2004-Fez conference on Differential Equations and Mechanics
Electronic Journal of Differential Equations, Conference 11, 2004, pp. 129–134.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

STABILITY OF THE PRINCIPAL EIGENVALUE OF A NONLINEAR ELLIPTIC SYSTEM

ABDELOUAHED EL KHALIL, MOHAMMED OUANAN,
ABDELFATTAH TOUZANI

ABSTRACT. This paper concerns some special properties of the principal eigenvalue of nonlinear elliptic systems with Dirichlet boundary conditions. We study the stability with respect to the exponents p and q; and the dependence on the domain variations.

1. INTRODUCTION

In this work, we survey two results concerning the nonlinear elliptic system

$$\begin{aligned} -\Delta_p u &= \lambda |u|^\alpha |v|^\beta v \quad \text{in } \Omega \\ -\Delta_q v &= \lambda |u|^\alpha |v|^\beta u \quad \text{in } \Omega \\ (u,v) &\in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, ($N \geq 3$), is a bounded domain not necessary regular; α, β, p, q are reel numbers such that $p > 1$, $q > 1$, $\alpha \geq 0$, $\beta \geq 0$ satisfying

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1. \tag{1.2}$$

The principal eigenvalue $\lambda_1(p,q)$ of (1.1) is defined as

$$\lambda_1 = \inf\big\{A(u,v) : (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega),\, B(u,v) = 1\big\}, \tag{1.3}$$

where

$$A(u,v) = \frac{\alpha+1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q\,dx,$$
$$B(u,v) = \int_\Omega |u|^\alpha |v|^\beta uv\,dx.$$

It is known (see [2]) that this eigenvalue is simple, isolated, and can be expressed as (1.3). It is also known that the associated principal eigenfunction (u,v) can be chosen strictly positive in Ω. This eigenfunction is a regular pair of functions where

2000 *Mathematics Subject Classification.* 35P20, 35J70, 35J20.
Key words and phrases. p-Laplacian operator; principal eigenvalue; stability; segment property; dependence on the domain.
©2004 Texas State University - San Marcos.
Published October 15, 2004.

the infimum of the energy functional A is attained on the C^1-manifold $\{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : B(u,v) = 1\}$.

The operator $-\Delta_p$ arises in problems from pure mathematics, such as the theory of quasiregular and quasiconformal mapping, as well as in problems from a variety of applications, e.g. non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, petroleum extraction, astronomy, etc.

The main purpose of this work is to prove that the mapping $(p,q) \mapsto \lambda_1(p,q)$ is continuous (stable) on the set

$$I_{\alpha,\beta} = \{(p,q) \in (]1,+\infty[)^2 \text{ such that (1.2) is satisfied}\},$$

for any bounded domain Ω having the segment property. Recall that an open subset Ω of $\mathbb{R}^N$ is said to have the segment property if, given any $x \in \partial\Omega$, there exist an open set G_x in $\mathbb{R}^N$ with $x \in G_x$ and y_x of $\mathbb{R}^N \setminus \{0\}$ such that, if $z \in \overline{\Omega} \cap G_x$ and $t \in]0,1[$, then $z + ty_x \in \Omega$.

This property allows us by a translation to push the support of a function u in Ω. This class of domains for which the boundary is sufficiently regular to guarantee that

$$W^{1,t}(\Omega) \cap_{1<r<t} W_0^{1,r}(\Omega) = W_0^{1,t}(\Omega),$$

for any $t > 1$. See [3] for more details about this property.

The difficulty is that as the exponents (p,q) change, the appropriate spaces $L^p(\Omega)$ and $L^q(\Omega)$ also change. In the case of equation $\Delta_p u + \lambda |u|^{p-2}u = 0$, the same result is studied by Lindqvist [5] but in a regular bounded domain. El Khalil et al. [3] studied the equation $\Delta_p u + \lambda g|u|^{p-2}u = 0$ with $g \in L^\infty_{\rm loc}(\Omega) \cap L^r(\Omega)$ and Ω having the segment property. Note that the stability was partially studied in [4].
For the dependence of λ_1 upon variations of the domain, instead of the exponents p and q, we prove that $\lambda_1^{(p,q)}(\Omega_j) \to \lambda_1^{(p,q)}(\Omega)$ as $j \to +\infty$, where $\Omega = \cup_{j\in\mathbb{N}^*}\Omega_j$. In the case of only one equation, see [1] for contracting domain, and see [6] for variations of the domain.

