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ON THE SOLVABILITY OF DEGENERATED QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. In this article, we study the quasilinear elliptic problem

$$\begin{gathered} Au = -\operatorname{div}(a(x,u,\nabla u)) = f(x,u,\nabla u) \quad \text{in } \mathcal{D}'(\Omega) \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ to its dual $W^{-1,p'}(\Omega, w^*)$. We show that there exists a solution in $W_0^{1,p}(\Omega, w)$ provided that

$$|f(x,r,\xi)| \leq \sigma^{1/q}[g(x) + |r|^\eta \sigma^{\eta/q} + \sum_{i=1}^N w_i^{\delta/p}(x)|\xi_i|^\delta],$$

where $g(x)$ is a positive function in $L^{q'}(\Omega)$ and $\sigma(x)$ is weight function and $0 \leq \eta < \min(p-1, q-1)$, $0 \leq \delta < (p-1)/q'$.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$, and p be a real number such that $1 < p < \infty$. Let $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector weight functions on Ω; i.e., each $w_i(x)$ is a measurable a.e. strictly positive function on Ω, satisfying some integrability conditions (see section2). Let us consider the problem

$$\begin{gathered} Au = f(x,u,\nabla u) \quad \text{in } \mathcal{D}'(\Omega) \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where A is a Leray-Lions operator $Au = -\operatorname{div}(a(x,u,\nabla u))$ and $f(x,r,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function. Boccardo, Murat and Puel in [3] studied the problem (1.1) in the non weighted case, with f satisfying the condition

$$|f(x,r,\xi)| \leq h(|r|)(1 + |\xi|^p),$$

where h is increasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$. The existence result is proved assuming the existence of the subsolution and supersolution in $W^{1,\infty}(\Omega)$, which play an important roll in their work. Further in [2] the author's studied the problem

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(1.1) with f satisfies the hypotheses

$$|f(x,r,\xi)| \leq \beta[g(x) + |r|^{p-1} + |\xi|^{p-1}], \tag{1.2}$$

$$f(x,r,\xi)r \geq \alpha |r|^p. \tag{1.3}$$

Recently, Tsang-Hai Kuo and Chiung-Chion Tsai [8] proved an existence result under the assumption

$$|f(x,r,\xi)| \leq c(1 + |r|^\delta + |\xi|^\eta).$$

Our objective in this paper, is to study the problem (1.1) in weighted Sobolev spaces where f satisfying only the growth condition

$$|f(x,r,\xi)| \leq \sigma^{1/q}[g(x) + |r|^\eta \sigma^{\frac{\eta}{q}} + \sum_{i=1}^N w_i^{\delta/p}(x)|\xi_i|^\delta],$$

where $g(x)$ is a positive function in $L^{q'}(\Omega)$, σ is a weight function, and

$$0 \leq \eta < \min(p-1, q-1), \quad 0 \leq \delta < \frac{p-1}{q'}.$$

Note that we obtain the existence result without assuming the condition (1.3) and without knowing a priori the existence of subsolutions and supersolutions. Let us point out that this work can be see as a generalization of the work in [2] and [8].

2. Preliminaries and Basic Assumptions

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$, and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions; i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that

$$w_i \in L^1_{\rm loc}(\Omega), \tag{2.1}$$

$$w_i^{-1/(p-1)} \in L^1_{\rm loc}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega, w)$ the space of real-valued functions $u \in L^p(\Omega, w_0)$ such that their derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for } i = 1, \dots, N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subspace of $W^{1,p}(\Omega, w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega, w)$ of $W^{1,p}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 1, \dots, N\}$ and p' is the conjugate of p, i.e. $p' = \frac{p}{p-1}$. For more details we refer the reader to [5]. We start by stating the following assumptions:

(H1) The expression
$$|||u||| = \Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{2.4}$$
is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to the norm (2.3). And there exist a weight function σ on Ω and a parameter $0 < q < \infty$, such that the Hardy inequality
$$\Big(\int_\Omega |u(x)|^q \sigma(x)\,dx\Big)^{1/q} \le c\Big(\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{2.5}$$
holds for every $u \in W_0^{1,p}(\Omega, w)$ with a constant $c > 0$. Moreover, the imbedding
$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{2.6}$$
is compact.

