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EXISTENCE AND REGULARITY OF ENTROPY SOLUTIONS FOR SOME NONLINEAR ELLIPTIC EQUATIONS

LAHSEN AHAROUCH, ELHOUSSINE AZROUL

Abstract. This paper concerns the existence and regularity of entropy solutions to the Dirichlet problem

$$
\begin{gathered}
Au = -\operatorname{div}(a(x,u,\nabla u)) = f - \operatorname{div}\phi(u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

In particular, we show the $L^{\bar{q}}$-regularity of the solution to this boundary-value problem.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let p be a real number such that $2 - \frac{1}{N} < p \leq N$. Consider a Leray Lions operator

$$
Au = -\operatorname{div}(a(x,u,\nabla u)),
$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \neq \bar{\xi} \in \mathbb{R}^N$ the conditions

$$
|a(x,s,\xi)| \leq \beta[c(x) + |s|^{p-1} + |\xi|^{p-1}] \tag{1.1}
$$

$$
a(x,s,\xi).\xi \geq \alpha|\xi|^p \tag{1.2}
$$

$$
\langle a(x,s,\xi) - a(x,s,\bar{\xi}), \xi - \bar{\xi}\rangle > 0. \tag{1.3}
$$

Here $\alpha > 0$, $\beta \geq 0$ and $c(x) \in L^{p'}(\Omega)$. In the present paper, we study the boundary-value problem

$$
\begin{gathered}
Au := -\operatorname{div} a(x,u,\nabla u) = f - \operatorname{div}\phi(u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.4}
$$

where the right hand side is assumed to satisfy

$$
f \in L^1(\Omega), \tag{1.5}
$$

$$
\phi \in C^0(\mathbb{R}, \mathbb{R}^N). \tag{1.6}
$$

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Recall that, since no growth hypothesis is assumed on the function ϕ, the term $\operatorname{div}\phi(u)$ may be meaningless, even as a distribution for a function $v \in W_0^{1,r}(\Omega)$, $r > 1$ (see [4] and [7]).

Definition A function u is called an entropy solution of the Dirichlet problem (1.4) if,

$$\begin{gathered} u \in W_0^{1,q}(\Omega), \quad 1 < q < \bar{q} = \frac{N(p-1)}{N-1}, \\ T_k(u) \in W_0^{1,p}(\Omega), \quad \forall k > 0, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx \le \int_\Omega f T_k(u-\varphi)\,dx + \int_\Omega \phi(u)\nabla T_k(u-\varphi)\,dx, \\ \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \end{gathered}$$

where $T_k(s)$ is the truncation operator at height $k > 0$ defined on $\mathbb{R}$.

When $\phi = 0$ and f is a bounded Radon measure, it is known that (1.4) admits a weak solution u in $W_0^{1,q}(\Omega)$ with $1 < q < \bar{q}$; see for example [5, 6, 9]. It also have been shown there, that if f lies in the Orlicz space LLogL(Ω), then the critical regularity $W_0^{1,\overline{q}}(\Omega)$ is attained. Further contributions in this sense can be founded in the work [3] where the authors have replaced the hypotheses (1.1) and (1.2) by some general assumptions.

When $\phi \neq 0$ and $f \in L^1(\Omega)$, L. Boccardo proved in [4, Theorem 2.1] that the boundary-value problem (1.4) admits an entropy solution (in the sense of the definition 1.7) which belongs to $W_0^{1,q}(\Omega), 1 < q < \overline{q}$. Moreover, the author showed that if $f \in \mathrm{LLog}(1+L)(\Omega)$, then the solution belongs to $W_0^{1,\overline{q}}(\Omega)$.

Our objective in this paper, is to prove the existence and $L^{\overline{q}}$-regularity of an entropy solution to the boundary value problem (1.4), when $\phi \neq 0$ and $f \in L^1(\Omega)$. This is possible by replacing (1.1)–(1.3) by the following assumption.

