2005-Oujda International Conference on Nonlinear Analysis.
*Electronic Journal of Differential Equations*, Conference 14, 2006, pp. 9–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# EXISTENCE RESULT FOR VARIATIONAL DEGENERATED PARABOLIC PROBLEMS VIA PSEUDO-MONOTONICITY

LAHSEN AHAROUCH, ELHOUSSINE AZROUL, MOHAMED RHOUDAF

Abstract. In this paper, we study the existence of weak solutions for the initial-boundary value problems of the nonlinear degenerated parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div} a(x,t,u,\nabla u) + a_0(x,t,u,\nabla u) = f,$$

where $Au = -\operatorname{div} a(x,t,u,\nabla u)$ is a classical divergence operator of Leray-lions acting from $L^p(0,T,W_0^{1,p}(\Omega,w))$ to its dual. The source term $f$ is assumed to belong to $L^{p'}(0,T,W^{-1,p'}(\Omega,w^*))$.

## 1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $Q$ be the cylinder $\Omega \times (0,T)$ with some given $T > 0$. Consider the parabolic initial-boundary value problem

$$\begin{gathered}\frac{\partial u}{\partial t} + A(u) = f \quad \text{in } Q \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \quad \text{in } \Omega,\end{gathered} \tag{1.1}$$

where $Au = -\operatorname{div} a(x,t,u,\nabla u)$ is a classical divergence operator of Leray-lions form with respect to the Sobolev space $L^p(0,T,W_0^{1,p}(\Omega))$ for some $1 < p < \infty$. The right-hand side $f$ is supposed lying in $L^{p'}(0,T,W_0^{-1,p'}(\Omega))$.

We consider, first, the case where $A$ satisfies the classical Leray-lions conditions, in particular the classical coercivity

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p. \tag{1.2}$$

Then $A$ is a bounded pseudo-monotone and coercive operator from the space $L^p(0,T,W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T,W_0^{-1,p'}(\Omega))$. In this setting, problems of the form (1.1) were solved by Lions [16] and Breszis-Browder [7] in the case $p \geq 2$ and by Landes [12] and Landes-Mustonen [13] when $1 < p < 2$ (see also [5],[6],[8]

2000 *Mathematics Subject Classification.* 35J60.
*Key words and phrases.* Weighted Sobolev spaces; boundary value problems; truncations; parabolic problems.

Published September 20, 2006.

for related topics ). When the classical coercivity (1.2) is replaced by the more general condition

$$a(x,t,s,\xi)\xi \geq c\sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{1.3}$$

where now $w(x) = \{w_i(x),\ 1 \leq i \leq N\}$ is a family of weight functions on $\Omega$, the problem (1.1) can not be solved in the classical Sobolev settings $L^p(0,T,W_0^{1,p}(\Omega))$. However, to do this, we must to change this classical setting by the general one $L^p(0,T,W_0^{1,p}(\Omega,w))$ related to the so-called weighted Sobolev space $W_0^{1,p}(\Omega,w)$. In this direction, we list in particular the work [17] where the authors have studied the existence of weak solution of the variational parabolic boundary-value problems

$$\begin{gathered}\frac{\partial u}{\partial t} + A(u) + A_0(x,t,u,\nabla u) = f \quad \text{in } Q\\ u(x,t) = u_0(x) \quad \text{on } \partial\Omega\times(0,T)\\ u(x,t) = 0 \quad \text{in } \Omega,\end{gathered} \tag{1.4}$$

but under more restrictions on the weight family $w$ (compare with Remark 2.1).

Note that, little information is known for the degenerate parabolic. Similar problems for degenerate nonlinear elliptic equations have been studied in [9] and [2]. Our aim of this paper is to study the same variational degenerate parabolic problems (1.1) in some general case of weight. For that some important lemmas is firstly proved and the approach of pseudo-monotonicity is used. A simple model of our problem is as follows

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}(|x|^s|Du|^{p-2}Du) + \sigma(x)|u|^{p-2}u = f \quad \text{in } Q\\ u(x,t) = u_0(x) \quad \text{on } \partial\Omega\times(0,T)\\ u(x,t) = 0 \quad \text{in } \Omega.\end{gathered}$$

The present paper is organized as follows: We start with the introduction of a basic assumptions and main result in section 2, which is proved in section 3. Finally, we give an example in section 4.