Let A be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by
$$A(u) = -\operatorname{div}(a(x, u, \nabla u)),$$
where $a(x, r, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-valued function that satisfies the following assumption:

(H2) For $i = 1, \dots, N$,
$$|a_i(x, r, \xi)| \le \beta w_i^{1/p}(x)[k(x) + \sigma^{\frac{1}{p'}}|r|^{q/p'} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}|\xi_j|^{p-1}] \tag{2.7}$$
$$[a(x, r, \xi) - a(x, r, \eta)](\xi - \eta) > 0 \quad \text{for all } \xi \ne \eta \in \mathbb{R}^N; \tag{2.8}$$
$$a(x, r, \xi)\xi \ge \alpha \sum_{i=1}^{N} w_i|\xi_i|^p, \tag{2.9}$$
where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are strictly positive constants.

Let $f(x, r, \xi)$ is a Carathéodory function satisfying the following assumptions:

(H3)
$$|f(x, r, \xi)| \le \sigma^{1/q}[g(x) + |r|^\eta \sigma^{\frac{\eta}{q}} + \sum_{i=1}^{N} w_i^{\delta/p}(x)|\xi_i|^\delta], \tag{2.10}$$
where $g(x)$ is a positive function in $L^{q'}(\Omega)$, and
$$0 \le \eta < \min(p-1, q-1), \quad 0 \le \delta < \frac{p-1}{q'}. \tag{2.11}$$

3. Main result

Consider the problem
$$\begin{gathered} -\operatorname{div}\ a(x, u, \nabla u) = f(x, u, \nabla u) \quad \text{in } D'(\Omega) \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{3.1}$$

Theorem 3.1. *Under hypotheses (H1)-(H3), there exist at least one solution to (3.1).*

We first give some definition and some lemmas that will be used in the proof of this theorem.

Definition Let Y be a separable reflexive Banach space, the operator B from Y to its dual Y^* is called of the calculus of variations type, if B is bounded and is of the from,

$$B(u) = B(u, u), \tag{3.2}$$

where $(u, v) \to B(u, v)$ is an operator $Y \times Y$ into Y^* satisfying the following properties:

$$\begin{aligned} &\text{For } u \in Y, \text{ the mapping } v \mapsto B(u, v) \text{ is bounded and hemicontinuous} \\ &\text{from } Y \text{ to } Y^* \text{ and } (B(u, u) - B(u, v), u - v) \geq 0; \end{aligned} \tag{3.3}$$

for $v \in Y$, the mapping $u \mapsto B(u, v)$ is bounded and hemicontinuous from Y to Y^*;

$$\begin{aligned} &\text{If } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } (B(u_n, u_n) - B(u_n, u), u_n - u) \to 0, \\ &\text{then } B(u_n, v) \rightharpoonup B(u, v) \text{ weakly in } Y^*, \text{ for all } v \in Y; \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\text{If } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } B(u_n, v) \rightharpoonup \psi \text{ weakly in } Y^*, \\ &\text{then } (B(u_n, v), u_n) \to (\psi, u). \end{aligned} \tag{3.5}$$

Lemma 3.2 ([1])**.** *Let $g \in L^q(\Omega, \gamma)$, $g_n \in L^q(\Omega, \gamma)$, and $\|g_n\|_{q,\gamma} \leq c$ $(1 < q < \infty)$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^q(\Omega, \gamma)$, where γ is a weight function on Ω.*

Lemma 3.3. *If $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, w)$ and $v \in W_0^{1,p}(\Omega, w)$, then $a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v)$ in $L^{p'}(\Omega, w_i^*)$.*