There exist two N-functions P, M with $P << M$; six positive real numbers $\alpha, \delta, k_1, k_2, k_3, k_4$; and a function C in $E_{\overline{M}}$ such that

$$|a(x,s,\zeta)| \le C(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|) \tag{1.7}$$

$$\langle a(x,s,\zeta) - a(x,s,\xi), \zeta - \xi \rangle > 0 \tag{1.8}$$

$$a(x,s,\zeta)\zeta \ge \alpha M(\frac{|\zeta|}{\delta}), \tag{1.9}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and all $\xi \in \mathbb{R}^N$

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. Equivalently, M admits the representation:

$$M(t) = \int_0^t a(s)\,ds$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The conjugate of M is also an N-function and it is defined by $\overline{M} = \int_0^t \bar{a}(s)\,ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is the function $\bar{a}(t) = \sup\{s : a(s) \le t\}$.

An N-function M is said to satisfy the Δ_2-condition if, for some k,

$$M(2t) \le kM(t) \quad \forall t \ge 0. \tag{2.1}$$

When (2.1) holds only for $t \ge t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$. Moreover, we have the following Young's inequality

$$st \le M(t) + \overline{M}(s), \quad \forall s, t \ge 0.$$

Given two N-functions, we write $P << Q$ to indicate P grows essentially less rapidly than Q; i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if

$$\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\,dx < +\infty \quad (\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})\,dx < +\infty \text{ for some } \lambda > 0).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\big\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})\,dx \le 1\big\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.(for more details see [1]).

We recall some lemmas introduced in [2] which will be used later.

Lemma 2.1. *A domain Ω has the segment property if for every $x \in \partial\Omega$ there exists an open set G_x and a nonzero vector y_x such that $x \in G_x$ and if $z \in \overline{\Omega} \cap G_x$, then $z + ty_x \in \Omega$ for all $0 < t < 1$.*

Lemma 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let M be an N-function and let $u\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u)\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases}F'(u)\frac{\partial}{\partial x_i}u & \text{a.e. in } \{x\in\Omega : u(x)\notin D\},\\ 0 & \text{a.e. in } \{x\in\Omega : u(x)\notin D\}\end{cases}$$

Lemma 2.3. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F:W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continous with respect to the weak* topology $\sigma(\prod L_M,\prod E_{\overline{M}})$.*

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [2]).

Lemma 2.4. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M,P,Q be N-functions such that $Q<<P$, and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that, for a.e. $x\in\Omega$ and all $s\in\mathbb{R}$:*

$$|f(x,s)|\le c(x)+k_1P^{-1}M(k_2|s|),$$

where k_1,k_2 are real constants and $c(x)\in E_Q(\Omega)$. Then the Nemytskii operator N_f defined by $N_f(u)(x)=f(x,u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega),\frac{1}{k_2})=\{u\in L_M(\Omega): d(u,E_M(\Omega))<\frac{1}{k_2}\}$ into $E_Q(\Omega)$.

3. Main results

In the sequel we assume that Ω is an open bounded subset of $\mathbb{R}^N$, $N\ge 2$, with the segment property, and that M is an N-functions satisfying the Δ_2-condition near infinity. We shall prove the following existence theorems.

Theorem 3.1. *Assume that (1.7)–(1.9) hold, $2-\frac{1}{N}<p<N$, $f\in L^1(\Omega)$, $\phi\in C^0(\mathbb{R},\mathbb{R}^N)$, $\frac{t^p}{M(t)}$ is nondecreasing near infinity and $\int_.^\infty \frac{t^{p-1}}{M(t)}\,dt<\infty$. Then the problem,*

$$\begin{gathered}T_k(u)\in W_0^1L_M(\Omega),\quad \forall k>0\\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx\le\int_\Omega fT_k(u-\varphi)\,dx+\int_\Omega \phi(u)\nabla T_k(u-\varphi)\,dx,\\ \forall\,\varphi\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)\end{gathered}\tag{3.1}$$

admits at least one solution $u\in W_0^{1,\overline{q}}(\Omega)$.

When $p=N$ we assume, in addition, that There exists an N-function H such that $H(t^N)$ is equivalent to $M(t)$.

