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A COMMON FIXED POINT THEOREM FOR COMMUTING EXPANDING MAPS ON NILMANIFOLDS

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Abstract. A self-map f of a compact connected manifold M is expanding if it locally expands distances with respect to some metric. We consider the case when M is a nilmanifold and we discuss a new common fixed point theorem for two expanding maps which commute.

1. Introduction

In fixed point theory there are several results that concern the existence of common fixed points for families of commuting maps. By the commuting property the set of the fixed points of one map is invariant with respect to another map, but this property alone is not a sufficient condition and usually one need more information about the maps and the set where they operate. We would like to mention the Markov-Kakutani theorem [5] for affine maps of a compact convex set and the Behan-Shields theorem [1] for analytic maps of the unit disc. These theorems and their subsequent generalizations assume that the set is convex. Here we present a common fixed point theorem where the set is not convex.

We start with the following very simple example which motivates the main ideas of this paper. Let f and g be two endomorphism of the unit circle

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{Ax+a} & \mathbb{R} \\ \pi\downarrow & & \downarrow\pi \\ S^1 & \xrightarrow{f} & S^1 \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{Bx+b} & \mathbb{R} \\ \pi\downarrow & & \downarrow\pi \\ S^1 & \xrightarrow{g} & S^1 \end{array}$$

where $A, B \in \mathbb{Z}$, $a, b \in \mathbb{R}$ and $\pi(x) = \exp(2\pi i x)$ is the natural projection from the universal covering $\mathbb{R}$ to the unit circle S^1. We assume that f and g locally expand distances and therefore the integers A and B are of modulus greater than 1. If f and g commute, i. e. $f \circ g = g \circ f$ on S^1, then there is an integer r such that

$$A(Bx + b) + a = B(Ax + a) + b + r \quad \text{for all } x \in \mathbb{R}$$

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that is $r = (A-1)b - (B-1)a$. Moreover, $\pi(x_0)$ is a common fixed point of f and g if and only if there are $p, q \in \mathbb{Z}$ such that

$$Ax_0 + a = x_0 + p \quad \text{and} \quad Bx_0 + b = x_0 + q.$$

If we eliminate x_0 we obtain the following diophantine equation in the unknowns p and q

$$(A-1)q - (B-1)p = (A-1)b - (B-1)a = r$$

which is solvable if and only if the greatest common divisor of $A-1$ and $B-1$ divides the integer r. It is easy to see that the positive integers $n(f) = |A-1|$ and $n(g) = |B-1|$ are respectively the number of fixed points of f and the number of fixed points of g. Hence it follows that if $n(f)$ and $n(g)$ are relatively prime then the maps f and g have a (unique) common fixed point. In the rest of this paper we will discuss a similar statement in a more general setting where instead of the circle we have a nilmanifold.

2. Expanding maps

Definition 2.1. Let M be a closed manifold. A C^1-map $f : M \to M$ is called an *expanding map* if there exist constants $c > 0$ and $\lambda > 1$ such that at any point $x \in M$

$$\|D_x f^n(v)\| \geq c\lambda^n \|v\| \quad \forall n \geq 1 \text{ and } \forall v \in T_x M$$

for some riemannian metric $\|\cdot\|$ on M.

These maps were introduced in 1969 by Shub in [8] and they are the simplest examples of non-invertible hyperbolic maps. In [3] it has been shown that any expanding map of a compact manifold is topologically conjugate to an expanding endomorphism on an infra-nilmanifold. Not all infra-nilmanifolds admit an expanding map.

Let $\widetilde{M}$ be the universal covering space of the manifold M then there exists a lifting $\widetilde{f} : \widetilde{M} \to \widetilde{M}$ which makes the following diagram commutative

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{f}} & \widetilde{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{f} & M \end{array}$$

The induced homomorphism $\widetilde{f}^*$ on the deck trasformation group Γ defined by

$$\widetilde{f}(\gamma(y)) = \widetilde{f}^*(\gamma)(\widetilde{f}(y))$$

determines the topological properties of f. More precisely the following *conjugation theorem* holds.