Proof. From (H2), it follows that

$$\begin{aligned} |a_i(x, u_n, \nabla v)|^{p'} w_i^{\frac{-p'}{p}} &\leq \beta[k(x) + |u_n|^{\frac{q}{p'}} \sigma^{\frac{1}{p'}} + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^{p-1} w_j^{\frac{1}{p'}}]^{p'} \\ &\leq \gamma[k(x)^{p'} + |u_n|^q \sigma + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^p w_j], \end{aligned} \tag{3.6}$$

where β and γ are positive constants. Since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and $W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma)$, it follows that $u_n \to u$ strongly in $L^q(\Omega, \sigma)$ and $u_n \to u$ a.e. in Ω; hence

$$|a_i(x, u_n, \nabla v)|^{p'} w_i^* \to |a_i(x, u, \nabla v)|^{p'} w_i^* \quad \text{a.e. in } \Omega, \tag{3.7}$$

and

$$\gamma\Big[k(x)^{p'} + |u_n|^q \sigma + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^p w_i\Big] \to \gamma\Big[k(x)^{p'} + |u|^q \sigma + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^p w_j\Big]$$

a.e. in Ω. Then, By Vitali's theorem,

$$a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \quad \text{strongly in } L^{p'}(\Omega, w_i^*), \text{ as } n \to +\infty. \tag{3.8}$$

□

Lemma 3.4 ([1])**.** *Assume that (H1)–(H2) are satisfied, and let (u_n) be a sequence in $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)] \nabla(u_n - u)\, dx \to 0.$$

Then, $u_n \to u$ *in* $W_0^{1,p}(\Omega, w)$.

For $v \in W_0^{1,p}(\Omega, w)$, we associate the Nemytskii operator F with respect to f,

$$F(v, \nabla v)(x) = f(x, v, \nabla v) \text{ a.e.}, x \in \Omega.$$

Lemma 3.5. *The mapping* $v \mapsto F(v, \nabla v)$ *is continuous from the space* $W_0^{1,p}(\Omega, w)$ *to* $L^{q'}(\Omega, \sigma^{1-q'})$.

Proof. By hypothesis (H3), we have

$$|f(x, r, \xi)| \le \sigma^{1/q}[g(x) + |r|^{\eta}\sigma^{\frac{\eta}{q}} + \sum_{i=1}^{N} w_i^{\delta/p}(x)|\xi_i|^{\delta}].$$

Thanks to Young's inequality,

$$|r|^{\eta}\sigma^{\eta/q} \le [\frac{\eta}{q-1}|r|^{q-1}\sigma^{(q-1)/q} + 1] \le [|r|^{q-1}\sigma^{q'} + 1],$$
$$w_i^{\sigma/p}|\xi_i|^{\sigma} \le [w_i^{1/q'}|\xi_i|^{p/q'} + 1],$$

which implies

$$|f(x, r, \xi)| \le \sigma^{1/q}[(N+2) + g(x) + |r|^{q-1}\sigma^{1/q'} + \sum_{i=1}^{N} w_i^{1/q'}|\xi_i|^{p/q'}]\,.$$

Then

$$|f(x, r, \xi)|^{q'}\sigma^{-q'/q} \le c_2[c_1 + g(x)^{q'} + |r|^{(q-1)q'}\sigma + \sum_{i=1}^{N} w_i|\xi_i|^{p}]\,.$$

Since f is a Carathéodory, and for all subset E measurable, such that $|E| < \eta$, we have

$$\int_E |f(x, v, \nabla v)|^{q'}\sigma^{\frac{-q'}{q}}\,dx \le c_2[c_3 + \int_E |v|^q \sigma\,dx + \int_E \sum_{i=1}^{N} w_i|\frac{\partial v}{\partial x_i}|^p\,dx]\,.$$

Then by Vitali's theorem, we deduce the continuous of the operator F. Moreover,

$$\Big(\int_\Omega |f(x, v, \nabla v)|^{q'}\sigma^{\frac{-q'}{q}}\,dx\Big)^{1/q'} \le c_2[c + |||v|||^{q/q'} + |||v|||^{p/q'}]. \tag{3.9}$$