2. Stability

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ having the segment property. Then, the function $(p,q) \mapsto \lambda_1(p,q)$ is continuous from $I_{\alpha,\beta}$ to $\mathbb{R}^+$. where*

$$I_{\alpha,\beta} = \{(p,q) \in (]1,+\infty[)^2 \textit{ such that (1.2) is satisfied }\}.$$

Proof. Let $(t_n)_{n\geq 1}$, $t_n = (p_n, q_n)$ be a sequence in $I_{\alpha,\beta}$ converging at $t = (p,q) \in I_{\alpha,\beta}$. We will prove that

$$\lim_{n\to+\infty} \lambda_1(p_n,q_n) = \lambda_1(p,q).$$

Indeed, let $(\phi,\psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ such that $B(\phi,\psi) > 0$; hence,

$$\lambda_1(p_n,q_n) \leq \frac{\frac{\alpha+1}{p_n}\|\nabla\phi\|_{p_n}^{p_n} + \frac{\beta+1}{q_n}\|\nabla\psi\|_{q_n}^{q_n}}{B(\phi,\psi)},$$

since $\lambda_1(p_n,q_n)$ being the infimum. Letting n tend to infinity, we deduce from Lebesgue's theorem

$$\limsup_{n\to+\infty} \lambda_1(p_n,q_n) \leq \frac{\frac{\alpha+1}{p}\|\nabla\phi\|_p^p + \frac{\beta+1}{q}\|\nabla\psi\|_q^q}{B(\phi,\psi)}. \tag{2.1}$$

Then,

$$\limsup_{n\to+\infty} \lambda_1(p_n,q_n) \le \lambda_1(p,q). \tag{2.2}$$

On the other hand, let $\{(p_{n_k},q_{n_k})\}_{k\geq 1}$ be a subsequence of $(t_n)_n$ such that

$$\lim_{k\to+\infty} \lambda_1(p_{n_k},q_{n_k}) = \liminf_{n\to+\infty} \lambda_1(p_n,q_n).$$

Let us fix $\varepsilon_0>0$ small enough, so that for all $\varepsilon\in(0,\varepsilon_0)$, we have

$$1<\min(p-\varepsilon,q-\varepsilon), \tag{2.3}$$

$$\max(p+\varepsilon,q+\varepsilon)<\min((p-\varepsilon)^*,(q-\varepsilon)^*). \tag{2.4}$$

For each $k\in\mathbb{N}^*$, let $(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})})\in W_0^{1,p_{n_k}}(\Omega)\times W_0^{1,q_{n_k}}(\Omega)$ be a principal eigenfunction of $(S_{p_{n_k},q_{n_k}})$ related with $\lambda_1(p_{n_k},q_{n_k})$. Then, by Hölder's inequality, for $\varepsilon\in(0,\varepsilon_0)$, the following inequalities hold

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\varepsilon} \le \|\nabla u_{(p_{n_k},q_{n_k})}\|_{p_{n_k}} |\Omega|^{\frac{p_{n_k}-p+\varepsilon}{p_{n_k}(p-\varepsilon)}}, \tag{2.5}$$

$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\varepsilon} \le \|\nabla v_{(p_{n_k},q_{n_k})}\|_{q_{n_k}} |\Omega|^{\frac{q_{n_k}-q+\varepsilon}{q_{n_k}(q-\varepsilon)}}. \tag{2.6}$$

Combining these two inequalities and using the variational characterization of λ_1, we have

$$\|\nabla u_{(p_{n_k},q_{n_k})}\|_{p-\varepsilon} \le \big\{\frac{p_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\alpha+1}\big\}^{\frac{1}{p_{n_k}}} |\Omega|^{\frac{p_{n_k}-p+\varepsilon}{p_{n_k}(p-\varepsilon)}} \tag{2.7}$$

$$\|\nabla v_{(p_{n_k},q_{n_k})}\|_{q-\varepsilon} \le \big\{\frac{q_{n_k}\lambda_1(p_{n_k},q_{n_k})}{\beta+1}\big\}^{\frac{1}{q_{n_k}}} |\Omega|^{\frac{q_{n_k}-q+\varepsilon}{q_{n_k}(q-\varepsilon)}}. \tag{2.8}$$

Therefore, via (2.3) and (2.4), for a subsequence

$$\begin{gathered}(u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})}) \rightharpoonup (u,v) \quad \text{weakly in } W_0^{1,p-\varepsilon}(\Omega)\times W_0^{1,q-\varepsilon}(\Omega),\\ (u_{(p_{n_k},q_{n_k})},v_{(p_{n_k},q_{n_k})}) \to (u,v) \quad \text{strongly in } L^{p+\varepsilon}(\Omega)\times L^{q+\varepsilon}(\Omega).\end{gathered}$$