Theorem 2.2. *Let f and Φ be two expanding maps. If there exists two liftings $\widetilde{f}$ and $\widetilde{\Phi}$ such that $\widetilde{f}^* = \widetilde{\Phi}^*$ then f and Φ are topologically conjugate, that is there is a homeomorphism h of M such that*

$$h \circ f \circ h^{-1} = \Phi.$$

Proof. We give a sketch of the proof because we will need it later. Let S be the set of all continuous maps $\widetilde{h} : \widetilde{M} \to \widetilde{M}$ which are liftings of continuous maps $h : M \to M$ such that $\widetilde{h}^*$ is the identity homomorphism. In S we define the following metric

$$D(\widetilde{h}_1, \widetilde{h}_2) = \sup_{y \in \widetilde{M}} d(\widetilde{h}_1(y), \widetilde{h}_2(y)).$$

Then S is complete with respect to this metric and the map

$$\mathcal{F}(\widetilde{h}) = \widetilde{\Phi}^{-1} \circ \widetilde{h} \circ \widetilde{f}$$

is a contraction in S. The projection of the unique fixed point in S of the map $\mathcal{F}$ is the homeomorphism h which realizes the conjugation. □

3. Commuting expanding maps on nilmanifolds

Definition 3.1. Let N be connected, simply connected nilpotent Lie group with a left invariant riemannian metric and let Γ be a uniform and discrete subgroup of N then the quotient $M = N/\Gamma$ is called *nilmanifold*.

We consider N as acting on himself by left translation and we indicate by $\mathrm{Aut}(N)$ and $\mathfrak{N}$ respectively the group of all continuous automorphisms of N and the Lie algebra over $\mathbb{R}$. If $\varphi : \Gamma :\to \Gamma$ is an injective homomorphism then it extends uniquely by [6] to an automorphism A of N. The corresponding projected map $\Phi_A : N/\Gamma \to N/\Gamma$ is called *endomorphism*

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{A_*} & \mathfrak{N} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ N & \xrightarrow{A} & N \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ N/\Gamma & \xrightarrow{\Phi_A} & N/\Gamma \end{array} .$$

The endomorphism Φ_A is an expanding map if the eigenvalues of the corresponding automorphism A_* of the Lie algebra $\mathfrak{N}$ are of modulus greater than 1.

Remark 3.2. If f is an expanding map on a nilmanifold then the induced homomorphism $\widetilde{f}^*$ is injective and therefore there is an expanding endomorphism Φ_A such that $\widetilde{f}^* = \widetilde{\Phi}_A^*$. Hence f and Φ_A are topologically conjugate.

Example 3.3. The simplest compact nilmanifold is the n-dimensional torus $\mathbb{T}^n$ which is the quotient of the abelian group $\mathbb{R}^n$ and the uniform discrete subgroup $\mathbb{Z}^n$. In this case any expanding map f is conjugate to a toral endomorphism Φ_A where $A = A_*$ is an integral matrix with all eigenvalues of modulus greater than 1.

Example 3.4. It is known (see for example Chapter 4 in [2]) that any compact nilmanifold of dimension ≤ 3 is diffeomorphic to a torus or to the quotient $N_3(\mathbb{R})/\Gamma_{1,1,k}$ where k is a positive integer, $N_3(\mathbb{R})$ is the non-abelian group of all unipotent upper triangular real matrices and the uniform discrete subgroup $\Gamma_{1,1,k}$ consist of all matrices of the form

$$\begin{bmatrix} 1 & p_1 & p_3/k \\ 0 & 1 & p_2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with } p_1, p_2, p_3 \in \mathbb{Z}.$$

Assigning to any unipotent upper triangular real matrix

$$\begin{bmatrix} 1 & y_1 & y_3 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{bmatrix}$$

the real vector (y_1, y_2, y_3), $N_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^3$ with the binary operation

$$(y_1, y_2, y_3) \cdot (y_1', y_2', y_3') = (y_1 + y_1', y_2 + y_2', y_3 + y_3' + y_1 y_2').$$