□

Proof of Theorem 3.1. **Step (1)** We will show that the operator $B : W_0^{1,p}(\Omega, w) \to W^{1,p'}(\Omega, w^*)$ defined by $B(v) = A(v) - f(x, v, \nabla v)$ is a calcul of variational.
Assertion 1. Let

$$B(u, v) = -\sum_{i=1}^{N} \frac{\partial a_i(x, u, \nabla v)}{\partial x_i} - f(x, u, \nabla u).$$

Then $B(v, v) = B(v)$ for all $v \in W_0^{1,p}(\Omega, w)$.
Assertion 2. We claim that the operator $v \to B(u, v)$ is bounded for all $u \in W_0^{1,p}(\Omega, w)$. Let $\psi \in W_0^{1,p}(\Omega, w)$, we have

$$\langle B(u, v), \psi)\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla v)\frac{\partial \psi}{\partial x_i} - \int_\Omega f(x, u, \nabla u)\psi\,dx.$$

From Hölder's inequality, the growth condition (2.7) and the compact imbedding (2.6), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla v)\frac{\partial\psi}{\partial x_i}\\
&\leq \sum_{i=1}^{N}\Big(\int_\Omega |a_i(x,u,\nabla v)|^{p'} w_i^{\frac{-p'}{p}}\,dx\Big)^{1/p'}\Big(\int_\Omega |\frac{\partial\psi}{\partial x_i}|^p w_i\,dx\Big)^{1/p}\\
&\leq c_4|||\psi|||\sum_{i=1}^{N}\Big(\int_\Omega k(x)^{p'}+|u|^q\sigma+\sum_{j=1}^{N}|\frac{\partial v}{\partial x_j}|^p w_j\,dx\Big)^{1/p'}\\
&\leq c_5|||\psi|||\big[c_6+|||u|||^{\frac{q}{p'}}+|||v|||^{p-1}\big].
\end{aligned}\tag{3.10}$$

Similarly,

$$\int_\Omega f(x,u,\nabla u)\psi\,dx \leq \Big(\int_\Omega |f(x,u,\nabla u)|^{q'}\sigma^{\frac{-q'}{q}}\,dx\Big)^{1/q'}\Big(\int_\Omega |\psi|^q\sigma\,dx\Big)^{1/q},$$

by (2.5) and (3.9), we have,

$$\int_\Omega f(x,u,\nabla u)\psi\,dx \leq c|||\psi|||\big[c_7+|||u|||^{q-1}+|||u|||^{p/q'}\big].\tag{3.11}$$

Since u and v belong to $W_0^{1,p}(\Omega,w)$ and in view of (3.10) and (3.11), we deduce that $\langle B(u,v),\psi\rangle$ is bounded in $W_0^{1,p}(\Omega,w)\times W_0^{1,p}(\Omega,w)$.

We claim that the operator $v\to B(u,v)$ is hemicontinuous for all $u\in W_0^{1,p}(\Omega,w)$, i.e., the operator $\lambda\to\langle B(u,v_1+\lambda v_2),\psi\rangle$ is continuous for all $v_1,v_2,\psi\in W_0^{1,p}(\Omega,w)$. Since a_i is a Carathéodory function,

$$a_i(x,u,\nabla(v_1+\lambda v_2))\to a_i(x,u,\nabla v_1)\quad\text{a.e. in }\Omega\text{ as }\lambda\to 0.$$

Further, we know from (2.7) that $(a_i(x,u,\nabla(v_1+\lambda v_2))_\lambda$ is bounded in $L^{p'}(\Omega,w_i^*)$; thus, by Lemma 3.2, we conclude

$$a_i(x,u,\nabla(v_1+\lambda v_2))\rightharpoonup a_i(x,u,\nabla v_1)\quad\text{weakly in } L^{p'}(\Omega,w_i^*),\ \text{as }\lambda\to 0\,.\tag{3.12}$$