## 2. Assumptions and Main results

**Hypotheses.** Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $p$ be a real number such that $2 < p < \infty$ and $w = \{w_i(x) : 1 \leq i \leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in $\Omega$. Further, we suppose in all our considerations that,

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{2.1}$$

$$w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega), \tag{2.2}$$

for any $0 \leq i \leq N$. We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \quad \text{for } i = 1,\dots,N.$$

This is a Banach space under the norm

$$\|u\|_{1,p,w} = \Big[\int_\Omega |u(x)|^p w_0\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, the condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ i = 0,\dots,N\}$ and where $p'$ is the conjugate of $p$ i.e. $p' = \frac{p}{p-1}$. For more details, we refer the reader to [10].

Now we state the some assumptions.

(H1) For $2 \le p < \infty$, the expression

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{2.4}$$

is a norm on $W_0^{1,p}(\Omega,w)$ and it's equivalent to (2.3). There exists a weight function $\sigma$ on $\Omega$ such that

$$\sigma \in L^1(\Omega) \quad \text{and} \quad \sigma^{\frac{-1}{p-1}} \in L^1_{\rm loc}(\Omega). \tag{2.5}$$

The Hardy inequality

$$\Big(\int_\Omega |u(x)|^p \sigma\,dx\Big)^{1/p} \le c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}, \tag{2.6}$$

holds for every $u \in W_0^{1,p}(\Omega,w)$ with a constant $c > 0$ independent of $u$. Moreover, the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega,\sigma) \tag{2.7}$$

expressed by the inequality (2.6) is compact.

Note that $(W_0^{1,p}(\Omega,w), |||.|||)$ is a uniformly convex (and thus reflexive) Banach space.

**Remark 2.1.** Assume that $w_0(x) \equiv 1$ and there exists $\nu \in ]\frac{N}{P},+\infty[\cap[\frac{1}{P-1},+\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \quad \text{for all } i = 1,\dots,N. \tag{2.8}$$

Note that the assumptions (2.1) and (2.8) imply that,

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{2.9}$$

is a norm defined on $W_0^{1,p}(\Omega,w)$ and it's equivalent to (2.3) and that, the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow L^p(\Omega) \tag{2.10}$$

is compact [10, pp 46]. Thus the hypothesis $(H_1)$ is satisfied for $\sigma \equiv 1$.

(H2) For $i = 1, \dots, N$,

$$|a_0(x,t,s,\xi)| \le \beta\sigma^{1/p}(x) \quad [c_0(x,t) + \sigma^{\frac{1}{p'}}|s|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{2.11}$$

$$|a_i(x,t,s,\xi)| \le \beta w_i^{1/p}(x) \quad [c_1(x,t) + \sigma^{\frac{1}{p'}}|s|^{p-1} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \tag{2.12}$$

$$\sum_{i=1}^{N}[a_i(x,t,s,\xi) - a_i(x,t,s,\eta)](\xi_i - \eta_i) > 0 \quad \forall \xi \ne \eta \in \mathbb{R}^N, \tag{2.13}$$

$$a_0(x,t,s,\xi).s + \sum_{i=1}^{N} a_i(x,t,s,\xi).\xi_i \ge \alpha \sum_{i=1}^{N} w_i|\xi_i|^p, \tag{2.14}$$

where $c_0(x,t)$ and $c_1(x,t)$ are some positive functions in $L^{p'}(Q)$, and $\alpha$ and $\beta$ are some strictly positive constants.