The subgroup $\Gamma_{1,1,k}$ is generated by

$$\gamma_1 = (1, 0, 0), \quad \gamma_2 = (0, 1, 0), \quad \gamma_3 = (0, 0, 1/k).$$

In order to classify the expanding maps on $N_3(\mathbb{R})/\Gamma_{1,1,k}$ we have to give a description of the automorphisms A of $N_3(\mathbb{R})$ which left $\Gamma_{1,1,k}$ invariant. So let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a map such that

$$A(\gamma_j) = (a_{1j}, a_{2j}, a_{3j}/k) \quad \text{for } j = 1, 2, 3$$

with $a_{ij} \in \mathbb{Z}$. Since the center group $Z(\Gamma_{1,1,k})$ is generated by γ_3 then $A(\gamma_3)) = \gamma_3^{a_{33}}$ and $a_{13} = a_{23} = 0$. Moreover, since the Lie bracket $[\gamma_1, \gamma_2] = \gamma_3^k$, then

$$A([\gamma_1, \gamma_2]) = [A(\gamma_1), A(\gamma_2)] = A(\gamma_3)^k = (0, 0, a_{33})$$

and after some routine calculations we find that

$$[A(\gamma_1), A(\gamma_2)] = (0, 0, a_{11}a_{22} - a_{12}a_{21})$$

that is $a_{33} = a_{11}a_{22} - a_{12}a_{21}$.

$$(p_1, p_2, p_3/k) = \gamma_1^{p_1} \cdot \gamma_2^{p_2} \cdot \gamma_3^{p_3 - p_1 p_2}.$$

By using the exponential map

$$\exp(x_1, x_2, x_3) = (x_1, x_2, x_3 + \frac{k}{2}\, x_1 x_2)$$

the group $N_3(\mathbb{R})$ may be identified with the Lie algebra $\mathfrak{N}_3$ endowed with the multiplication given by the Campbell-Hausdorff formula

$$x \star y = x + y + \frac{k}{2}\, [x, y].$$

Therefore, an expanding map f on the nilmanifold $N_3(\mathbb{R})/\Gamma_{1,1,k}$ is topologically conjugate to an expanding endomorphism Φ_A with associated automorphism

$$\begin{aligned} A(x_1, x_2, x_3) &= A_*(x_1, x_2, x_3) + \frac{k}{2}\, (0, 0, a_{11}a_{21}x_1^2 + a_{21}a_{22}x_2^2 + 2a_{12}a_{21}x_1x_2) \\ &= \exp(A_* \exp^{-1}(x_1, x_2, x_3)) \end{aligned}$$

where A_* is a matrix

$$A_* = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} - \frac{k}{2}\, a_{11}a_{21} & a_{32} - \frac{k}{2}\, a_{12}a_{22} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

with $a_{ij} \in \mathbb{Z}$ such that the eigenvalues of A_* are of modulus greater than 1.

Definition 3.5. If f and g commute then for any liftings $\widetilde{f}$ and $\widetilde{g}$ there is $r \in \Gamma$ such that

$$\widetilde{f} \circ \widetilde{g} = r \cdot \widetilde{g} \circ \widetilde{f}.$$

We say that f and g are *super-commuting* if there are two liftings such that r belongs to the center $Z(N)$.

Remark 3.6. Note that if N is abelian then commuting property is equivalent to super-commuting one. Furthermore if $x \in N/\Gamma$ is a fixed point of f and $y \in \pi^{-1}(x)$ then we can choose a lifting $\widetilde{f}$ such that $\widetilde{f}(y) = y$ (otherwise $\widetilde{f}(y) = p \cdot y$ for some $p \in \Gamma$ and we can replace $\widetilde{f}$ with the lifting $p^{-1} \cdot \widetilde{f}$). Therefore, if x is a common fixed point of f and g there are two liftings such that $\widetilde{f}(y) = y$ and $\widetilde{g}(y) = y$. Hence

$$y = (\widetilde{f} \circ \widetilde{g})(y) = r \cdot (\widetilde{g} \circ \widetilde{f})(y) = r \cdot y$$

and it follows that r is the identity map of N that is the maps f and g are super-commuting.

