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ASYMPTOTIC BEHAVIOR OF A NON-NEWTONIAN FLOW WITH STICK-SLIP CONDITION

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Abstract. This paper concerns the asymptotic behavior of solutions of the 3D non-newtonian fluid flow with slip condition (Tresca's type) imposed in a part of the boundary domain. Existence of at least one weak solution is proved. We study the limit when the thickness tends to zero and we prove a convergence theorem for velocity and pressure in appropriate functional spaces. The limit of slip condition is obtained. Besides, the uniqueness of the velocity and the pressure limits are also proved.

1. Introduction

In the case of polymer fluids the no slip condition on the fluid-solid interface is not always satisfied. This boundary condition is sometimes overpassed and we must deal with slip at the wall. This phenomenon has been studied in a lot of mechanical papers related to non newtonian fluids (see [7, 12]). For polymer fluids, slip at the wall is not surprising : entangled polymer have a mixed fluid and solid dynamic behavior.

We consider the incompressible isothermal viscous flow of a non newtonian fluid through a thin slab. The viscosity of fluid follows the power law (see [4]). On the part of the boundary we consider the stick-slip condition given by Tresca law. We suppose that the flow is steady and the Reynolds number is proportional to $\varepsilon^{-\gamma}$. The inertia effects are neglected, this condition is proved in [2] for different cases corresponding to various values of γ and of the power r of the Carreau law. It is know that for polymer (non newtonian) flow through a thin slab the Hele-Shaw equation is used. Our goal is to give mathematical foundation for the nonlinear averaged momentum equation with stick-slip condition.

Let ω be a bounded open set of $\mathbb{R}^2$ with sufficiently smooth boundary. The domain is thin slab defined by:

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3, (x_1, x_2) \in \omega,\ \text{and } 0 < x_3 < \varepsilon h(x_1, x_2)\}$$

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Where $h : \omega \longrightarrow \mathbb{R}_+^*$, is a C^1. The incompressibility equation is

$$\operatorname{div} v^\varepsilon = v^\varepsilon_{i,i} = 0 \quad \text{in } \Omega_\varepsilon. \tag{1.1}$$

For simplicity we take the constant density $\rho = 1$. Then the equation of motion is

$$-\sigma^\varepsilon_{ij,j} = f_i \quad \text{in } \Omega_\varepsilon \quad (i,j = 1,2,3). \tag{1.2}$$

The constitutive law is

$$\sigma^\varepsilon = -p^\varepsilon I + 2\eta_0 (D_{II})^{\frac{r-2}{2}} D(v^\varepsilon), \tag{1.3}$$

where $v^\varepsilon = (v^\varepsilon_i)$ represents the velocity field, $\sigma^\varepsilon = \sigma^\varepsilon_{ij}$ the stress tensor, $f = (\hat{f}(x_1,x_2), f_3(x_1,x_2))$ the body forces, $D = D_{ij}$ the rate of strain tensor, given by $D_{ij}(v^\varepsilon) = \frac{1}{2}(v^\varepsilon_{i,j} + v^\varepsilon_{j,i})$, $D_{II} = D_{ij}(v^\varepsilon)D_{ij}(v^\varepsilon)$, p^ε denotes the pressure, η_0 the viscosity and $r > 1$ the power of law which may represent a pseudo-plastic fluid if $1 < r < 2$, a dilatant fluid if $r > 2$.

We consider the following conditions on $\partial\Omega_\varepsilon = \omega \cup \Gamma^\varepsilon_a$:

- On ω: $v^\varepsilon.n = 0$ and

$$\begin{gathered} |\tau^\varepsilon_t| < g(x_1,x_2) \;\Rightarrow\; v^\varepsilon_t(x_1,x_2) = 0 \\ |\tau^\varepsilon_t| = g(x_1,x_2) \;\Rightarrow\; \exists\, \lambda \geq 0 \text{ such that } v^\varepsilon_t = -\lambda \tau^\varepsilon_t \end{gathered} \tag{1.4}$$

