frac{1}{p'} = 1$.
We obtain that

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{r} \leq \frac{1}{p}. \tag{3.4}$$

By Hölder's inequality, we arrive at

$$\Big|\int_\Omega \rho(x)|u(x)|^{p-2}u(x)v(x)dx\Big| \leq \|\rho\|_r \|u\|_p^{p-1}\|v\|_q,$$

for any $u, v \in W_0^{2,p}(\Omega)$. Then B' is also well defined.
Third case: $\frac{N}{p} < 2$ and $r = 1$. In this case $W_0^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \cap L^\infty(\Omega)$. Then for any $u, v \in W_0^{2,p}(\Omega)$, we have

$$\Big|\int_\Omega \rho(x)|u(x)|^{p-2}u(x)v(x)dx\Big| < \infty,$$

with $\rho \in L^1(\Omega)$, and B' is well defined.

Step 2. B' is completely continuous. Let $(u_n) \subset W_0^{2,p}(\Omega)$ be a sequence such that $u_n \to u$ weakly in $W_0^{2,p}(\Omega)$. We have to show that $B'(u_n) \to B'(u)$ strongly in $W_0^{2,p}(\Omega)$, i.e.,

$$\sup_{v \in W_0^{2,p}(\Omega)\ \|\Delta v\|_p \leq 1} \Big|\int_\Omega \rho[|u_n|^{p-2}u_n - |u|^{p-2}u]v\, dx\Big| \to 0, \quad \text{as } n \to +\infty.$$

For this end, we distinguish three cases as in step 1 above. For $\frac{N}{p} > 2$, and

$r > \frac{N}{2p}$. Let s be as in (3.3). Then

$$\begin{aligned}
&\sup_{v\in W_0^{2,p}(\Omega),\, \|\Delta v\|_p\le 1} \Big|\int_\Omega \rho\big[|u_n|^{p-2}u_n - |u|^{p-2}u\big] v dx\Big| \\
&\le \sup_{v\in W_0^{2,p}(\Omega),\, \|\Delta v\|_p\le 1} \big[\|\rho\|_r \big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_{\frac{s}{p-1}} \|v\|_{p_2}\big] \\
&\le c\|\rho\|_r \big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_{\frac{s}{p-1}},
\end{aligned}$$

where c is the constant of Sobolev's embedding [1]. On the other hand, the Nemytskii's operator $u \mapsto |u|^{p-2}u$ is continuous from $L^s(\Omega)$ into $L^{\frac{s}{p-1}}(\Omega)$, and $u_n \to u$ weakly in $W_0^{2,p}(\Omega)$. So, we deduce that $u_n \to u$ strongly in $L^s(\Omega)$ because $s < p_2$. Hence,

$$\big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_{\frac{s}{p-1}} \to 0, \quad \text{as } n \to +\infty.$$

This completes the proof of the claim.
If $\frac{N}{p} = 2$ then

$$\Big|\int_\Omega \rho\big[|u_n|^{p-2}u_n - |u|^{p-2}u\big] v\, dx\Big| \le \|\rho\|_r \big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_p^{p-1} \|v\|_q,$$

where q is given by (3.4). By Sobolev's embedding there exist $c > 0$ such that

$$\|v\|_q \le c\|\Delta v\|_p, \quad \forall v \in W_0^{2,p}(\Omega).$$

Thus

$$\sup_{\substack{v\in W_0^{2,p}(\Omega)\\ \|\Delta v\|_p\le 1}} \Big|\int_\Omega \rho[|u_n|^{p-2}u_n - |u|^{p-2}u] v\, dx\Big| \le c\|\rho\|_r \big\||u_n|^{p-2}u_n - |u|^{p-2}u\big\|_p^{p-1}.$$