Theorem 3.2. *Assume that for $p=N$ the above hypothesis hold, (1.7)–(1.9) hold, $f\in L^1(\Omega)$, $\phi\in C^0(\mathbb{R},\mathbb{R}^N)$, $\int_.^\infty\frac{t^{N-1}}{M(t)}\,dt<\infty$ and $\frac{t^N}{\overline{H}^{-1}(e^{t^{N'}})}$ remains bounded near infinity. Then (3.1) admits at least one solution in $W_0^{1,N}(\Omega)$.*

Proof of Theorems 3.1 and 3.2.
Step 1 The approximate problem and a priori estimate. Let f_n be a

sequence in $W^{-1}E_{\overline{M}}(\Omega) \cap L^1(\Omega)$ such that $f_n \to f$ in $L^1(\Omega)$, and $\|f_n\|_1 \le \|f\|_1$. Consider the approximate problem

$$\begin{gathered} Au_n = f_n - \operatorname{div} \phi_n(u_n) \\ u_n \in W_0^1 L_M(\Omega) \end{gathered} \tag{3.2}$$

where $\phi_n(x) = \phi(T_n(x))$. From the work [8], there exists at least one solution u_n of the approximate problem (3.2). Moreover, as in [3], there exists a constant $C = C(p, \alpha, \|f\|_1)$ such that

$$\|\nabla u_n\|_{L_{\overline{q}}(\Omega)} \le C,$$

which implies that u_n is bounded in $W_0^{1,\overline{q}}(\Omega)$. Then there exists $u \in W_0^{1,\overline{q}}(\Omega)$ and a subsequence still denoted by u_n such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,\overline{q}}(\Omega) \\ u_n \to u \quad \text{strongly in } L^{\overline{q}}(\Omega) \text{ and a.e. in } \Omega. \end{gathered} \tag{3.3}$$

Moreover, the use of $T_k(u_n)$ as test function in (3.2) implies that the sequence $T_k(u_n)$ is bounded in $W_0^1 L_M(\Omega)$, then there exists a subsequence of $T_k(u_n)$ still denoted by $T_k(u_n)$ such that

$$\begin{gathered} T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}) \\ T_k(u_n) \to T_k(u) \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{gathered} \tag{3.4}$$

Step 2 Convergence of the gradient. Let $\Omega_r = \{x \in \Omega : |\nabla T_k(u(x))| \le r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\operatorname{meas}(\Omega \backslash \Omega_r) \to 0$ as $r \to \infty$.
Fix r and let $s \ge r$. We have,

$$\begin{aligned} 0 &\le \int_{\Omega_r} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &\le \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\,dx \\ &= \int_{\Omega_s} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \\ &\le \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx. \end{aligned}$$

On the other hand, let $h > k$ and $M = 4k + h$. If one takes $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ as test function in (3.2), it is easy to see that $\nabla w_n = 0$ when $|u_n| > M$. We can write

$$\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \nabla w_n\,dx = \int_\Omega f_n w_n\,dx + \int_\Omega \phi_n(u_n) \nabla w_n\,dx.$$

We have

$$\begin{aligned} &\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx \\ &\ge \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u))\,dx \\ &\quad - \int_{|u_n| > k} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\,dx \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\\
&\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u) - \nabla T_k(u)\chi_s)\,dx\\
&\quad - \int_{|u_n|>k} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\chi_s\,dx\\
&\quad - \int_{|u_n|>k} |a(x, T_M(u_n), \nabla T_M(u_n))|(|\nabla T_k(u)| - |\nabla T_k(u)|\chi_s)\,dx\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx\\
&\geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\\
&\quad - \int_{\Omega\setminus\Omega_s} a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)\,dx\\
&\quad - \int_{|u_n|>k} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\chi_s\,dx\\
&\quad - \int_{\Omega\setminus\Omega_s} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\,dx\,.
\end{aligned}
$$

From this inequality, it follows

$$
\begin{aligned}
&\int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\\
&\leq \int_{|u_n|>k} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\chi_s\,dx\\
&\quad + \int_{\Omega\setminus\Omega_s} a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u)\,dx\\
&\quad + \int_{\Omega\setminus\Omega_s} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\,dx\\
&\quad + \int_\Omega f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx\\
&\quad + \int_\Omega \phi_n(u_n))\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))\,dx\\
&\quad - \int_\Omega a(x, T_k(u_n), \nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx
\end{aligned}
\tag{3.5}
$$