- On Γ^ε_a: $v^\varepsilon = 0$

Here $n = (n_i)$ is the unit outward normal to $\partial\omega_\varepsilon$, and

$$\begin{gathered} v^\varepsilon_t = v^\varepsilon - v^\varepsilon_n n, \quad v^\varepsilon_n = v^\varepsilon_i n_i \\ \tau^\varepsilon_{ti} = \sigma^\varepsilon_{ij} n_j - \sigma^\varepsilon_n n_i, \quad \sigma^\varepsilon_N = \sigma^\varepsilon_{ij} n_i n_j \end{gathered}$$

are, respectively, the tangential velocity, normal velocity the components of tangential stress tensor and the normal stress. $g(x_1,x_2)$ is a positive function in $L^\infty(\omega)$ and f in $L^{r'}(\Omega)$. We use the re-scaling $z = \frac{x_3}{\varepsilon}$ and the notation $v(\varepsilon)(x_1,x_2,z) = v^\varepsilon(x_1,x_2,\varepsilon z)$, $p(\varepsilon)(x_1,x_2,z) = p^\varepsilon(x_1,x_2,\varepsilon z)$. Hence, $v(\varepsilon)$ is sequence of functions defined on fixed domain Ω, then the system (1.1)-(1.4) can be written

$$\begin{gathered} -\varepsilon^\gamma \operatorname{div}_\varepsilon(|D_\varepsilon(v(\varepsilon))|^{r-2} D_\varepsilon(v(\varepsilon))) + \nabla_\varepsilon p(\varepsilon) = f \quad \text{in } \Omega \\ \operatorname{div}_\varepsilon(v(\varepsilon)) = 0 \quad \text{in } \Omega \end{gathered} \tag{1.5}$$

On Γ_a: $v(\varepsilon) = 0$
On ω: $v(\varepsilon).n = 0$ and

$$\begin{gathered} |\tau_t(\varepsilon)| < g(x_1,x_2) \;\Rightarrow\; v_t(\varepsilon)(x_1,x_2) = 0 \\ |\tau_t(\varepsilon)| = g(x_1,x_2) \;\Rightarrow\; \exists\, \lambda \geq 0 \quad \text{such that } v_t(\varepsilon) = -\lambda \tau_t(\varepsilon)\,. \end{gathered}$$

Here ∇_ε, D_ε, $\operatorname{div}_\varepsilon$ are the corresponding rescaled differential operators defined by

$$\begin{gathered} (\nabla_\varepsilon v)_{i,j} = \frac{\partial v_i}{\partial x_j} \quad \text{for } i = 1,2,3;\; j = 1,2; \\ (\nabla_\varepsilon v)_{i,3} = \frac{1}{\varepsilon}\frac{\partial v_i}{\partial z} \quad \text{for } i = 1,2,3; \\ D_\varepsilon(v) = \frac{1}{2}((\nabla_\varepsilon v) + (\nabla_\varepsilon v)^t), \quad \operatorname{div}_\varepsilon = \nabla_\varepsilon\,. \end{gathered}$$

Our main aim in this paper is to prove the existence of weak solution $(v(\varepsilon), p(\varepsilon))$ of problem (1.5) and to study the limit when the small thickness of the slab tends to zero.

2. Functional framework and existence

To formulate the notion of weak solution of the problem (1.5), we recall some Sobolev spaces

$$\begin{gathered} W^{1,r}(\Omega)=\{v\in L^r(\Omega) \text{ and } \frac{\partial v}{\partial x_i}\in L^r(\Omega),\ i=1,2,3\},\\ V^r=\{v\in (W^{1,r}(\Omega))^3,\ v=0 \text{ on } \Gamma_a,\ v.n=0 \text{ on } \omega\},\\ V^r_{\rm div}=\{v\in V^r, \operatorname{div}(v)=0 \text{ in } \Omega\}. \end{gathered}$$

On V^r, we define the functional $j: V^r\to\mathbb{R},\quad v\mapsto \int_\omega g(s)|v_t(s)|ds$. Note that j is continuous convex, but non differentiable. The problem (2.1) has a variational formulation (see [5]) written as follos:
Find $(v(\varepsilon),p(\varepsilon))\in V^r_{\rm div}\times L_0^{r'}(\Omega)$ such that

$$\begin{aligned} &\varepsilon^\gamma\int_\Omega |D_\varepsilon(v(\varepsilon))|^{r-2}D_\varepsilon(v(\varepsilon))D_\varepsilon(w-v(\varepsilon))dx+\int_\Omega p(\varepsilon)div_\varepsilon(w)dx+j(w)-j(v(\varepsilon))\\ &\geq\int_\Omega f(w-v(\varepsilon))dx,\quad \forall w\in V^r \end{aligned} \tag{2.1}$$

For $v\in W^{1,r}(\Omega)$, we define the functional

$$F_r^\varepsilon(v)=\frac{\varepsilon^\gamma}{r}\int_\Omega |D(v)|^r dx-\int_\Omega fv\,dx. \tag{2.2}$$

Note that F_r^ε is Gateaux-differentiable and strictly convex, and that for every v,w in $W^{1,r}(\Omega)$,

$$\langle DF_r^\varepsilon(v),w\rangle=\varepsilon^\gamma\int_\Omega |D(v)|^{r-2}D(v)D(w)dx-\int_\Omega fw\,dx, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes duality of pairing $W^{-1,r'}(\Omega)\times W^{1,r}(\Omega)$, and r' is the conjugate number of r,$(\frac{1}{r}+\frac{1}{r'}=1)$.