Lemma 3.7. *If f and g are super-commuting expanding maps on a nilmanifold N/Γ then the associated automorphisms A and B commute.*

Proof. Let $p \in \Gamma$ and $y \in N$

$$(\widetilde{f} \circ \widetilde{g})(p \cdot y) = (A \circ B)(p) \cdot (\widetilde{f} \circ \widetilde{g})(y)$$

On the other hand since $r \in Z(N)$

$$r \cdot (\widetilde{g} \circ \widetilde{f})(p \cdot y) = r \cdot (B \circ A)(p) \cdot (\widetilde{g} \circ \widetilde{f})(y) = (B \circ A)(p) \cdot r \cdot (\widetilde{g} \circ \widetilde{f})(y).$$

Hence we have that $A \circ B = B \circ A$ in Γ and therefore in N. $\square$

As we have seen in the previous section, an expanding map is topologically conjugate to an expanding endomorphism. This conjugation is a powerful tool because allow us to work with a *simpler* map that has many algebraic properties. The following theorem establishes that when two expanding maps on a nilmanifold are super-commuting then there is a homeomorphism which realizes both conjugations at the same time.

Theorem 3.8. *If f and g are super-commuting expanding maps on a nilmanifold N/Γ then there exist a homeomorphism h and an element $\theta \in Z(N)$ such that*

$$\begin{aligned} h \circ f \circ h^{-1} &= \Phi_A \\ h \circ g \circ h^{-1} &= R_\theta \circ \Phi_B \end{aligned} \tag{3.1}$$

where R_θ is the map induced on N/Γ by the left translation by θ. Furthermore the endomorphism Φ_A commutes with both maps R_θ and Φ_B.

Proof. We define the following two maps

$$\mathcal{F}(\widetilde{h}) = A^{-1} \circ \widetilde{h} \circ \widetilde{f}, \quad \mathcal{G}_\theta(\widetilde{h}) = B^{-1} \circ \theta^{-1} \cdot \widetilde{h} \circ \widetilde{g},$$

where the parameter θ will be chosen later in $Z(N)$. We already know that $\mathcal{F}$ is a contraction in S which has exactly one fixed point $\widetilde{h}$ in S that induces a homeomorphism h on N/Γ. The map $\mathcal{G}_\theta$ is a contraction in S too because the left translation by θ sends S in itself ($\theta \in Z(N)$) and it is an isometry with respect to the metric D. Now, it will be shown that we can pick the parameter θ in such a way that the maps $\mathcal{G}_\theta$ and $\mathcal{F}$ commute and thus they have the same fixed point $\widetilde{h}$. First note that for all $\widetilde{h} \in S$

$$\begin{aligned} (\mathcal{F} \circ \mathcal{G}_\theta)(\widetilde{h}) &= A^{-1} \circ \mathcal{G}_\theta(\widetilde{h}) \circ \widetilde{f} \\ &= A^{-1} \circ B^{-1} \circ \theta^{-1} \cdot \widetilde{h} \circ \widetilde{g} \circ \widetilde{f} \\ &= A^{-1}(B^{-1}(\theta^{-1})) \cdot A^{-1} \circ B^{-1} \circ \widetilde{h} \circ \widetilde{g} \circ \widetilde{f} \end{aligned}$$