Passing to the limit in (2.7) and (2.8), respectively as $k\to\infty$ and as $\epsilon\to 0^+$, we have

$$\begin{gathered}\|\nabla u\|_p^p \le \frac{p}{\alpha+1}\lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k})<+\infty,\\ \|\nabla v\|_q^q \le \frac{q}{\beta+1}\lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k})<+\infty.\end{gathered}$$

Then,

$$u\in W_0^{1,p-\varepsilon}(\Omega)\cap W^{1,p}(\Omega)=W_0^{1,p}(\Omega), \quad v\in W_0^{1,q-\varepsilon}(\Omega)\cap W^{1,q}(\Omega)=W_0^{1,q}(\Omega),$$

because Ω satisfies the segment property.

On the other hand, from the variational characterization of $\lambda_1(p_{n_k},q_{n_k})$, (2.5), (2.6), and using weak lower semi continuity of the norm; it follows that

$$\frac{1}{|\Omega|^{\frac{\varepsilon}{p-\varepsilon}}}\frac{\alpha+1}{p}\|\nabla u\|_{p-\varepsilon}^p + \frac{1}{|\Omega|^{\frac{\varepsilon}{q-\varepsilon}}}\frac{\beta+1}{q}\|\nabla v\|_{q-\varepsilon}^q \le \lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k}).$$

Letting $\varepsilon\to 0^+$ in (2.10), the Fatou lemma yields,

$$\frac{\alpha+1}{p}\|\nabla u\|_p^p + \frac{\beta+1}{q}\|\nabla v\|_q^q \le \lim_{k\to+\infty}\lambda_1(p_{n_k},q_{n_k}).$$

Since $B(u_{(p_{n_k},q_{n_k})}, v_{(p_{n_k},q_{n_k})}) = 1$ via compactness of B, (u,v) is admissible in the variational characterization of $\lambda_1(p,q)$; hence

$$\lambda_1(p,q) \leq \lim_{k\to+\infty} \lambda_1(p_{n_k}, q_{n_k}) = \liminf_{n\to+\infty} \lambda_1(p_n, q_n).$$

This and (2.2) complete the proof. □

Remark 2.2. Observe that the segment property is used only for to prove

$$\lambda_1(p,q) \leq \liminf_{n\to+\infty} \lambda_1(p_n, q_n).$$

3. Dependence with variations domain

Theorem 3.1. *Let $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \dots$ be an exhaustion of Ω such that*

$$\Omega = \bigcup_{j\in\mathbb{N}^*} \Omega_j.$$

Then

$$\lim_{j\to+\infty} \lambda_1^{(p,q)}(\Omega_j) = \lambda_1^{(p,q)}(\Omega), \tag{3.1}$$

$$\lim_{j\to+\infty} \int_\Omega |\nabla \widetilde{u_j} - \nabla u|^p \,dx = \lim_{j\to+\infty} \int_\Omega |\nabla \widetilde{v_j} - \nabla v|^q \,dx = 0, \tag{3.2}$$

where (u,v) is a positive principal eigenfunction of (1.1) related to $\lambda_1^{(p,q)}(\Omega)$; (u_j,v_j) is the principal eigenfunction related to $\lambda_1^{(p,q)}(\Omega_j)$ and $(\widetilde{u_j}, \widetilde{v_j})$ is the extended by 0 in $\Omega\backslash\Omega_j$ of (u_j,v_j).

Proof. To prove (3.1), we note that

$$\lambda_1^{(p,q)}(\Omega_1) \geq \lambda_1^{(p,q)}(\Omega_2) \geq \dots \geq \lambda_1^{(p,q)}(\Omega).$$

For each $\varepsilon > 0$, there is a pair of function $(\varphi_\varepsilon, \psi_\varepsilon) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ such that

$$\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon \psi_\varepsilon \,dx > 0$$

$$\lambda_1^{(p,q)}(\Omega) > \frac{\frac{\alpha+1}{p}\int_\Omega |\nabla\varphi_\varepsilon|^p \,dx + \frac{\beta+1}{q}\int_\Omega |\nabla\psi_\varepsilon|^q \,dx}{\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon \,dx} - \varepsilon,$$

since $\lambda_1^{(p,q)}(\Omega)$ is the infimum.