The operators DF_r^ε is strictly monotone bounded coercive and hemicontinuous (see [14]). Now, we introduce an auxiliary problem.
Find $v(\varepsilon)\in V^r_{div}$ such that

$$\langle DF_r^\varepsilon(v(\varepsilon)),w-v(\varepsilon)\rangle+j(w)-j(v(\varepsilon))\geq\int_\Omega f(w-v(\varepsilon))dx,\quad \forall w\in V^r_{div}\,. \tag{2.4}$$

Then we define the associated minimization problem: Find $u\in V^r_{\rm div}$ such that

$$\phi(u)=\inf_{w\in V^r_{div}}[\phi(w)], \tag{2.5}$$

where $\phi(w)=F_r^\varepsilon(w)+j(w)$.

Lemma 2.1. *Problems* (2.4) *and* (2.5) *are equivalent.*

Proof. The proof use the monotonicity of the operators DF_r^ε and the convexity of j (see [14]). □

Theorem 2.2. *For $r>1$ and fixed ε,* (1.5) *has a unique solution $(v(\varepsilon),p(\varepsilon))$ in $V^r_{\rm div}\times L_0^{r'}(\Omega)$.*

Proof. From nonlinear operator theory we deduce that (2.4) has a unique solution $v(\varepsilon) \in V^r_{div}$ (see [9, 14]). Using Lemma 2.1 we have that $v(\varepsilon)$ is also a unique solution of the problem (2.5).

Let $Y = L^r(\omega) \times L^r(\Omega)$, we introduce the following indicator functionals

$$w \in L^r(\Omega); \quad \psi(w) = \begin{cases} 0 & \text{if } w = 0 \\ +\infty & \text{otherwise} \end{cases}$$
$$v \in V^r; \quad \mathcal{G}(v) = (v/\omega, \operatorname{div}(v)) \in Y,$$
$$q = (q_1, q_2) \in Y; \quad \varphi(q) = j(q_1) \ + \ \psi(q_2).$$

The unique solution $v(\varepsilon)$ of (2.4), satisfies

$$v(\varepsilon) := \inf_{w \in V^r} [F^\varepsilon_r(w) + \varphi(\mathcal{G}(w))] \tag{2.6}$$

The existence of pressure p is assured by using the dual problem. The problem dual to (2.6) can be written as

$$p^* = (p^*_1, p^*_2) \in Y; \quad p* := \sup_{q^* \in Y^*} [-F^{\varepsilon^*}_r(\mathcal{G}^*(q^*)) - \varphi^*(-q^*)] \tag{2.7}$$

$v(\varepsilon)$ and p^* are solutions of (2.6) and (2.7) verify the following extremality relation (see [6]):

$$F^\varepsilon_r(v(\varepsilon)) + \varphi(\mathcal{G}(v(\varepsilon))) + F^{\varepsilon^*}_r(\mathcal{G}^*(p^*)) + \varphi^*(-p^*) = 0 \tag{2.8}$$

where $F^{\varepsilon^*}_r$, φ^* denote the conjugates functionals of F^ε_r and φ defined by

$$F^{\varepsilon^*}_r(\mathcal{G}^*(p^*)) = \sup_{u \in V^r} \{< \mathcal{G}^*(p^*), u > -F(u)\} \tag{2.9}$$

$$\begin{aligned} \varphi^*(-q^*) &= \sup_{q \in Y} \{\langle -q^*, q\rangle - \varphi(q)\} \\ &= \sup_{q_1 \in L^r(\omega)} \{\langle -q^*_1, q_1\rangle - j(q_1)\} + \sup_{q_2 \in L^r(\Omega)} \{\langle -p^*_2, q_2\rangle - \psi(q_2)\} \end{aligned} \tag{2.10}$$

Observing that $\sup_{q_2 \in L^r(\Omega)}\{\langle -p^*_2, q_2\rangle - \psi(q_2)\} = 0$ and replacing in (2.8) q by $\mathcal{G}(u)$, we obtain that $v(\varepsilon)$ satisfies

$$F^\varepsilon_r(v(\varepsilon)) - F^\varepsilon_r(u) + j(\mathcal{G}_1(v(\varepsilon))) - j(\mathcal{G}_1(u)) + \psi(\mathcal{G}_2(v(\varepsilon))) - \langle p^*_2, div(u)\rangle \le 0, \quad \forall u \in V^r.$$