Hence,

$$\begin{aligned}
&\lim_{\lambda\to 0}\langle B(u,v_1+\lambda v_2),\psi\rangle\\
&=\lim_{\lambda\to 0}\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla(v_1+\lambda v_2))\frac{\partial\psi}{\partial x_i}\,dx-\int_\Omega f(x,u,\nabla u)\psi\,dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla v_1)\frac{\partial\psi}{\partial x_i}\,dx-\int_\Omega f(x,u,\nabla u)\psi\,dx\\
&=\langle B(u,v_1),\psi\rangle\quad\text{for all } v_1,v_2,\psi\in W_0^{1,p}(\Omega,w).
\end{aligned}\tag{3.13}$$

Similarly, we show that $u\to\langle B(u,v),\psi\rangle$ is bounded and hemicontinuous for all $v\in W_0^{1,p}(\Omega,w)$. Indeed. By (3.9), we have $f((x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)))_\lambda$ is bounded in $L^{q'}(\Omega,\sigma^{1-q'})$ and as f is a Carathéodory function then

$$f(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2))\to f(x,u_1,\nabla u_1)\quad\text{a.e. in }\Omega.$$

Hence, Lemma 3.2 gives,

$$f(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \rightharpoonup f(x,u_1,\nabla u_2) \quad \text{weakly in } L^{q'}(\Omega,\sigma^{1-q'}) \text{ as } \lambda\to 0, \tag{3.14}$$

On the other hand, as in (3.12), we have

$$a_i(x,u_1+\lambda u_2,\nabla v) \rightharpoonup a_i(x,u_1,\nabla v) \quad \text{in } L^{p'}(\Omega,w_1^*), \text{ as } \lambda\to 0. \tag{3.15}$$

Combining (3.14) and (3.15), we conclude that, $u\to B(u,v)$ is bounded and hemicontinuous.

Assertion 3. From (2.8), we have,

$$\langle B(u,u)-B(u,v),u-v\rangle = \sum_{i=1}^N\int_\Omega (a_i(x,u,\nabla u)-a_i(x,u,\nabla v))\,(\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i})\ge 0$$

Assertion 4. Assume that $u_n\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and $\langle B(u_n,u_n)-B(u_n,u),u_n-u\rangle\to 0$, we claim that $B(u_n,v)\rightharpoonup B(u,v)$ weakly in $W^{-1,p'}(\Omega,w^*)$.

We can write $\langle B(u_n,u_n)-B(u_n,u),u_n-u\rangle\to 0$ as $n\to\infty$,

$$\begin{aligned}
&\Big\langle\sum_{i=1}^N -\big[\frac{\partial}{\partial x_i}a_i(x,u_n,\nabla u_n)-\frac{\partial}{\partial x_i}a_i(x,u_n,\nabla u)\big],u_n-u\Big\rangle\\
&=\sum_{i=1}^N\int_\Omega [a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u)]\,\frac{\partial}{\partial x_i}(u_n-u)\,dx\to 0
\end{aligned}$$

Then, by Lemma 3.4, we have $u_n\to u$ strongly in $W_0^{1,p}(\Omega,w)$ and it follows from Lemma 3.5 that

$$f(x,u_n,\nabla u_n)\to f(x,u,\nabla u) \quad \text{in } L^{q'}(\Omega,\sigma^{1-q'}). \tag{3.16}$$

Since $u_n\to u$ in $L^p(\Omega,w)$ and by (2.7) and $W_0^{1,p}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, we can obtain from Lemma 3.3 that

$$a_i(x,u_n,\nabla v)\to a_i(x,u,\nabla v) \quad \text{in } L^{p'}(\Omega,w_i^*)\,. \tag{3.17}$$

This implies

$$\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial\psi}{\partial x_i}\,dx\to\int_\Omega a_i(x,u,\nabla v)\frac{\partial\psi}{\partial x_i}\,dx. \tag{3.18}$$