Since $\widetilde{f} \circ \widetilde{g} = r \cdot \widetilde{g} \circ \widetilde{f}$, $\widetilde{h}^*$ is the identity map and A and B commute by Lemma 3.7. Therefore, we have that for all $\widetilde{h} \in S$,

$$
\begin{aligned}
(\mathcal{G}_\theta \circ \mathcal{F})(\widetilde{h}) &= B^{-1} \circ \theta^{-1} \cdot \mathcal{F}(\widetilde{h}) \circ \widetilde{g} \\
&= B^{-1} \circ \theta^{-1} \cdot A^{-1} \circ \widetilde{h} \circ \widetilde{f} \circ \widetilde{g} \\
&= B^{-1} \circ \theta^{-1} \cdot A^{-1} \circ \widetilde{h} \circ r \cdot \widetilde{g} \circ \widetilde{f} \\
&= B^{-1} \circ \theta^{-1} \cdot A^{-1} \circ r \cdot \widetilde{h} \circ \widetilde{g} \circ \widetilde{f} \\
&= B^{-1}(\theta^{-1} \cdot A^{-1}(r)) \cdot B^{-1} \circ A^{-1} \circ \widetilde{h} \circ \widetilde{g} \circ \widetilde{f} \\
&= B^{-1}(\theta^{-1} \cdot A^{-1}(r)) \cdot A^{-1} \circ B^{-1} \circ \widetilde{h} \circ \widetilde{g} \circ \widetilde{f}
\end{aligned}
$$

Thus, comparing the last two equations, $\mathcal{F}$ and $\mathcal{G}_\theta$ commute if and only if

$$A^{-1}(B^{-1}(\theta)) = B^{-1}(\theta \cdot A^{-1}(r)).$$

that is $A(\theta) = r \cdot \theta$ because $r \in Z(N)$ and $A \circ B = B \circ A$.

The map $\varphi(y) = r^{-1} \cdot A(y)$ is an expanding diffeomorphism of N and therefore θ is the unique fixed point of the contraction map $\varphi^{-1}(y) = A^{-1}(r \cdot y)$ and it is the limit of the sequence $\theta_k = \varphi^{-k}(r)$ for $k \geq 0$. Since $Z(N)$ is closed, in order to prove that $\theta \in Z(N)$ it suffices to show by induction that each $\theta_k \in Z(N)$. The first element is $\theta_0 = e \in Z(N)$. If $\theta_k \in Z(N)$, then, for all $y \in N$,

$$
\begin{aligned}
\theta_{k+1} \cdot y &= \varphi^{-1}(\theta_k) \cdot y \\
&= A^{-1}(r \cdot \theta_k) \cdot y \\
&= A^{-1}(r \cdot \theta_k \cdot A(y)) \\
&= A^{-1}(A(y) \cdot r \cdot \theta_k) \\
&= y \cdot \varphi^{-1}(\theta_k) \\
&= y \cdot \theta_{k+1}.
\end{aligned}
$$

□

4. A common fixed point theorem on nilmanifolds

Now we consider the set of fixed points of an expanding map f on a nilmanifold N/Γ. By the expanding property the number of such fixed points, $n(f)$, is a positive integer number and it is just equal to the Nielsen number of the map f. According to [7], there is an *algorithm* to compute $n(f)$ based on the fact that a nilmanifold can be decomposed into tori through a sequence of fibrations. Here we are not going to discuss this algorithm and we will only recall some of its properties.

Example 4.1. If f is an expanding map on the n-dimensional torus $\mathbb{T}^n$ then it is conjugate to an expanding endomorphism Φ_A and the number of fixed points of f is given by the following formula

$$n(f) = |\det(A - I)|.$$

Example 4.2. If f is an expanding map on $N_3(\mathbb{R})/\Gamma_{1,1,k}$ then it is conjugate to an expanding endomorphism Φ_A and

$$n(f) = |\det(A_2 - I)| \cdot |\det(a_{33} - 1)|$$

where

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad a_{33} = a_{11}a_{22} - a_{12}a_{21}.$$

The following preliminary statement can be found in [9].