Finally, we get the result by using lemma 2.1. □

3. Convergence results

The limit model is obtained thanks to an asymptotic analysis. The used techniques emanate from the ones used in homogenization. In this section we adopt the following notation:

$$v(\varepsilon) = (\hat{v}(\varepsilon), v_3(\varepsilon)), \quad \hat{v}(\varepsilon) = (v_1(\varepsilon), v_2(\varepsilon)),$$
$$x = (x', z), \quad x' = (x_1, x_2), \quad \delta = \frac{r-\gamma}{r-1}.$$

Also, we use the functional space

$$\chi^r = \{v \in L^r(\Omega) \text{ and } \frac{\partial v}{\partial z} \in L^r(\Omega)\}.$$

and we will need the following results given by lemmas 3.1 and 3.2. For $v^\varepsilon \in (L^r(\Omega_\varepsilon))^3$, $1 \le r < +\infty$, we have for every $v \in (L^r(\Omega_\varepsilon))^3$: $\|v^\varepsilon\|_{(L^r(\Omega_\varepsilon))^3} = \varepsilon^{\frac{1}{r}}\|v(\varepsilon)\|_{(L^r(\Omega))^3}$.

Lemma 3.1 (Poincaré inequality)**.** *For $v \in (W_0^{1,r}(\Omega_\varepsilon))^3$, $1 < r < +\infty$,*

$$\|v^\varepsilon\|_{(L^r(\Omega_\varepsilon))^3} \le \varepsilon \|\frac{\partial v^\varepsilon}{\partial x_3}\|_{(L^r(\Omega_\varepsilon))^3}. \tag{3.1}$$

Lemma 3.2 (Korn inequality)**.** *For $v \in (W_0^{1,r}(\Omega_\varepsilon))^3$, $1 < r < +\infty$,*

$$\|\nabla v^\varepsilon\|_{(L^r(\Omega_\varepsilon))^9} \le C.\|D(v^\varepsilon)\|_{(L^r(\Omega_\varepsilon))^9}. \tag{3.2}$$

where C is a positive constant independent of ε and v^ε.

The proof of the above lemma can be found in [8] and [10].

Proposition 3.3. *Let $(v(\varepsilon), p(\varepsilon))$, be a sequence solution to problem (2.1), we have*

$$\hat{u}(\varepsilon) = \varepsilon^{-\delta}\hat{v}(\varepsilon) \rightharpoonup \hat{u} \quad in\ (\chi^r)^2, \tag{3.3}$$

$$u_3(\varepsilon) = \varepsilon^{-\delta}\hat{v}_3(\varepsilon) \rightharpoonup 0 \quad in\ \chi^r, \tag{3.4}$$

$$p(\varepsilon) \rightharpoonup p(x') \quad in\ L_0^{r'}(\Omega). \tag{3.5}$$

Proof. Estimates for velocity and pressure are obtained from (2.1), by using the Poincaré and Korn inequalities. Taking $w = -v(\varepsilon)$ in (2.1) we obtain with Schwartz inequality

$$\varepsilon^\gamma \int_\Omega |D_\varepsilon(v(\varepsilon))|^r dx \le c\|v(\varepsilon)\|_{L^r(\Omega)}. \tag{3.6}$$

Using (3.1) and (3.2), we deduce $\|v(\varepsilon)\|_{(L^r(\Omega))^3} \le \|\frac{\partial v(\varepsilon)}{\partial z}\|_{(L^r(\Omega))^3}$ and

$$\|v(\varepsilon)\|_{(L^r(\Omega))^3} \le C\varepsilon\|D_\varepsilon(v(\varepsilon))\|_{(L^r(\Omega))^{3\times 3}}$$

which with (3.6) give

$$\begin{gathered}\|v(\varepsilon)\|_{(L^r(\Omega))^3} \le C.\varepsilon^\delta; \quad \|\nabla_{x'}v(\varepsilon)\|_{(L^r(\Omega))^3} \le C.\varepsilon^{\delta-1}, \\ \|\frac{\partial v(\varepsilon)}{\partial z}\|_{(L^r(\Omega))^3} \le C.\varepsilon^\delta.\end{gathered} \tag{3.7}$$