**Some lemmas.** In this subsection we establish some imbedding and compactness results in weighted Sobolev Spaces which allow in particular to extend in the settings of weighted Sobolev spaces.
Let $V = W_0^{1,p}(\Omega, w)$, $H = L^2(\Omega, \sigma)$ and let $V^* = W^{-1,p'}(\Omega, w^*)$, with $(2 \le p < \infty)$. Let $X = L^p(0,T,V)$. The dual space of $X$ is $X^* = L^{p'}(0,T,V^*)$ where $\frac{1}{p'} + \frac{1}{p} = 1$ and denoting the space $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$ endowed with the norm

$$\|u\|_{w_p^1} = \|u\|_X + \|u'\|_{X^*}, \tag{2.15}$$

is a Banach space. Here $u'$ stands for the generalized derivative of $u$; i.e.,

$$\int_0^T u'(t)\varphi(t)\,dt = -\int_0^T u(t)\varphi'(t)\,dt \quad \text{for all } \varphi \in C_0^\infty(0,T).$$

**Lemma 2.2.** *The Banach space $H$ is an Hilbert space and its dual $H'$ can be identified with him self; i.e., $H' \simeq H$*

**Lemma 2.3.** *The evolution triple $V \subseteq H \subseteq V^*$ is verified.*

**Lemma 2.4.** *Let $g \in L^r(Q,\gamma)$ and let $g_n \in L^r(Q,\gamma)$, with $\|g_n\|_{L^r(Q,\gamma)} \le c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in $Q$, then $g_n \rightharpoonup g$ in $L^r(Q,\gamma)$, where $\rightharpoonup$ denotes weak convergence and $\gamma$ is a weight function on $Q$.*

**Lemma 2.5.** *Assume that (H1) and (H2) are satisfied and let $(u_n)$ be a sequence in $L^p(0,T,W_0^{1,p}(\Omega,w))$ such that $u_n \rightharpoonup u$ weakly in $L^p(0,T,W_0^{1,p}(\Omega,w))$ and*

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\nabla u)][\nabla u_n - \nabla u]\,dt\,dx \to 0. \tag{2.16}$$

*Then $u_n \to u$ in $L^p(0,T,W_0^{1,p}(\Omega,w))$.*

Now we recall the well-known general Sobolev imbedding theorems for evolution equations.

**Lemma 2.6** ([18]). *Let $V \subseteq H \subseteq V^*$ be an evolution triple. Then the imbedding*

$$W_p^1(0,T,V,H) \subseteq C([0,T]),H)$$

*is continuous.*

**Lemma 2.7** ([18]). *Let $Z_1, Y, Z_2$ be real reflexive Banach spaces. Assume that the imbeddings $Z_1 \subseteq Y \subseteq Z_2$ are continuous, and the imbedding $Z_1 \subseteq Y$ is compact, $0 < T < \infty$, $1 < p,\ q < \infty$. Then $W = \{u \in L^p(0,T,Z_1) : u' \in L^q(0,T,Z_2)\}$ equipped with the norm*

$$\|u\|_w = \|u\|_{L^p(0,T,Z_1)} + \|u'\|_{L^q(0,T,Z_2)}$$

*is a Banach space and the imbedding $W \subseteq L^p(0,T,Y)$ is compact.*

**Existence results.**

**Definition 2.8.** A monotone map $T : D(T) \to X^*$ is called maximal monotone if its graph

$$G(T) = \{(u, T(u)) \in X \times X^* \text{for all } u \in D(T)\}$$

is not a proper subset of any monotone set in $X \times X^*$.

Let us consider the operator $\frac{\partial}{\partial t}$ which induces a linear map $L$ from the subset $D(L) = \{v \in X : v' \in X^*, v(0) = 0\}$ of $X$ into $X^*$ by

$$\langle Lu, v\rangle_X = \int_0^T \langle u'(t), v(t)\rangle_V \, dt \quad u \in D(L),\ v \in X. \tag{2.17}$$

**Lemma 2.9** ([18]). *$L$ is a closed linear maximal monotone map.*

In our study we deal with mappings of the form $F = L + S$ where $L$ is a given linear densely defined maximal monotone map from $D(L) \subset X$ to $X^*$ and $S$ is a bounded demicontinuous map of monotone type from $X$ to $X^*$.