From the continuity of $u \mapsto |u|^{p-1}u$ from $L^p(\Omega)$ into $L^{p'}(\Omega)$, and from the compact embedding of $W_0^{2,p}(\Omega)$ in $L^p(\Omega)$, we have the desired result.
If $\frac{N}{p} < 2$ and $r = 1$, $W_0^{2,p}(\Omega) \subset C(\overline{\Omega})$, then we obtain

$$\sup_{\substack{v\in W_0^{2,p}(\Omega)\\ \|\Delta v\|_p\le 1}} \Big|\int_\Omega \rho\big[|u_n|^{p-2}u_n - |u|^{p-2}u\big] v\, dx\Big| \le c\|\rho\|_1 \sup_{\overline{\Omega}} \big||u_n|^{p-2}u_n - |u|^{p-2}u\big|,$$

where c is the constant given by embedding of $W_0^{2,p}(\Omega)$ in $C(\overline{\Omega}) \cap L^\infty(\Omega)$. It is clear that

$$\sup_{\overline{\Omega}} \big||u_n|^{p-2}u_n - |u|^{p-2}u\big| \to 0, \quad \text{as } n \to +\infty.$$

Hence B' is completely continuous, also in this case.
(ii) $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$. Hence without loss of generality, we can assume that u_n converges weakly in $W_0^{2,p}(\Omega)$ for some function $u \in W_0^{2,p}(\Omega)$

and $\|\Delta u_n\|_p \to c$. For the rest we distinct two cases:
If $c = 0$ then u_n converges strongly to 0 in $W_0^{2,p}(\Omega)$.

If $c \neq 0$, then we argue as follows:

$$\langle \Delta_p^2 u_n, u_n - u\rangle = \|\Delta u_n\|_p^p - \langle \Delta_p^2(u_n), u\rangle.$$

Applying ϵ_n of (3.1) to u, we deduce that

$$\Theta_n := \langle \Delta_p^2(u_n), u\rangle - \|\Delta u\|_p^p \langle B'(u_n), u\rangle \to 0 \quad \text{as } n \to +\infty. \tag{3.5}$$

Thus

$$\langle \Delta_p^2 u_n, u_n - u\rangle = \|\Delta u_n\|_p^p - \Theta_n - \|\Delta u_n\|_p^p \langle B'(u_n), u\rangle.$$

That is,

$$\langle \Delta_p^2 u_n, u_n - u\rangle = \|\Delta u_n\|_p^p (1 - \langle B'(u_n), u\rangle) - \Theta_n.$$

Hence,

$$\limsup_{n\to+\infty} \langle \Delta_p^2 u_n, u_n - u\rangle \leq c^p \limsup_{n\to+\infty} (1 - \langle B'(u_n), u\rangle).$$

On the other hand, from (i) $B'(u_n) \to B'(u)$ in $W^{-2,p'}(\Omega)$ and $pB(u) = 1$, because $pB(u_n) = 1$ for all $n \in \mathbb{N}^*$. So $pB(u) = \langle B'(u), u\rangle = 1$. This yields that

$$1 - \langle B'(u_n), u\rangle = \langle B'(u), u > - < B'(u_n), u\rangle \leq \quad \|B'(u) - B'(u_n)\|_* \|\Delta u\|_p,$$

where $\|.\|_*$ is the dual norm associated with $\|\Delta.\|_p$.

From (i) again $B'(u_n) \to B'(u)$ in $W_0^{-2,p'}(\Omega)$, we deduce that

$$\limsup_{n\to+\infty} \langle \Delta_p^2 u_n, u_n - u\rangle \leq 0 \tag{3.6}$$

We can write $\Delta_p^2 u_n = \|\Delta u_n\|_p^{p-2} J(u_n)$, since $\|\Delta u_n\|_p \neq 0$ for n large enough. Therefore,

$$\limsup_{n\to+\infty} \langle \Delta_p^2 u_n, u_n - u\rangle = c^{p-2} \limsup_{n\to+\infty} \langle J u_n, u_n - u\rangle.$$

According to (3.5) we conclude that

$$\limsup_{n\to+\infty} \langle J u_n, u_n - u\rangle \leq 0$$

J being a duality mapping, thus it satisfies the condition S^+. Therefore, $u_n \to u$ strongly in $W_0^{2,p}(\Omega)$. This achieves the proof of the lemma. □

Remark 3.4 A' is continuous, odd, (p-1)-homogeneous, continuously invertible and $\|A'(u)\|_* = \|\Delta u\|_p^{p-1}$, $\forall u \in W_0^{2,p}(\Omega)$.