We have, with incompressibility condition, for any $\varphi \in W_0^{1,r'}(\Omega)$,

$$\int_\Omega div_\varepsilon(v(\varepsilon))\varphi dx'dz = \int_\Omega div_{x'}v(\varepsilon)\nabla_{x'}\varphi dx'dz \ + \ \frac{1}{\varepsilon}\int_\Omega \frac{\partial v_3(\varepsilon)}{\partial z}\varphi dx = 0$$

using Green's formula, we obtain

$$|\int_\Omega \frac{\partial v_3(\varepsilon)}{\partial z}\varphi dx| \le \varepsilon\|v(\varepsilon)\|_{L^r(\Omega)}\|\varphi\|_{W_0^{1,r'}(\Omega)}$$

which with (3.7) give

$$\|\frac{\partial v_3(\varepsilon)}{\partial z}\|_{W^{-1,r}(\Omega)} \le C\varepsilon^{\delta+1}. \tag{3.8}$$

Let $\varphi \in (W_0^{1,r}(\Omega)^3$, multiplying, the first equation of (1.5), by φ and integrating over Ω, we obtain with Green's formula and Schwartz inequality

$$\langle \nabla_\varepsilon p(\varepsilon), \varphi\rangle \le \varepsilon^\gamma(\int_\Omega |D_\varepsilon(v(\varepsilon))|^r)^{\frac{1}{r'}}\|D_\varepsilon(\varphi)\|_{(L^r(\Omega))^3} \ + \ C\|\varphi\|_{L^r(\Omega)}.$$

Using the inequality $\|D_\varepsilon(\varphi)\|_{(L^r(\Omega))^{3\times 3}} \le C.\frac{1}{\varepsilon}.\|\nabla\varphi\|_{(L^r(\Omega))^{3\times 3}}$ and (3.7) we deduce

$$\langle \nabla_\varepsilon p(\varepsilon), \varphi\rangle \le C\varepsilon^{\gamma-1}\|D_\varepsilon(v(\varepsilon))\|^{\frac{r}{r'}}\|\varphi\|_{W_0^{1,r}(\Omega)} + \ C\|\varphi\|_{W_0^{1,r}(\Omega)}$$

which gives

$$\|p(\varepsilon)\|_{L_0^{r'}(\Omega)} \le C; \quad \text{and} \quad \|\frac{\partial p(\varepsilon)}{\partial z}\|_{W^{-1,r'}(\Omega)} \le C\varepsilon. \tag{3.9}$$

Finally (3.3), (3.4) and (3.5) are direct consequence of the *a priori* estimates (3.7), (3.8) and (3.9). ☐

Proposition 3.4. *The function $\overline{u}(x')$ defined by $\overline{u}(x') = \int_0^{h(x')} \hat{u}(x', z)dz$ satisfies*

$$\begin{aligned} \operatorname{div}_{x'}(\overline{u}(x')) = 0 \quad \text{in } \omega, \\ \nu.\overline{u}(x') = 0 \quad \text{on } \partial\omega. \end{aligned} \tag{3.10}$$

where ν is the unit outward normal to $\partial\omega$.

Proof. Let $\varphi \in C_0^\infty(\omega)$, using the incompressibility condition and Green's formula, we obtain

$$\begin{aligned} &\int_\Omega \nabla_\varepsilon.u(\varepsilon)(x', z)\varphi(x')dx'dz \\ &= -\int_\Omega u(\varepsilon)(x', z)\nabla_{x'}\varphi(x')dx'dz + \int_{\partial\Omega} u(\varepsilon)(x', z).n\ \varphi(x')d\gamma, \end{aligned}$$

as $u(\varepsilon).n = 0$ on $\partial\Omega$, we deduce

$$\nabla_{x'}.(\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz) = 0. \tag{3.11}$$

$$\begin{aligned} \|\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz\|^r_{(L^r(\omega))^2} &= \int_\omega |\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz|^r dx' \\ &\le C(\int_\Omega |\hat{u}(\varepsilon)(x', z)|^r dx'dz). \end{aligned}$$

This implies

$$\|\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz\|_{(L^r(\omega))^2} \le C\,. \tag{3.12}$$

Let $\varphi \in W^{1,r'}(\omega)$, multiplying (3.11) by φ and integrating over ω, we obtain with Green's formula

$$\begin{gathered} |\int_{\partial\omega} \nu.(\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz).\varphi(x')d\gamma| \le C.\|\varphi\|_{W^{1,r'}(\omega)}, \\ \|\nu.(\int_0^{h(x')} \hat{u}(\varepsilon)(x', z)dz)\|_{W^{-\frac{1}{r},r}(\omega)} \le C. \end{gathered}$$