**Definition 2.10.** A mapping $S$ is pseudomonotone with respect to $D(L)$, if for any sequence $\{u_n\}$ in $D(L)$ with $u_n \rightharpoonup u$, $Lu_n \rightharpoonup Lu$ and $\lim_{n\to\infty} \sup\langle S(u_n), u_n - u\rangle \le 0$, we have $\lim_{n\to\infty}\langle S(u_n), u_n - u\rangle = 0$ and $S(u_n) \rightharpoonup S(u)$ as $n \to \infty$.

Consider the non linear parabolic problem

$$\begin{gathered}\frac{\partial u}{\partial t} + A(u) + A_0(x,t,u,\nabla u) = f \quad \text{in } Q\\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T)\\ u(x,0) = u_0 \quad \text{in } \Omega.\end{gathered}$$

**Definition 2.11.** A function $u$ is said to be a weak solution of the initial-boundary value problem (1.4) if $u \in C([0,T],H) \cap L^p(0,T,V)$, $\frac{\partial u}{\partial t} \in L^{p'}(0,T,V^*)$ and $u$ satisfies the equation

$$\frac{\partial u}{\partial t} + Au + A_0 u = f \qquad 0 < t < T, u(0) = u_0,$$

where the operator $A + A_0 : X \to X^*$ is defined by

$$\langle (A + A_0)(u), v\rangle = \int_Q a(x,t,u,\nabla u)\nabla v \, dx\, dt + \int_Q a_0(x,t,u,\nabla u) v \, dx\, dt$$

**Proposition 2.12.** *The operator $A + A_0 : X \to X^*$ is :*

(a) *bounded and demicontinuous;*
(b) *pseudomonotone with respect to $D(L)$*
(c) *strongly coercive, i.e.,*

$$\frac{\langle (A + A_0)(u), u\rangle_X}{\|u\|_X} \to +\infty, \quad \text{as } \|u\|_X \to +\infty.$$

We first consider the Zero-initial value problem,

$$\begin{gathered}\frac{\partial u}{\partial t}+A(u)+A_0(x,t,u,\nabla u)=f \quad \text{in } Q\\ u(x,t)=0 \quad \text{on } \partial\Omega\times(0,T)\\ u(x,0)=0 \quad \text{in } \Omega,\end{gathered} \tag{2.18}$$

**Theorem 2.13.** *Assume that the conditions (H1)-(H2) hold, then problem* (2.18) *admits a weak solution for any* $f\in X^*$.

**Theorem 2.14.** *Assume that the conditions (H1)-(H2) hold and* $u_0\in W_0^{1,p}(\Omega,w)$, *then the initial-boundary value problem* (1.4) *admits a weak solution for any* $f\in X^*$.

## 3. Proofs of Main results

*Proof of lemma 2.2.* Let the map $F: H\times H\to\mathbb{R}$ be defined by

$$F(f,g)=\int_\Omega fg\sigma\,dx.$$

Note that $F$ is a symmetric bilinear form, which is also continuous and defined positively, since

$$\int_\Omega fg\sigma\,dx=\int_\Omega f\sigma^{1/2}g\sigma^{1/2}\,dx\le\Big(\int_\Omega|f|^2\sigma\,dx\Big)^{1/2}\Big(\int_\Omega|g|^2\sigma\,dx\Big)^{1/2}.$$

Then the Banach space $H$ is an Hilbert space. Finally by a standard argument, we can identify $H$ with its dual $H'$; i.e., $H'\simeq H$. □

*Proof of lemma 2.3.* By the imbedding (2.7) and the fact that $2\le p<\infty$, and $\sigma\in L^1(\Omega)$, we can write

$$W_0^{1,p}(\Omega,w)\hookrightarrow\hookrightarrow L^p(\Omega,\sigma)\hookrightarrow H\simeq H'\hookrightarrow W^{-1,p'}(\Omega,w^*).$$

□

*Proof of lemma 2.4.* Since $g_n\gamma^{1/r}$ is bounded in $L^r(Q)$ and $g_n(x)\gamma^{1/r}(x)\to g\gamma^{1/r}$, a.e. in $Q$, then by [15, lemma 3.2], we have

$$g_n\gamma^{1/r}\rightharpoonup g\gamma^{1/r}\quad\text{in } L^r(Q).$$

Moreover, for all $\varphi\in L^{r'}(Q,\gamma^{1-r'})$, we have $\varphi\gamma^{\frac{-1}{r}}\in L^{r'}(Q)$. Then

$$\int_Q g_n\varphi\,dx\to\int_Q g\varphi\,dx,\quad\text{i.e. } g_n\rightharpoonup g\quad\text{in } L^r(Q,\gamma).$$