The support of φ_ε and ψ_ε being compact sets, are covered a finite number of the sets Ω_j; hence, there exist $j_0 \in \mathbb{N}^*$ such that

$$\operatorname{supp}\varphi_\varepsilon \subset \Omega_j \quad \text{and} \quad \operatorname{supp}\psi_\varepsilon \subset \Omega_j \quad \forall j \geq j_0.$$

Then,

$$\begin{aligned}
\lambda_1^{(p,q)}(\Omega_j) &\leq \frac{\frac{\alpha+1}{p}\int_{\Omega_j} |\nabla\varphi_\varepsilon|^p \,dx + \frac{\beta+1}{q}\int_{\Omega_j} |\nabla\psi_\varepsilon|^q \,dx}{\int_{\Omega_j} |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon \,dx} \\
&= \frac{\frac{\alpha+1}{p}\int_\Omega |\nabla\varphi_\varepsilon|^p \,dx + \frac{\beta+1}{q}\int_\Omega |\nabla\psi_\varepsilon|^q \,dx}{\int_\Omega |\varphi_\varepsilon|^\alpha |\psi_\varepsilon|^\beta \varphi_\varepsilon\psi_\varepsilon \,dx} \\
&\leq \lambda_1^{(p,q)}(\Omega) + \epsilon
\end{aligned}$$

for all large j. It is clear that $\lambda_1^{(p,q)}(\Omega) \geq \lim_{j\to+\infty} \lambda_1^{(p,q)}(\Omega_j)$. which proves the desired result.

For the strong convergence (3.2), we proceed as follows: First, we have $(\widetilde{u_j},\widetilde{v_j})\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$; $\widetilde{u_j}\geq 0$, $\widetilde{v_j}\geq 0$ a.e in Ω;

$$\int_\Omega |\widetilde{u_j}|^\alpha|\widetilde{v_j}|^\beta\widetilde{u}_j\widetilde{v}_j\,dx=1,$$

because

$$\int_\Omega |\widetilde{u_j}|^\alpha|\widetilde{v_j}|^\beta\widetilde{u_j}\widetilde{v_j}\,dx=\int_{\Omega_j}|u_j|^\alpha|v_j|^\beta u_jv_j\,dx=1;$$

and $\nabla\widetilde{u_j}=\widetilde{\nabla u_j}$ and $\nabla\widetilde{v_j}=\widetilde{\nabla v_j}$ a.e. in Ω. Hence, $(\widetilde{u_j},\widetilde{v_j})$ is admissible function in definition of $\lambda_1^{(p,q)}(\Omega)$. Which implies

$$\lambda_1^{(p,q)}(\Omega)\leq\frac{\alpha+1}{p}\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\widetilde{v_j}|^q\,dx=\lambda_1^{(p,q)}(\Omega_j).$$

Therefore, via (3.1) and for a subsequence,

$$\begin{gathered}(\widetilde{u_j},\widetilde{v_j})\rightharpoonup(\varphi,\psi)\quad\text{weakly in } W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega),\\ (\widetilde{u_j},\widetilde{v_j})\to(u,v)\quad\text{strongly in } L^p(\Omega)\times L^q(\Omega),\\ \varphi\geq 0,\quad\psi\geq 0\quad\text{a.e in }\Omega\end{gathered}$$

Since B is compact, we have $\int_\Omega\varphi^{\alpha+1}\psi^{\beta+1}dx=1$. Then (φ,ψ) is admissible function in definition of $\lambda_1^{(p,q)}(\Omega)$. We deduce that

$$\begin{aligned}\lambda_1^{(p,q)}(\Omega)&\leq\frac{\alpha+1}{p}\int_\Omega|\nabla\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\psi|^q\,dx\\&\leq\liminf_{j\to\infty}\Big(\frac{\alpha+1}{p}\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\widetilde{v_j}|^q\,dx\Big)\\&=\liminf_{j\to\infty}\lambda_1^{(p,q)}(\Omega_j).\end{aligned}$$

Then

$$\lambda_1^{(p,q)}(\Omega)=\frac{\alpha+1}{p}\int_\Omega|\nabla\phi|^p\,dx+\frac{\beta+1}{q}\int_\Omega|\nabla\psi|^q\,dx$$

which implies (φ,ψ) is positive eigenfunction related to $\lambda_1^{(p,q)}(\Omega)$ that it is simple (see [2]) and $\int_\Omega|\varphi|^\alpha|\psi|^\beta\varphi\psi dx=1$, hence $\phi\equiv u$ and $\psi\equiv v$. Also, (u,v) is independent of the subsequence. Then $\{(\widetilde{u_j},\widetilde{v_j})\}_j$ converge to (u,v) least in $L^p(\Omega)\times L^q(\Omega)$.