Finally, we prove (3.10) passing to the limit in (3.11) ☐

4. Limit problem

Theorem 4.1. *The cluster point* $(\hat{u}(x',z),p(x'))$ *defined by* (3.3) *and* (3.5) *verify the following limit problem*

$$\begin{gathered}\frac{-1}{2^{r/2}}\frac{\partial}{\partial z}(|\frac{\partial \hat{u}}{\partial z}|^{r-2}\frac{\partial \hat{u}}{\partial z})+\nabla_{x'}p=\hat{f}\quad \text{in } W^{-1,r'}(\Omega),\\ |\frac{\partial \hat{u}}{\partial z}|<(g(x'))^{\frac{1}{r-1}}\ \Rightarrow\ \hat{u}(x',0)=0,\\ |\frac{\partial \hat{u}}{\partial z}|=(g(x'))^{\frac{1}{r-1}}\ \Rightarrow\ \exists\,\lambda\geq 0 \text{ such that } \hat{u}(x',0)=\lambda\hat{\tau}(x').\end{gathered}\tag{4.1}$$

where $\hat{\tau}(x')=|\frac{\partial \hat{u}}{\partial z}(x',0)|^{r-2}\frac{\partial \hat{u}}{\partial z}(x',0)$.

Proof. To linearize the problem we use the Minty's lemma (see[6]), we obtain that (2.1) is equivalent to

$$\begin{aligned}&\varepsilon^{\gamma}\int_\Omega |D_\varepsilon(w)|^{r-2}D_\varepsilon(w)D_\varepsilon(w-v(\varepsilon))dx+\int_\Omega p(\varepsilon)\operatorname{div}_\varepsilon(w)dx+j(w)-j(v(\varepsilon))\\ &\geq \int_\Omega f(w-v(\varepsilon))dx,\quad \forall w\in\ V^r\end{aligned}\tag{4.2}$$

as $u(\varepsilon)=\varepsilon^{-\delta}v(\varepsilon)$, we take in (4.2), $w=(\hat{w},w_3)=\varepsilon^{\delta}w$ and we divide the inequality by ε^{δ}, we obtain that $u(\varepsilon)$ satisfies

$$\begin{aligned}&\varepsilon^{r}\int_\Omega |D_\varepsilon(w)|^{r/2}D_\varepsilon(w)D_\varepsilon(w-u(\varepsilon))dx+\int_\Omega p(\varepsilon)\operatorname{div}_\varepsilon(w)dx+j(w)-j(u(\varepsilon))\\ &\geq \int_\Omega f(w-u(\varepsilon))dx,\quad \forall w\in V^r\end{aligned}\tag{4.3}$$

Using proposition 3.1, we pass to the limit in the first term of (4.3), and we obtain that

$$\varepsilon^{r}\int_\Omega |D_\varepsilon(w)|^{r/2}D_\varepsilon(w)D_\varepsilon(w-u(\varepsilon))dx$$

converges to

$$\int_\Omega\Big(\frac12\sum_{i=1}^{2}(\frac{\partial w_i}{\partial z})^2+(\frac{\partial w_3}{\partial z})^2\Big)^{\frac{r-2}{2}}\Big(\frac12\sum_{i=1}^{2}\frac{\partial w_i}{\partial z}\frac{\partial(w_i-u_i)}{\partial z}+\frac{\partial w_3}{\partial z}\frac{\partial(w_3-u_3)}{\partial z}\Big)dx.$$

With (2.9) we have that $u_3=0$, then we can choose $w_3=0$, using (3.3), (3.5) and the fact j is convex and continuous, when passing to limit in (4.3), we get to $\hat{u}$ and p are the solution of the following inequality

$$\begin{aligned}&\frac{1}{2^{r/2}}\int_\Omega|\frac{\partial \hat{w}}{\partial z}|^{r-2}\frac{\partial \hat{w}}{\partial z}\frac{\partial(\hat{w}-\hat{u})}{\partial z}dx+\langle\nabla_{x'}p,\hat{w}-\hat{u}\rangle+j(\hat{w})-j(\hat{u})\\ &\geq\int_\Omega\hat{f}(\hat{w}-\hat{u})dx,\quad\forall\hat{w}\in V^r\end{aligned}\tag{4.4}$$

Using again Minty's lemma, the last inequality is equivalent to

$$\begin{aligned}&\frac{1}{2^{r/2}}\int_\Omega|\frac{\partial \hat{u}}{\partial z}|^{r-2}\frac{\partial \hat{u}}{\partial z}\frac{\partial(\hat{w}-\hat{u})}{\partial z}dx\ +\ \langle\nabla_{x'}p,\hat{w}-\hat{u}\rangle+j(\hat{w})-j(\hat{u})\\ &\geq\int_\Omega\hat{f}(\hat{w}-\hat{u})dx,\quad\forall\hat{w}\in V^r\end{aligned}\tag{4.5}$$

Now, we use the density result shown in ([3]): there exists a sequence of functions in V^r which has $\hat{u}$ as a limit in χ^r, then we can take in (4.5) $\hat{w} = \hat{u} \pm \varphi$, where $\varphi \in (W_0^{1,r}(\Omega))^2$, and we get the first equation of (4.1).