□

*Proof of lemma 2.5.* Let $D_n=[a(x,t,u_n,\nabla u_n)-a(x,t,u_n,\nabla u)][\nabla u_n-\nabla u]$. Then by (2.13), $D_n$ is a positive function and by (2.16), $D_n\to 0$ in $L^1(Q)$. Extracting a subsequence, still denoted by $u_n$, and using (2.7) we can write

$$u_n\to u\quad\text{a.e. in } Q,\qquad D_n\to 0\quad\text{a.e. in } Q.$$

Then, there exists a subset $B$ of $Q$, of zero measure such that for $(t,x) \in Q \setminus B$, $|u_n(x,t)| < \infty$, $|\nabla u(x,t)| < \infty$, $|c_1(x,t)| < \infty$, $w_i(x) > 0$ and $u_n(x,t) \to u(x,t), D_n(x,t) \to 0$. We set $\epsilon_n = \nabla u_n(x,t)$ and $\epsilon = \nabla u(x,t)$. Then

$$\begin{aligned} D_n(x,t) &= [a(x,t,u_n,\epsilon_n) - a(x,t,u_n,\epsilon)](\epsilon_n - \epsilon) \\ &\geq \alpha \sum_{i=1}^N w_i |\epsilon_n^i|^p + \alpha \sum_{i=1}^N w_i |\epsilon^i|^p \\ &\quad - \sum_{i=1}^N \beta w_i^{1/p} \Big[ c_1(x,t) + \sigma^{\frac{1}{p'}} |u_n|^{p-1} + \sum_{j=1}^N w_j^{\frac{1}{p'}} |\epsilon_n^j|^{p-1} \Big] |\epsilon^i| \\ &\quad - \sum_{i=1}^N \beta w_i^{1/p} \Big[ c_1(x,t) + \sigma^{\frac{1}{p'}} |u_n|^{p-1} + \sum_{j=1}^N w_j^{\frac{1}{p'}} |\epsilon^j|^{p-1} \Big] |\epsilon_n^i|; \end{aligned}$$

i.e,

$$D_n(x,t) \geq \alpha \sum_{i=1}^N w_i |\epsilon_n^i|^p - c_{x,t} \Big[ 1 + \sum_{j=1}^N w_j^{\frac{1}{p'}} |\epsilon_n^j|^{p-1} + \sum_{i=1}^N w_i^{1/p} |\epsilon_n^i| \Big], \tag{3.1}$$

where $c_{x,t}$ is a constant which depends on $x$, but does not depend on $n$. Since $u_n(x,t) \to u(x,t)$, we have $|u_n(x,t)| \leq M_{x,t}$, where $M_{x,t}$ is some positive constant. Then by a standard argument $|\epsilon_n|$ is bounded uniformly with respect to $n$. Indeed, (3.1) becomes

$$D_n(x,t) \geq \sum_{i=1}^N |\epsilon_n^i|^p \Big( \alpha w_i - \frac{c_{x,t}}{N |\epsilon_n^i|^p} - \frac{c_{x,t} w_i^{\frac{1}{p'}}}{|\epsilon_n^i|} - \frac{c_{x,t} w_i^{1/p}}{|\epsilon_n^i|^{p-1}} \Big).$$

If $|\epsilon_n| \to \infty$ (for a subsequence) there exists at least one $i_0$ such that $|\epsilon_n^{i_0}| \to \infty$, which implies that $D_n(x,t) \to \infty$ which gives a contradiction.