Now, we study each term of the right hand side of the above inequality. We denote by $\varepsilon_i(t)$ $(i = 1, 2, 3, \dots)$ various sequences of real numbers which tends to 0 when t tends to infinity. Remark that $a(x, T_\mu(u_n), \nabla T_\mu(u_n))$ is bounded in $L_{\overline{M}}(\Omega)$ for all $\mu > 0$. Let $\varepsilon > 0$, we have

$$
M\Big(\frac{|\nabla T_k(u)|\chi_s\chi_{\{|u_n|>k\}}}{\varepsilon}\Big) \leq M\Big(\frac{s}{\varepsilon}\Big) \in L^1(\Omega)
$$

and

$$
|\nabla T_k(u)|\chi_s\chi_{\{|u_n|>k\}} \to 0 \quad \text{a.e.}
$$

Then by the Lebesgue dominated convergence theorem we deduce that

$$|\nabla T_k(u)|\chi_s\chi_{\{|u_n|>k\}} \to 0 \quad \text{in } L_M(\Omega),$$

which implies that the first term in the right hand side of (3.5) tends to 0 as n tends to ∞. Concerning the second and third terms on the right hand side of (3.5), since $|a(x,T_M(u_n),\nabla T_M(u_n))|$ and $|a(x,T_k(u_n),\nabla T_k(u_n))|$ are bounded in $L_{\overline{M}}(\Omega)$ then there exist two functions φ and ψ in $L_{\overline{M}}(\Omega)$ such that

$$\begin{gathered}|a(x,T_M(u_n),\nabla T_M(u_n))| \to \varphi \quad \text{for } \sigma(L_{\overline{M}},E_M)\\ |a(x,T_k(u_n),\nabla T_k(u_n))| \to \psi \quad \text{for } \sigma(L_{\overline{M}},E_M)\,.\end{gathered} \tag{3.6}$$

This implies

$$\int_{\Omega\setminus\Omega_s} |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\,dx \to \int_{\Omega\setminus\Omega_s}\varphi|\nabla T_k(u)|\,dx \tag{3.7}$$

and

$$\int_{\Omega\setminus\Omega_s} |a(x,T_k(u_n),\nabla T_k(u_n))||\nabla T_k(u)|\,dx \to \int_{\Omega\setminus\Omega_s}\psi|\nabla T_k(u)|\,dx\,. \tag{3.8}$$

On the other hand,

$$\lim_{n\to\infty}\int_\Omega f_nT_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx = \int_\Omega fT_{2k}(u-T_h(u))\,dx = \varepsilon_3(h)$$

and, for n large enough, one can write.

$$\begin{aligned}&\int_\Omega \phi_n(u_n)\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\ &= \int_\Omega \phi(T_{4k+h}(u_n))\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx,\end{aligned}$$

which yields,

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega \phi_n(u_n)\nabla T_{2k}(u_n-T_h(u_n)+T_k(u_n)-T_k(u))\,dx\\ &= \int_\Omega \phi(u)\nabla T_{2k}(u-T_h(u)\,dx = 0.\end{aligned}$$

The right-most term in (3.5) tends to 0: Since $a(x,T_k(u_n),\nabla T_k(u)\chi_s)$ converges strongly to $a(x,T_k(u),\nabla T_k(u)\chi_s)$ in $(E_{\overline{M}}(\Omega))^N$, using Lemma 2.4 while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$ by (3.3). We conclude then that

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\int_{\Omega_r}\big[a(x,T_k(u_n),\nabla T_k(u_n))\\ &\quad - a(x,T_k(u_n),\nabla T_k(u)][\nabla T_k(u_n)-\nabla T_k(u)\big]\,dx\\ &\le \int_{\Omega\setminus\Omega_s}\varphi|\nabla T_k(u)|\,dx + \int_{\Omega\setminus\Omega_s}\psi|\nabla T_k(u)|\,dx + \int_\Omega fT_{2k}(u-T_h(u))\,dx\,.\end{aligned}$$

Letting s and h approach infinity we get,

$$\int_{\Omega_r}[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)][\nabla T_k(u_n)-\nabla T_k(u)]\,dx \to 0$$

as $n\to\infty$. Passing to a subsequence if necessary, we can assume that

$$[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)][\nabla T_k(u_n)-\nabla T_k(u)] \to 0$$

a.e. in Ω_r. As in [2], we deduce that there exists a subsequence still denoted by u_n such that $\nabla u_n \to \nabla u$ a.e. in Ω.