Remark 3.5 We can give another method to prove that the functional A satisfies the Palais-Smale condition on $\mathcal{M}$.

Indeed, $\{u_n\}$ is bounded in $W_0^{2,p}(\Omega)$, we can assume for a subsequence if necessary that u_n converges weakly in $W_0^{2,p}(\Omega)$. The claim is to prove that u_n is of Cauchy in $W_0^{2,p}(\Omega)$. Set

$$G(u_n, u_m) = \int_\Omega (|\Delta u_n|^{p-2}\Delta u_n - |\Delta u_m|^{p-2}\Delta u_m).\Delta(u_n - u_m).$$

From (ii) of the proposition 2.1, Δ_p^2 is a monotonous operator on $W_0^{2,p}(\Omega)$. So that

$$\begin{aligned} 0 \le G(u_n, u_m) =& \langle \Delta_p^2 u_n - \Delta_p^2 u_m, u_n - u_m\rangle \\ =& \langle \epsilon_n - \epsilon_m, u_n - u_m\rangle + \langle h_n - h_m, u_n - u_m\rangle, \end{aligned}$$

with ϵ_n defined as in (3.1) and $h_n = \|\Delta u_n\|_p^p B'(u_n)$.

$$G(u_n, u_m) \le \|\epsilon_n - \epsilon_m\|_* \|\Delta u_n - \Delta u_m\|_p + \|h_n - h_m\|_* \|\Delta u_n - \Delta u_m\|_p.$$

Or h_n converges for a subsequence if necessary in $W_0^{2,p}(\Omega)$. Indeed, from (i) of Lemma 3.3 $B' : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$ is completely continuous. On the other hand, for a subsequence if necessary $\|\Delta u_n\| \to c \ge 0$. It follows that $(h_n)_{n\ge 0}$ is convergent in $W^{-2,p'}(\Omega)$. Then

$$G(u_n, u_m) \to 0, \quad \text{as } n \to +\infty. \tag{3.7}$$

From [4], we have the following inequality

$$|t_1 - t_2|^p \le c\{(|t_1|^{p-2}t_1 - |t_2|^{p-2}t_2).(t_1 - t_2)\}^{\frac{\gamma}{2}}(|t_1|^p + |t_2|^p)^{1-\frac{\gamma}{2}},$$

for any $t_1, t_2 \in \mathbb{R}$, with $\gamma = p$ if $1 < p < 2$ and $\gamma = 2$ if $p \ge 2$. By applying Hölder's inequality, we deduce that

$$\|\Delta u_n - \Delta u_m\|_p^p \le c\{G(u_n, u_m)\}^{\frac{\gamma}{2}}(\|\Delta u_n\|_p^p + \|\Delta u_m\|_p^p)^{1-\frac{\gamma}{2}} \tag{3.8}$$

for some positive constante c independent of n and m. According to (3.7), (3.8) shows that $(u_n)_n$ is a Cauchy's sequence in $W_0^{2,p}(\Omega)$. This proves the claim. □

Set

$$\Gamma_k = \{K \subset \mathcal{M} : \text{ K is symmetric,compact and } \gamma(\mathcal{K}) \ge \|\},$$

where $\gamma(K) = k$ is the genus of k, i.e., the smallest integer k such that there exists an odd continuous map from K to $\mathbb{R}^k - \{0\}$.

Now, by the Ljusternick-Schnirelmann theory, see e.g. [6], we have our main result formulated as follows.