Let now $\epsilon^*$ be a cluster point of $\epsilon_n$. We have $|\epsilon^*| < \infty$ and by the continuity of a with respect to the two last variables we obtain

$$(a(x,t,u(x,t),\epsilon^*) - a(x,t,u(x,t),\epsilon))(\epsilon^* - \epsilon) = 0.$$

In view of (2.13) we have $\epsilon^* = \epsilon$. The uniqueness of the cluster point implies

$$\nabla u_n(x,t) \to \nabla u(x,t) \quad \text{a.e. in } Q.$$

Since the sequence $a(x,t,u_n,\nabla u_n)$ is bounded in the space $\prod_{i=1}^N L^{p'}(Q, w_i^*)$ and $a(x,t,u_n,\nabla u_n) \to a(x,t,u,\nabla u)$ a.e. in $Q$, Lemma 2.4 implies

$$a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u) \quad \text{in } \prod_{i=1}^N L^{p'}(Q, w_i^*) \quad \text{and a.e. in } Q.$$

We set $\bar{y}_n = a(x,t,u_n,\nabla u_n)\nabla u_n$ and $\bar{y} = a(x,t,u,\nabla u)\nabla u$. As in [4, lemma lemma 5] we can write $\bar{y}_n \to \bar{y}$ in $L^1(Q)$. By (2.14), we have

$$\alpha \sum_{i=1}^N w_i \Big| \frac{\partial u_n}{\partial x_i} \Big| \leq a(x,t,u_n,\nabla u_n)\nabla u_n.$$

Let

$$z_n = \sum_{i=1}^{N} w_i |\frac{\partial u_n}{\partial x_i}|^p, \quad z = \sum_{i=1}^{N} w_i |\frac{\partial u}{\partial x_i}|^p,$$

$$y_n = \frac{\bar{y}_n}{\alpha}, \quad y = \frac{\bar{y}}{\alpha}.$$

Then, by Fatou's lemma we obtain

$$\int_Q 2y \, dxdt \leq \lim_{n\to\infty} \inf \int_Q y + y_n - |z_n - z| \, dx \, dt;$$

i.e., $0 \leq \lim_{n\to\infty} \sup \int_Q |z_n - z| \, dxdt$, hence

$$0 \leq \lim_{n\to\infty} \inf \int_Q |z_n - z| \, dxdt \leq \lim_{n\to\infty} \sup \int_Q |z_n - z| \, dxdt \leq 0.$$

This implies

$$\nabla u_n \to \nabla u \quad \text{in } \prod_{i=1}^{N} L^p(Q, w_i),$$

which with (2.4) completes the present proof. $\square$

*Proof of proposition 2.12.* (a) We set $B = A + A_0$. Using (2.11), (2.12) and Hölder's inequality we can show that $B$ is bounded. For showing that $B$ is demicontinuous, let $v_\epsilon \to v$ in X as $\epsilon \to 0$, and prove that,

$$\langle B(v_\epsilon), \varphi\rangle \to \langle B(v), \varphi\rangle \quad \text{for all } \varphi \in X.$$

Since, $a_i(x,t,v_\epsilon,\nabla v_\epsilon) \to a_i(x,t,v,\nabla v)$ as $\epsilon \to 0$, for a.e. $x \in \Omega$, by the growth conditions (2.12), (2.11) and lemma 2.4 we get

$$a_i(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup a_i(x,t,v,\nabla v) \quad \text{in } L^{p'}(Q, w_i^{1-p'}) \quad \text{as } \epsilon \to 0$$

for $(i = 1, \dots, N)$ and

$$a_0(x,t,v_\epsilon,\nabla v_\epsilon) \rightharpoonup a_0(x,t,v,\nabla v) \quad \text{in } L^{p'}(Q, \sigma^{1-p'}) \quad \text{as } \epsilon \to 0.$$

Finally for all $\varphi \in X$,

$$\langle B(v_\epsilon), \varphi\rangle \to \langle B(v), \varphi\rangle \quad \text{as } \epsilon \to 0$$

(since $\varphi \in L^p(Q, \sigma)$ for all $\varphi \in X$).

(b) Suppose that $\{u_j\}$ is a sequence in $D(L)$ with
- (i) $u_j \rightharpoonup u$ weakly in $X$
- (ii) $Lu_j \to Lu$ weakly in $X^*$,
- (iii) $\limsup\langle A + A_0(u_j), u_j - u\rangle_X \leq 0$.