Step 3 Passage to the limit. Let $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$, and set $M = k + \|\varphi\|_\infty$ with $k > 0$. We shall prove that

$$\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx \geq \int_\Omega a(x, u, \nabla u) \nabla T_k(u - \varphi)\, dx.$$

We have: If $|u_n| > M$ then $|u_n - \varphi| > k$ which implies

$$\begin{aligned}
&a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi) \\
&= a(x, T_M(u_n), \nabla T_M(u_n))(\nabla u_n - \nabla \varphi)\chi_{\{|u_n-\varphi|\leq k\}} \\
&= a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla \varphi)\chi_{\{|u_n-\varphi|\leq k\}}.
\end{aligned}$$

Let $\Omega_s = \{x \in \Omega : |\nabla\varphi| \leq s\}$ and denote by χ_s the characteristic function of Ω_s. Then

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx \\
&= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla\varphi)\chi_{\{|u_n-\varphi|\leq k\}}\, dx \\
&= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla\varphi\chi_s)\chi_{\{|u_n-\varphi|\leq k\}}\, dx \\
&\quad - \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla\varphi - \nabla\varphi\chi_s)\chi_{\{|u_n-\varphi|\leq k\}}\, dx,
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx \\
&\geq -\int_{\Omega\setminus\Omega_s} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla\varphi|\, dx \\
&\quad + \int_\Omega \big[a(x, T_M(u_n), \nabla T_M(u_n)) - a(x, T_M(u_n), \nabla\varphi\chi_s)\big] \\
&\quad \times [\nabla T_M(u_n) - \nabla\varphi\chi_s]\chi_{\{|u_n-\varphi|\leq k\}}\, dx \\
&\quad + \int_\Omega a(x, T_M(u_n), \nabla\varphi\chi_s)[\nabla T_M(u_n) - \nabla\varphi\chi_s]\chi_{\{|u_n-\varphi|\leq k\}}\, dx.
\end{aligned} \tag{3.9}$$

Similarly to the proof of (3.7), the first term in the right hand side of (3.9) is greater than a value $\varepsilon_6(s)$, which implies

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx \\
&\geq \lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla\varphi\chi_s)[\nabla T_M(u_n) - \nabla\varphi\chi_s]\chi_{\{|u_n-\varphi|\leq k\}}\, dx + \varepsilon_6(s) \\
&\quad + \int_\Omega [a(x, T_M(u), \nabla T_M(u)) - a(x, T_M(u), \nabla\varphi\chi_s)] \\
&\quad \times [\nabla T_M(u) - \nabla\varphi\chi_s]\chi_{\{|u-\varphi|\leq k\}}\, dx.
\end{aligned} \tag{3.10}$$

From Lemma 2.4, the first term in the right hand side of (3.10) is equal to

$$\int_\Omega a(x, T_M(u), \nabla\varphi\chi_s)[\nabla T_M(u) - \nabla\varphi\chi_s]\chi_{\{|u-\varphi|\leq k\}}\, dx + \varepsilon_6(s),$$

then

$$\liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx$$
$$\geq \int_\Omega a(x,T_M(u),\nabla T_M(u))[\nabla T_M(u)-\nabla\varphi\chi_s]\chi_{\{|u-\varphi|\leq k\}}\,dx+\varepsilon_6(s).$$

By letting $s\to+\infty$, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi)\,dx\\
&\geq \int_\Omega a(x,T_M(u),\nabla T_M(u))[\nabla T_M(u)-\nabla\varphi]\chi_{\{|u-\varphi|\leq k\}}\,dx\\
&=\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi)\,dx.
\end{aligned}$$

Now taking $T_k(u_n-\varphi)$ as test function in (3.7) and passing to the limit we deduce the desired statement. □

Remark 3.3. If M and $\overline{M}$ satisfy the Δ_2 condition, instead of (1.7) we can assume the condition:

$$|a(x,s,\xi)|\leq c(x)+k_1\overline{M}^{-1}M(k_2|s|)+k_3\overline{M}^{-1}M(k_4|\xi|). \tag{3.11}$$

Then we prove the same result as in Theorems 3.1 and 3.2.

Remark 3.4. If wf belongs to $W^{-1}L_{\overline{M}}(\Omega)$ the statements of Theorems 3.1 and 3.2 still hold.

Example. Let $2-\frac{1}{N}<p\leq N$, $(N\geq 2)$, and let the N-function be $M(t)=t^p\log^{\alpha p}(e+t)$ with $\alpha p>1$. Then it is easy to verify that $M(t)$ satisfies the condition of Theorems 3.1 and 3.2.

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