For the strong convergence (2.3), we use Clarckson inequality and distinguish three possible cases for $(p,q)\in I_{\alpha,\beta}$.

case 1: $p\geq 2$ and $q\geq 2$. For all $j\geq j_0$, we have

$$\begin{aligned}&\frac{\alpha+1}{p}\int_\Omega\Big|\frac{\nabla\widetilde{u_j}-\nabla u}{2}\Big|^p\,dx+\frac{\beta+1}{q}\int_\Omega\Big|\frac{\nabla\widetilde{v_j}-\nabla v}{2}\Big|^q\,dx\\&\leq\frac{\alpha+1}{p}\Big[-\int_\Omega\Big|\frac{\nabla\widetilde{u_j}+\nabla u}{2}\Big|^p\,dx+\frac12\int_\Omega|\nabla\widetilde{u_j}|^p\,dx+\frac12\int_\Omega|\nabla u|^p\,dx\Big]\\&\quad+\frac{\beta+1}{q}\Big[-\int_\Omega\Big|\frac{\nabla\widetilde{v_j}+\nabla v}{2}\Big|^p\,dx+\frac12\int_\Omega|\nabla\widetilde{v_j}|^q\,dx+\frac12\int_\Omega|\nabla v|^q\,dx\Big].\end{aligned}$$

Also, we have

$$\begin{aligned} &\lambda_1^{(p,q)}(\Omega)\int_\Omega \big(\frac{\widetilde{u_j}+u}{2}\big)^{\alpha+1}\big(\frac{\widetilde{v_j}+v}{2}\big)^{\beta+1}dx \\ &\leq \frac{\alpha+1}{p}\int_\Omega \big|\frac{\nabla\widetilde{u_j}+\nabla u}{2}\big|^p\,dx+\frac{\beta+1}{q}\int_\Omega \big|\frac{\nabla\widetilde{v_j}+\nabla v}{2}\big|^q\,dx. \end{aligned}$$

Combining the last inequality and using Lebegue's Theorem; we obtain

$$\limsup_{j\to+\infty}\big\{\frac{\alpha+1}{p}\int_\Omega \big|\frac{\nabla\widetilde{u_j}-\nabla u}{2}\big|^p\,dx+\frac{\beta+1}{q}\int_\Omega \big|\frac{\nabla\widetilde{v_j}-\nabla v}{2}\big|^q\,dx\big\}\leq 0,$$

Case 2: $1<p<2$ and $2<q$ or $1<q<2$ and $2<p$. Using the second Clarckson inequality, we prove that

$$\lim_{j\to+\infty}\int_\Omega |\nabla\widetilde{u_j}-\nabla u|^p\,dx=0.$$

Using first Clarckson inequality, we prove that

$$\lim_{j\to+\infty}\int_\Omega |\nabla\widetilde{v_j}-\nabla v|^q\,dx=0.$$

Which completes the proof. □

References

[1] A. Anane, *Etude des valeurs propres et de la résonance pour l'opérateur p-Laplacien,* thèse de Doctorat, U.L.B (1987 -1988).

[2] A. El Khalil, M. Ouanan, and A. Touzani; *Bifurcation of nonlinear elliptic system from the first eigenvalue*, *E. J. Qualitative Theory of Diff. Equ.*, No. **21**. (2003), pp. 1-18.

[3] A. El Khalil, P. Lindqvist, and A. Touzani, *On the stability of the eigenvalue of $A_pu+\lambda g(x)|u|^{p-2}u=0$, with varying p*, to appear in *Rendiconti di Matematica, Serie VII Volume 24, Roma (2004).*

[4] A. El Khalil, S. El Manouni, and M. Ouanan, *Simplicity and Stability of the first eigenvalue of a Nonlinear Elleptic System*, submitted in *International Journal of Mathematics and Mathematical Sciences.*

[5] P. Lindqvist, *On Nonlinear Rayleigh Quotients*, Potential Analysis **2**, pp. 199-218, (1993).

[6] P. Lindqvist, *On a Nonlinear eigenvalue problem,* Ber. Univer. Jyväskylä Math. Inst. **68** , pp. 33-54, (1995).

Abdelouahed El Khalil
Department of Mathematics and Industrial Genie, Ecole Polytechnic School Montreal, Montreal, Canada
E-mail address: lkhlil@hotmail.com

Mohammed Ouanan
Department of mathematics, Faculty of sciences Dhar-Mahraz, P.O. Box 1796 Atlas, Fez 30000, Morocco
E-mail address: m_ouanan@hotmail.com

Abdelfattah Touzani
Department of mathematics, Faculty of sciences Dhar-Mahraz, P.O. Box 1796 Atlas, Fez 30000, Morocco
E-mail address: atouzani@iam.net.ma