By the definition of the operator $L$ in (2.17), we obtain that $\{u_j\}$ is a bounded sequence in $W_p^1(0,T,V,H)$. By virtue of lemma 2.7, we get,

$$u_j \to u \quad \text{strongly in } L^p(Q, \sigma).$$

On the other hand,

$$\langle A_0 u_j, u_j - u\rangle = \int_Q a_0(x,t,u_j,\nabla u_j)(u_j - u) \, dx \, dt$$

Thus the Hölder's inequality and (i) imply

$$\begin{aligned}\langle A_0 u_j, u_j - u\rangle &\le \Big(\int_Q |a_0|^{p'}\sigma^{1-p'}\,dx\,dt\Big)^{1/p'}\|u_j-u\|_{L^p(Q,\sigma)}\\ &\le \|a_0\|_{L^{p'}(Q,\sigma*)}\|u_j-u\|_{L^p(Q,\sigma)},\end{aligned}$$

i.e, $\langle A_0u_j, u_j-u\rangle \to 0$ as $j\to\infty$. Combining the last convergence with (iii), we obtain

$$\lim_{j\to\infty}\sup\langle Au_j, u_j-u\rangle \le 0.$$

And by the pseudo-monotonicity of $A$ (see [9, Proposition 1]), we have

$$Au_j \rightharpoonup Au \quad\text{in } X^* \quad\text{and}\quad \lim_{j\to\infty}\langle Au_j, u_j-u\rangle = 0.$$

Then

$$\lim_{j\to\infty}\langle Au_j + A_0(u_j), u_j-u\rangle = 0.$$

On the other hand, $\lim_{j\to\infty}\langle Au_j, u_j-u\rangle = 0$, which implies

$$\begin{aligned}0 &= \lim_{j\to\infty}\int_Q a(x,t,u_j,\nabla u_j)\nabla(u_j-u)\,dx\,dt\\ &= \lim_{j\to\infty}\int_Q [a(x,t,u_j,\nabla u_j)-a(x,t,u_j,\nabla u)][\nabla u_j-\nabla u]\,dx\,dt\\ &\quad+\lim_{j\to\infty}\int_Q a(x,t,u_j,\nabla u)(\nabla u_j-\nabla u)\,dx\,dt.\end{aligned}$$

The last integral in the right hand tends to zero, since by the continuity of the Nemytskii operator, $a(x,t,u_j,\nabla u)\to a(x,t,u,\nabla u)$ in $\prod_{i=1}^N L^{p'}(Q,w_i^{1-p'})$ as $j\to+\infty$. So that

$$\lim_{j\to\infty}\int_Q [a(x,t,u_j,\nabla u_j)-a(x,t,u_j,\nabla u)][\nabla u_j-\nabla u]\,dx\,dt = 0.$$

By lemma 2.5 we have

$$\nabla u_j \to \nabla u \quad \text{a.e. in } Q.$$

Hence $a_0(x,t,u_j,\nabla u_j)\to a_0(x,t,u,\nabla u)$ a.e. in $Q$ as $j\to\infty$ and since

$$a_0(x,t,u_j,\nabla u_j)\in L^{p'}(Q,\sigma^{1-p'})$$

by Lemma 2.4, we obtain

$$a_0(x,t,u_j,\nabla u_j)\rightharpoonup a_0(x,t,u,\nabla u)\quad\text{in } L^{p'}(Q,\sigma^{1-p'}).$$

Finally,

$$B(u_j)\rightharpoonup B(u)\quad\text{in } X^*.$$

(c) The strongly coercivity follows from (2.14) □

*Proof of Theorem 2.13.* By proposition 2.12 the operator $A+A_0: X\to X^*$ is pseudomonotone with respect to $D(L)$, and the operator $A+A_0$ satisfies the strong coercivity condition which implies that both of the conditions (i) and (ii) in [3, theorem 4] hold. So all the conditions in [3, theorem 4] are met. Therefore, there exists a solution $u\in D(L)$ of the evolution equation