Theorem 3.6 *For any integer $k \in \mathbb{N}^*$,*

$$\lambda_k := \inf_{K\in\Gamma_k} \max_{u\in K} pA(u)$$

is a critical value of A restricted on $\mathcal{M}$. More precisely, there exists $u_k \in K_k \in \Gamma_k$ such that

$$\lambda_k = pA(u_k) = \sup_{u\in K_k} pA(u)$$

and (λ_k, u_k) is a solution of (1.1) associated with the positive eigenvalue λ_k. Moreover,

$$\lambda_k \to +\infty, \quad \text{as } k \to +\infty.$$

Proof We need only to prove that for any $k \in \mathbb{N}^*$, $\Gamma_k \neq \emptyset$ and the least assertion. Indeed, since $W_0^{2,p}(\Omega)$ is separable, there exist $(e_i)_{i\geq 1}$ linearly dense in $W_0^{2,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_j = \emptyset$ if $i \neq$j. We can assume that $e_i \in \mathcal{M}$. Let $k \in \mathbb{N}^*$, denote $F_k = \operatorname{span}\{e_1, e_2, \dots, e_k\}$. F_k is a vectorial subspace and $\dim F_k = k$. If $v \in F_k$, then there exist $\alpha_1, \dots \alpha_k$ in $\mathbb{R}$ such that $v = \sum_{i=1}^k \alpha_i e_i$. Thus $B(v) = \sum_{i=1}^k |\alpha_i|^p B(e_i) = \frac{1}{p}\sum_{i=1}^k |\alpha_i|^p$. It follows that the map $v \mapsto (pB(v))^{1/p} := \|v\|$ defines a norm on F_k. Consequently, there is a constant $c > 0$ such that

$$c\|\Delta u\|_p \leq \|v\| \leq \frac{1}{c}\|\Delta u\|_p.$$

This implies that the set

$$V = F_k \cap \{v \in W_0^{2,p}(\Omega) : B(v) \leq \frac{1}{p}\}$$

is bounded. Thus V is a symmetric bounded neighbourhood of $0 \in F_k$. By (f) in [6, Prop. 2.3], $\gamma(F_k \cap \mathcal{M}) = \|$. Because $F_k \cap \mathcal{M}$ is compact and $\Gamma_k \neq \emptyset$. Now, we claim that $\lambda_k \to +\infty$, as $k \to +\infty$. Indeed let be $(e_n, e_j^*)_{n,j}$ a bi-orthogonal system such that $e_n \in W_0^{2,p}(\Omega)$ and $e_j^* \in W^{-2,p'}(\Omega)$, the e_n are linearly dense in $W_0^{2,p}(\Omega)$; and the e_j^* are total for $W^{-2,p'}(\Omega)$, see e.g. [6]. For $k \in \mathbb{N}^*$, set

$$F_k = \operatorname{span}\{e_1, \dots, e_k\}, \quad F_k^{\perp} = \operatorname{span}\{e_{k+1,e_{k+2},\dots}\}.$$

By (g) of Proposition 2.3 in [6], we have for any A$\in \Gamma_k$, $A \cap F_{k-1}^{\perp} \neq \emptyset$. Thus

$$t_k := \inf_{A\in\Gamma_k} \sup_{u\in A\cap F_{k-1}^{\perp}} pA(u) \to +\infty.$$

Indeed, if not, for k is large, there exists $u_k \in F_{k-1}^{\perp}$ with $\|u_k\|_p = 1$ such that

$$t_k \leq pA(u_k) \leq M,$$

for some $M > 0$ independent of k. Thus $\|\Delta u_k\|_p \leq M$. This implies that $(u_k)_k$ is bounded in $W_0^{2,p}(\Omega)$. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converge weakly in $W_0^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$. By our choice of $F_{k-1}^{\perp}$, we have $u_k \hookrightarrow 0$ weakly in $W_0^{2,p}(\Omega)$. Because $\langle e_n^*, e_k\rangle = 0$, $\forall k \geq n$. This contradicts the fact that $\|u_k\|_p = 1\,\forall k$. Since $\lambda_k \geq t_k$, the claim is proved. This completes the proof. □

Corrolary 3.7 *(i)* $\lambda_1 = \inf\{\|\Delta v\|_p^p, v \in W_0^{2,p}(\Omega); \int_\Omega \rho(x)|v|^p dx = 1\}$.
(ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$.