On the other hand, by Hölders inequality,

$$\int_\Omega |f(x,u_n,\nabla u_n)\psi|\,dx\le\Big(\int_\Omega |f(x,u_n,\nabla u_n)|^{q'}\sigma^{1-q'}\,dx\Big)^{1/q'}\Big(\int_\Omega|\psi|^q\sigma\,dx\Big)^{1/q}.$$

Thanks to (3.16), (2.5), and Lebegue's dominated convergence theorem, we obtain

$$\int_\Omega f(x,u_n,\nabla u_n)\psi\,dx\to\int_\Omega f(x,u,\nabla u)\psi\,dx\,. \tag{3.19}$$

Then, we have

$$\begin{aligned}\lim_{n\to\infty}\langle B(u_n,v),\psi\rangle &= \lim_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial\psi}{\partial x_i}\,dx-\int_\Omega f(x,u_n,\nabla u_n)\psi\,dx\\ &=\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla v)\frac{\partial\psi}{\partial x_i}\,dx-\int_\Omega f(x,u,\nabla u)\psi\,dx\\ &=\langle B(u,v),\psi\rangle,\quad\text{for all }\psi\in W_0^{1,p}(\Omega,w).\end{aligned}$$

Assertion 5. Assume $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and $B(u_n,v)\rightharpoonup\psi$ weakly in $W^{-1,p'}(\Omega,w)$. We claim that $\langle B(u_n,v),u_n\rangle\to\langle\psi,u\rangle$. Thanks to $u_n\rightharpoonup u$ in $W_0^p(\Omega,w)$, we obtain by Lemma 3.3,

$$a_i(x,u_n,\nabla v)\to a_i(x,u,\nabla v)\quad\text{strongly in } L^{p'}(\Omega,w_i^*)\text{ as } n\to+\infty. \tag{3.20}$$

And so

$$\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u_n}{\partial x_i}\,dx\to\int_\Omega a_i(x,u,\nabla v)\frac{\partial u}{\partial x_i}\,dx. \tag{3.21}$$

Hence together with

$$\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u}{\partial x_i}\,dx-\int_\Omega f(x,u_n,\nabla v)u\,dx\to\langle\psi,u\rangle, \tag{3.22}$$

we have

$$\begin{aligned}\langle B(u_n,v),u_n)\rangle &= \sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u_n}{\partial x_i}\,dx-\int_\Omega f(x,u_n,\nabla u_n)u_n\,dx\\ &=\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i})\,dx+\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)\frac{\partial u}{\partial x_i}\,dx\\ &\quad-\int_\Omega f(x,u_n,\nabla u_n)u\,dx-\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,dx.\end{aligned}$$

But in view of (3.20) and (3.21), we obtain

$$\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla v)(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i})\,dx\to 0. \tag{3.23}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}&\int_\Omega |f(x,u_n,\nabla u_n)(u_n-u)|\,dx\\ &\le\Big(\int_\Omega|f(x,u_n,\nabla u_n)|^{q'}\sigma^{1-q'}\,dx\Big)^{1/q'}\Big(\int_\Omega|u_n-u|^q\sigma\,dx\Big)^{1/q}\\ &\le c\|u_n-u\|_{L^q(\Omega,\sigma)}\to\quad\text{as } n\to\infty\end{aligned}$$

i.e.,

$$\int_\Omega f(x,u_n,\nabla u_n)(u_n-u)\,dx\to 0\quad\text{as } n\to\infty. \tag{3.24}$$

Thanks to (3.22), (3.23) and (3.24), we obtain

$$\langle B(u_n,v),u_n\rangle\to\langle\psi,u\rangle.$$

Step 2. We claim that the operator B satisfies the coercivity condition

$$\lim_{|||v|||\to+\infty} \frac{\langle B(v),v\rangle}{|||v|||}=\infty. \tag{3.25}$$

Since

$$\langle Bv,v\rangle=\sum_{i=1}^N\int_\Omega a_i(x,v,\nabla v)\frac{\partial v}{\partial x_i}\,dx-\int_\Omega f(x,v,\nabla v)v\,dx.$$