From (4.5), the first equation of (4.1) and by using Green's formula, we show that $\hat{u}$ satisfies

$$\int_\omega (g|\hat{w}| - \hat{\tau}\hat{w})d\Gamma - \int_\omega (g|\hat{u}| - \hat{\tau}\hat{u})d\Gamma \geq 0, \quad \forall \hat{w} \in (W^{1,r}(\Omega))^2. \tag{4.6}$$

Choosing $\hat{w}$ in (4.6) such that $\hat{w} = \pm\lambda\hat{w}$, where $\lambda \geq 0$, we obtain that for all $\hat{w} \in (W_0^{1,r}(\Omega))^2$,

$$\int_\omega (g|\hat{w}| \pm \hat{\tau}\hat{w})d\Gamma \geq 0, \tag{4.7}$$

$$\int_\omega (g|\hat{u}| - \hat{\tau}\hat{u})d\Gamma \leq 0\,. \tag{4.8}$$

Now, we introduce the functional space

$$\mathcal{I} = \{\psi \in (W^{1-\frac{1}{r},r}(\partial\Omega)),\ \text{whith a compact support on } \omega\}.$$

From (4.7), we deduce that the function $\hat{w} \in \mathcal{I}$, $\hat{w} \to \int_\omega \hat{\tau}\hat{w}\, d\Gamma$ is continuous for the topology induced by $(L^r(\omega))^2$. Since g is in $L^\infty(\omega)$ and strictly positive, with (4.7) we have

$$\int_\omega (g^{-1}\hat{\tau})(g\hat{w})\, d\Gamma \leq \int_\omega g|\hat{w}|\, d\Gamma = \|g\hat{w}\|.$$

Since $\mathcal{I}$ is dense in $L^1(\omega)$, we get

$$g^{-1}\hat{\tau} \in L^\infty(\omega) \quad \text{and} \quad |\hat{\tau}| \leq g \text{ a.e. on } \omega. \tag{4.9}$$

Which with (4.9) and using inequality (4.8) implies the boundary conditions on ω. □

Theorem 4.2. *The limit problem* (4.1) *has a unique solution* $(\hat{u}(x',z), p(x'))$ *in* $(\chi^r)^2 \times L_0^{r'}(\omega)$.

Proof. The uniqueness will be proven by using proposition 3.2. As usual, we assume that the problem (4.1) has at least two solutions $(\hat{u}_1, p_1)$ and $(\hat{u}_2, p_2)$. Integrating the first equation with respect to z, we obtain

$$\frac{1}{2^{r/2}}\Big|\frac{\partial \hat{u}_1}{\partial z}\Big|^{r-2}\frac{\partial \hat{u}_1}{\partial z} = \tau_1(x') - z(\hat{f} - \nabla_{x'}p_1) \tag{4.10}$$

$$\frac{1}{2^{r/2}}\Big|\frac{\partial \hat{u}_2}{\partial z}\Big|^{r-2}\frac{\partial \hat{u}_2}{\partial z} = \tau_2(x') - z(\hat{f} - \nabla_{x'}p_2) \tag{4.11}$$

where $\tau_i(x') = |\frac{\partial \hat{u}_i}{\partial z}(x',0)|^{r-2}\frac{\partial \hat{u}_i}{\partial z}(x',0)$, $i = 1,2$. We consider the function η_r defined by $\xi \in \mathbb{R}^2$, $\eta_r(\xi) = |\xi|^{r-2}\xi$, which satisfies (see [14], [13]),

$$\begin{aligned}
&\big(\eta_r(\frac{\partial \hat{u}_2}{\partial z}) - \eta_r(\frac{\partial \hat{u}_1}{\partial z}), \frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}\big) \\
&\geq \begin{cases} (\frac{1}{2})^{r-1}\big|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}\big|^r & \text{if } r \geq 2 \\ (r-1)\big(|\frac{\partial \hat{u}_2}{\partial z}| + |\frac{\partial \hat{u}_1}{\partial z}|\big)^{r-2}\big|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}\big|^2 & \text{if } 1 < r < 2\,. \end{cases}
\end{aligned}$$