Proof (i) For $u \in \mathcal{M}$, we put $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, that A is even and that

$$pA(u) = \max_{K_1} pA \geq \inf_{K \in \Gamma_1} \max_{K} pA.$$

Hence

$$\inf_{u \in \mathcal{M}} pA(u) \geq \inf_{K \in \Gamma_1} \max_{K} pA = \lambda_1.$$

On the other hand, $\forall K \in \Gamma_1,\ \forall u \in K$,

$$\sup_K pA \geq pA(u) \geq \inf_{u \in \mathcal{M}} pA(u).$$

So

$$\inf_{K \in \Gamma_1} \max_{K} pA = \lambda_1 \geq \inf_{u \in \mathcal{M}} pA(u).$$

Thus

$$\lambda_1 = \inf_{u \in \mathcal{M}} pA(u) = \inf\{\|\Delta v\|_p^p, v \in W_0^{2,p}(\Omega) : \int_\Omega \rho(x)|v|^p dx = 1\}.$$

(ii) For all $i \geq j$, $\Gamma_i \subset \Gamma_j$. From the definition of $\lambda_i, i \in \mathbb{N}^*$, we have $\lambda_i \geq \lambda_j$. $\lambda_n \to +\infty$ is already proved in Theorem 3.6. Which completes the proof. □

Corrolary 3.8 *Assume that* $|\Omega_\rho^-| \neq 0$ *with* $\Omega_\rho^- = \{x \in \Omega : \rho(x) < 0\}$. *Then* Δ_p^2 *has a decreasing sequence of the negative eigenvalues* $(\lambda_{-n})(\rho)_{n\geq 0}$, *such that* $\lim_{n\to+\infty} \lambda_{-n} = -\infty$.

Proof First, remark that $\Omega_\rho^- = \Omega_{(-\rho)}^+$, so $|\Omega_{(-\rho)}^+| = |\Omega_\rho^-| \neq 0$. From Theorem 3.6, Δ_p^2 has an increasing sequence of the positive eigenvalues $\lambda_n(-\rho)$, such that $\lim_{n\to+\infty} \lambda_n(-\rho) = +\infty$. Note that $\lambda_n(-\rho)$ satisfies

$$\Delta_p^2 u = \lambda_n(-\rho)(-\rho)|u|^{p-2}u = -\lambda_n(-\rho)\rho|u|^{p-2}u,$$

for $u \in W_0^{2,p}(\Omega)$. Put $\lambda_{-n}(\rho) := -\lambda_n(-\rho)$ then $\lambda_n(-\rho)_{n\geq 0}$ is an increasing positive sequence so $(\lambda_{-n})(\rho)_{n\geq 0}$ is a negative decreasing sequence. On the other hand, $\lim_{n\to+\infty} \lambda_n(-\rho) = +\infty$. So

$$\lim_{n\to+\infty} \lambda_{-n}(\rho) = -\infty.$$

References

[1] R. Adams, *Sobolev spaces*, Academic Press, New-York (1975).

[2] P . Drábek and M. Ôtani, *Global bifurcation result for the p-biharmonic operator*, Electronic Journal of Differential Equations, Vol. 2001(2001), No. 48, 1-19.

[3] D. Gilbarg and Neil S. Trudinger, *Elliptic Partial Differential Equations of second order*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo (1983).

[4] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc., 109 (1990), 157-164.

[5] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires,* Dunod, Paris (1969).

[6] A. Szulkin, *Ljusternick-Schnirelmann theory on C^1-manifolds*, Ann. Inst. Henri Poincaré, Anal. Non., 5 (1988), 119-139.

Abdelouahed El Khalil (e-mail: lkhlil@hotmail.com)
Siham Kellati (e-mail: siham360@caramail.com)
Abdelfattah Touzani (e-mail: atouzani@iam.net.ma)
Department of Mathematics and Informatic,
Faculty of Sciences Dhar-Mahraz,
P.O. Box 1796 Atlas-Fez, Morocco