Then, using (2.9), we have

$$\langle Bv,v\rangle\ge\alpha\sum_{i=1}^N w_i|\frac{\partial v}{\partial x_i}|^p-\int_\Omega f(x,v,\nabla v)v\,dx. \tag{3.26}$$

Moreover,

$$\begin{aligned}&\int_\Omega f(x,v,\nabla v)v\,dx\\ &\le\int_\Omega\sigma^{1/q}g(x)v\,dx+\int_\Omega|v|^{\eta+1}\sigma^{(\eta+1)/q}\,dx+\int_\Omega\sum_{i=1}^N w_i^{\delta/p}|\frac{\partial v}{\partial x_i}|^\delta\sigma^{1/q}|v|\,dx.\end{aligned} \tag{3.27}$$

Thanks to Hölder's inequality and (2.5), we have

$$\int_\Omega\sigma^{1/q}g(x)v\,dx\le\Big(\int_\Omega|g(x)|^{q'}\,dx\Big)^{1/q'}\Big(\int_\Omega|v|^q\sigma\,dx\Big)^{1/q}\le c|||v|||. \tag{3.28}$$

On the other hand, by Hölder's inequality,

$$\sum_{i=1}^N w_i^{\delta/p}|\frac{\partial v}{\partial x_i}|^\delta\sigma^{1/q}|v|\le c\sum_{i=1}^N\Big(\int_\Omega w_i^{\frac{\delta q'}{p}}|\frac{\partial v}{\partial x_i}|^{\delta q'}\,dx\Big)^{1/q'}\Big(\int_\Omega|v|^q\sigma\,dx\Big)^{1/q}.$$

In view of (2.5), we have

$$\sum_{i=1}^N w_i^{\delta/p}|\frac{\partial v}{\partial x_i}|^\delta\sigma^{1/q}|v|\le c\sum_{i=1}^N\Big(\int_\Omega w_i^{\frac{\delta q'}{p}}|\frac{\partial v}{\partial x_i}|^{\delta q'}\,dx\Big)^{1/q'}|||v|||. \tag{3.29}$$

Since $0\le\frac{\delta q'}{p}<1$, hence by Hölder's inequality, we deduce

$$\Big(\int_\Omega w_i^{\delta/q'}p|\frac{\partial v}{\partial x_i}|^{\delta q'}\,dx\Big)^{1/q'}\le\Big(\int_\Omega w_i|\frac{\partial v}{\partial x_i}|^p\,dx\Big)^{\delta/p}, \tag{3.30}$$

remark that,

$$(a+b)^r\ge c(a^r+b^r)\quad\text{if } 0\le r<1. \tag{3.31}$$

Combining (3.29), (3.30) and (3.31), we conclude that

$$\sum_{i=1}^N w_i^{\delta/p}|\frac{\partial v}{\partial x_i}|^\delta\sigma^{1/q}|v|\le c|||v|||\Big(\sum_{i=1}^N\int_\Omega w_i|\frac{\partial v}{\partial x_i}|^p\,dx\Big)^{\delta/p}\le c|||v|||\;|||v|||^\delta. \tag{3.32}$$

Further, $0\le\frac{\eta+1}{q}<1$, then by Hölder's inequality and (2.6), we deduce

$$\int_\Omega|v|^{\eta+1}\sigma^{(\eta+1)/q}\,dx\le c|||v|||^{\eta+1}. \tag{3.33}$$

Then from (3.26), (3.28), (3.32) and (3.33), we deduce that

$$\langle Bv,v\rangle\ge\alpha|||v|||^{p-1}-c_1-c_2|||v|||^\eta-c_3|||v|||^{\delta-1}$$

and since $p-1>\eta$ and $p>\delta$, we conclude that $\frac{\langle Bv,v\rangle}{\|v\|}\to+\infty$. Finally, the proof of Theorem is complete thanks to the classical Theorem in [7]. $\square$