Let $\beta(x') = \tau_2(x') - \tau_1(x')$, we have

$$\begin{aligned}
&\big(\beta(x') + z\nabla_{x'}(p_2 - p_1), \frac{\partial}{\partial z}(\hat{u}_2 - \hat{u}_1)\big) \\
&\geq \begin{cases} (\frac{1}{2})^{r-1}\big|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}\big|^r & \text{if } r \geq 2 \\ (r-1)\big(|\frac{\partial \hat{u}_2}{\partial z}| + |\frac{\partial \hat{u}_1}{\partial z}|\big)^{r-2}\big|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}\big|^2 & \text{if } 1 < r < 2 \end{cases}
\end{aligned} \tag{4.12}$$

Using the boundary conditions on ω, we get

$$\int_\omega \big(\beta(x'), \hat{u}_2(x', 0) - \hat{u}_1(x', 0)\big) dx' \geq 0 \tag{4.13}$$

Integrating (4.12) over Ω, from Green's formula proposition 3.2 and (4.13), we have

$$|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}|^r = 0 \quad \text{if } r \geq 2 \quad \text{and} \tag{4.14}$$

$$(|\frac{\partial \hat{u}_2}{\partial z}| + |\frac{\partial \hat{u}_1}{\partial z}|)^{r-2}|\frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_1}{\partial z}|^2 = 0 \quad \text{if } 1 < r < 2\,. \tag{4.15}$$

Finally, since $\hat{u}_1(x', h(x')) = u_2(x', h(x')) = 0$, we deduce that $\hat{u}_2 = \hat{u}_1$ a.e. in Ω and $p_2 = p_1$ a.e. on ω. □

Theorem 4.3. *Let $\delta > 0$ and $0 < \delta \leq h(x') \leq 1$. Then $p(x')$, $\hat{\tau}(x')$, and $\bar{\hat{u}}(x') = \int_0^{h(x')} \hat{u}(x', z)\, dz$ satisfy $p(x') \in W^{1,r'}(\omega)$ and*

$$\begin{gathered}
\operatorname{div}_{x'}(\bar{\hat{u}}(x')) = 0 \quad \text{in } \omega, \\
\nu.\bar{\hat{u}}(x') = 0 \quad \text{on } \partial\omega.
\end{gathered}$$

where

$$\begin{aligned}
\bar{\hat{u}}(x') = &\int_0^{h(x')} \big(\int_0^z |\hat{\tau}(x') - \gamma(x')\xi|^{r'-2}(\hat{\tau}(x') - \gamma(x')\xi)d\xi\big)dz \\
&+ \ h(x') \int_0^{h(x')} |\gamma(x')\xi - \hat{\tau}(x')|^{r'-2}(\gamma(x')\xi - \hat{\tau}(x'))d\xi.
\end{aligned}$$

and $\gamma(x') = 2^{r/2}(\hat{f}(x') - \nabla_{x'}p(x'))$.

Proof. As a weak limit $\hat{u} \in (\chi^r)^2$, and then

$$|\frac{\partial \hat{u}}{\partial z}|^{r-2}\frac{\partial \hat{u}}{\partial z} \ \in \ (L^{r'}(\Omega)^2.$$

Taking the test function $\hat{\phi}(x', z) = \mathcal{G}(x')\psi(z)$, for all $\mathcal{G} \in \ (C_0^\infty(\omega))^2$, and for fixed $\psi \in \ C_0^\infty(0, \delta)$, $\psi \geq 0$, $\int_0^\delta \psi(z)\, dz = 1$, we have by first equation of (4.1),

$$\frac{1}{2^{r/2}} \int_\Omega \hat{w}\frac{\partial \hat{u}}{\partial z}\, dx'dz - \int_\omega p(x') \operatorname{div}_{x'}\mathcal{G}(x') = \int_\omega \hat{f}(x')\mathcal{G}(x')\, dx'. \tag{4.16}$$

Since f is regular, (4.16) shows that $\nabla_{x'}p \in L^{r'}(\Omega)$, and we have

$$\frac{-1}{2^{r/2}}\frac{\partial}{\partial z}\big(\big|\frac{\partial \hat{u}}{\partial z}\big|^{r-2}\frac{\partial \hat{u}}{\partial z}\big) + \nabla_{x'}p = \hat{f} \quad \text{in } L^{r'}(\Omega).$$

Integrating this equation twice with respect to z, we obtain

$$\hat{u}(x', z) = \int_0^z |\hat{\tau}(x') - \gamma(x')\xi|^{r'-2}\, (\hat{\tau}(x') - \gamma(x')\xi)\, d\xi \ + \ \hat{u}(x', 0).$$

Since $\hat{u}(x', h(x')) = 0$, we get

$$\hat{u}(x',0) = \int_0^{h(x')} |\gamma(x')\xi - \hat{\tau}(x')|^{r'-2}(\gamma(x')\xi - \hat{\tau}(x'))d\xi.$$

Finally, we obtain the result for proposition 3.2. □

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