Proposition 4.3. *Let f and g be two commuting expanding maps on $\mathbb{T}^n$. If $n(F)$ and $n(G)$ are relatively prime then there exists one and only one common fixed point.*

Proof. By Theorem 3.8 it suffices to find the common fixed points of Φ_A and $R_\theta \circ \Phi_B$ where $\theta \in \mathbb{R}^n$ satisfies the equation $A(\theta) + r = \theta$ with $r \in \mathbb{Z}^n$. So we have to find $y \in \mathbb{R}^n$ and $p, q \in \mathbb{Z}^n$ such that

$$A(y) = y + p, \quad B(y) + \theta = y + q.$$

Since A and B commute, eliminating y we find the following equation in p, q

$$(A - I)q - (B - I)p = (A - I)\theta = r.$$

Let D the left greatest common divisor integral matrix (see for example [4]) then there exist integral matrices P and Q such that

$$(A - I)Q - (B - I)P = D.$$

The integer $\det(D)$ divides $n(F) = |\det(A - I)| \geq 1$ and $n(G) = |\det(B - I)| \geq 1$ which are relatively prime. Hence $\det(D) = \pm 1$ and it follows that D^{-1} is an integral matrix too. Therefore

$$(A - I)Q(D^{-1}r) - (B - I)P(D^{-1}r) = D(D^{-1}r) = r$$

and $q = Q(D^{-1}r) \in \mathbb{Z}^n$, $p = P(D^{-1}r) \in \mathbb{Z}^n$ solve our equation. The common fixed point is $x = \pi((A - I)^{-1}(p))$. □

Here is the main result.

Theorem 4.4. *Let f and g be two commuting expanding maps on nilmanifold N/Γ. If $n(F)$ and $n(G)$ are relatively prime then there exists at least a common fixed point.*

Proof. By Theorem 3.8 we can assume that $f = \Phi_A$ and $g = R_\theta \circ \Phi_B$ with $A \circ B = B \circ A$, $A(\theta) = r \cdot \theta$ and $r \in \Gamma \cap Z(N)$, $\theta \in Z(N)$. In order to find a common fixed point we have to prove that there exist $p, q \in \Gamma$ and $y \in N$ such that

$$A(y) = p \cdot y \quad \text{and} \quad \theta \cdot B(y) = q \cdot y.$$

If we apply $r \cdot \theta \cdot B(\cdot)$ to the first equation and then we use the second one we get

$$r \cdot \theta \cdot B(A(y)) = r \cdot \theta \cdot B(p) \cdot B(y) = r \cdot B(p) \cdot \theta \cdot B(y) = r \cdot B(p) \cdot q \cdot y.$$

In a similar way, applying $A(\cdot)$ to the second equation we get

$$A(\theta \cdot B(y)) = A(q \cdot y) = A(q) \cdot A(y) = A(q) \cdot p \cdot y.$$

Since by hypothesis

$$r \cdot \theta \cdot B(A(y)) = A(\theta) \cdot A(B(y)) = A(\theta \cdot B(y))$$

we can eliminate y and we obtain the following equation in $p, q \in \Gamma$

$$A(q) \cdot p \cdot q^{-1} \cdot B(p^{-1}) = r \in Z(N) \cap \Gamma.$$

The common fixed point exists if and only if the above equation can be solved. Since $Z(N)/(Z(N) \cap \Gamma)$ is isomorphic to a torus $\mathbb{T}^d$ (d is less or equal to the dimension of the nilmanifold N/Γ) and the center $Z(N)$ is invariant with respect to A and B then the restrictions of the automorphisms A and B can be represented respectively by two $d \times d$ integer matrices A_1 and B_1. In $Z(N)$ the elements commute and the previous equation can be simplified and it becomes

$$(A_1 - I)q - (B_1 - I)p = r \quad \text{with } r, p, q \in \mathbb{Z}^d.$$

Note that Φ_{A_1} and Φ_{B_1} are two commuting expanding maps on a torus and, by the algorithm presented in [7], $n(\Phi_{A_1})$ divides $n(f)$ and $n(\Phi_{B_1})$ divides $n(g)$. Therefore, since $n(f)$ and $n(g)$ are relatively prime the same holds also for the numbers $n(\Phi_{A_1})$ and $n(\Phi_{B_1})$ and by Proposition 4.3 we have that the previous equation has a solution. □

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