4. Examples

Let us consider the Carathéodory functions

$$a_i(x,r,\xi)=w_i|\xi_i|^{p-1}\operatorname{sgn}(\xi_i)$$

Where $w_i(x)(i=1,\dots,N)$ are a given weight functions strictly positive almost everywhere in Ω. We shall assume that the weight function satisfies $w_i(x)=w(x)$, $x\in\Omega$ for $i=0,\dots,N$. It is easy to show that the $a_i(x,s,\xi)$ are Carathéodory function satisfying the growth condition (2.7) and the coercivity (2.9). On the other side, the monotonicity condition (2.8) is verified. In fact,

$$\begin{aligned}
&\sum_{i=1}^{N}(a_i(x,s,\xi)-a_i(x,s,\hat{\xi}))(\xi_i-\hat{\xi}_i)\\
&=w(x)\sum_{i=1}^{N-1}(|\xi_i|^{p-1}\operatorname{sgn}(\xi_i)-|\hat{\xi}_i|^{p-1}\operatorname{sgn}(\hat{\xi}_i))(\xi_i-\hat{\xi}_i)>0
\end{aligned}$$

for almost all $x\in\Omega$ and for all $\xi,\hat{\xi}\in\mathbb{R}^N$ with $\xi\neq\hat{\xi}$, since $w>0$ a.e. in Ω. We consider the Hardy inequality (2.5) in the form

$$\Big(\int_\Omega|u(x)|^q\sigma(x)\,dx\Big)^{1/q}\leq c\Big(\int_\Omega|\nabla u(x)|^p w(x)\,dx\Big)^{1/p},$$

where σ and q are defined in (2.5). In particular, let us use a special weight functions w and σ expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x)=\operatorname{dist}(x,\partial\Omega)$ and set

$$w(x)=d^\lambda(x),\quad \sigma(x)=d^\mu(x).$$

In this case, the Hardy inequality reads

$$\Big(\int_\Omega|u(x)|^q d^\mu(x)\,dx\Big)^{1/q}\leq c\Big(\int_\Omega|\nabla u(x)|^p d^\lambda(x)\,dx\Big)^{1/p}.$$

The corresponding imbedding is compact if:

(i) For, $1<p\leq q<\infty$,

$$\lambda<p-1,\quad \frac{N}{q}-\frac{N}{p}+1\geq 0,\quad \frac{\mu}{q}-\frac{\lambda}{p}+\frac{N}{q}-\frac{N}{p}+1>0. \tag{4.1}$$

(ii) For $1\leq q<p<\infty$,

$$\lambda<p-1,\quad \frac{\mu}{q}-\frac{\lambda}{p}+\frac{1}{q}-\frac{1}{p}+1>0. \tag{4.2}$$

Remarks. 1. Condition (4.1) or Condition (4.2) is sufficient for the compact imbedding (2.6) to hold; see for example [4, example 1], [5, example 1.5], and [6, Theorems 19.17, 19.22].

Let us consider the Carathéodory function

$$f(x,r,\xi)=d^{\frac{\mu}{q}}(x)\Big(d^{\frac{\mu\delta}{q}}(x)|r|^\eta+\sum_{i=1}^{N}d^{\frac{\lambda\delta}{p}}(x)|\xi_i|^\delta+g(x)\Big),$$

with $g \in L^{q'}(\Omega)$, $\sigma(x)$ is weight function and $0 \le \eta < \min(p-1, q-1)$, $0 \le \delta < \frac{p-1}{q'}$. Because of its definition, $f(x, r, \xi)$ satisfies the growth condition (2.10). Also the hypotheses of Theorem 3.1 are satisfied. Therefore, the problem

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \Big(d^\lambda(x) |\frac{\partial u}{\partial x_i}|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \Big)\, dx \\ &= \int_\Omega d^{\mu/q}(x) \Big(d^{\mu\delta/q}(x)|u|^\eta + \sum_{i=1}^{N} d^{\lambda\delta/p}(x)|\xi_i|^\delta + g(x) \Big) v\, dx\,, \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega, w)$, has at last one solution.

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