$$\frac{\partial u}{\partial t} + Au + A_0u = f$$

for any $f \in X^*$. In order to prove that $u$ is also a weak solution of the problem (2.18), we have to show that $u \in C([0,T],H)$. By the definition of $D(L)$ and lemma 2.6, we obtain

$$D(L) \subseteq W_p^1(0,T,V,H) \subseteq C([0,T],H).$$

Which implies that $u \in C([0,T],H)$. □

*Proof of Theorem 2.14.* Now we turn to problem (1.4). Assume that (H1)-(H2) hold and $u_0 \in W_0^{1,p}(\Omega,w)$. Let

$$\overline{a}_i(x,t,u,\nabla u) = a_i(x,t,u+u_0,\nabla u+\nabla u_0)$$

for all i = 0,…,N. Then it is easy to see that $\overline{a}_i$ also satisfies the conditions (H1)-(H2). But $\beta$ , $\alpha$ and the function $c_0(x,t)$ and $c_1(x,t)$ in (H1)-(H2) may depend on the function $u_0$. Analogously, $\overline{A}+\overline{A}_0 : X \to X^*$ is defined by

$$\langle(\overline{A}+\overline{A}_0)(u),v\rangle = \int_Q \overline{a}(x,t,u,\nabla u)\nabla v\,dx\,dt + \int_Q \overline{a}_0(x,t,u,\nabla u)v\,dx\,dt$$

for $u,v \in L^p(0,T,V)$, where $\overline{A} = -\mathrm{div}\overline{a}(x,t,u,\nabla u)$. Then, by Theorem 2.13, we have Theorem 2.14 □

## 4. An example

Let $\Omega$ be a bounded domain of $\mathbb{R}^N (N \geq 1)$, satisfying the cone condition. Let us consider the Carathéodory functions

$$a_i(x,t,s,\epsilon) = w_i|\epsilon_i|^{p-1}\operatorname{sgn}(\epsilon_i) \quad \text{for } i = 1,\dots,N,$$
$$a_0(x,t,s,\epsilon) = \rho\sigma(x)s|s|^{p-2}, \quad \rho > 0,$$

where $\sigma$ and $w_i(x)$ ($i =,1,\dots,N$) are given weight functions, strictly positive almost everywhere in $\Omega$. We shall assume that the weight functions satisfy, $w_i(x) = w(x)$, $x \in \Omega$, for all $i = 1,\dots,N$. Then, we can consider the Hardy inequality (2.6) in the form

$$\Big(\int_Q |u(x,t)|^p\sigma(x)\,dx\Big)^{1/p} \leq c\Big(\int_Q |\nabla u(x,t)|^p w\,dx\Big)^{1/p}.$$

It is easy to show that the functions $a_i(x,t,s,\epsilon)$ are Carathéodory functions satisfying the growth condition (2.12) and the coercivity (2.13). On the other hand, the monotonicity condition is satisfied, in fact,

$$\begin{aligned}&\sum_{i=1}^N (a_i(x,t,s,\epsilon) - a_i(x,t,s,\hat{\epsilon}))(\epsilon_i - \hat{\epsilon}_i)\\ &= w(x)\sum_{i=1}^N (|\epsilon_i|^{p-1}\operatorname{sgn}\epsilon_i - |\hat{\epsilon}_i|^{p-1}\operatorname{sgn}\hat{\epsilon}_i)(\epsilon_i - \hat{\epsilon}_i) > 0\end{aligned}$$

for almost all $(x,t) \in Q$ and for all $\epsilon,\hat{\epsilon} \in \mathbb{R}^N$ with $\epsilon \neq \hat{\epsilon}$, since $w > 0$ a.e. in $\Omega$. In particular, let us use the special weight functions $w$ and $\sigma$ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \mathrm{dist}(x,\partial\Omega)$ and set

$$w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\Big(\int_Q |u(x,t)|^p d^\mu(x)\,dx\Big)^{1/p} \leq \Big(c\int_Q |\nabla u(x,t)|^p d^\lambda(x)\,dx\Big)^{1/p}.$$

For $\lambda < p-1$, $\frac{\mu-\lambda}{p}+1>0$; See for example [9]

**Corollary 4.1.** *The parabolic initial-boundary value problem*

$$\begin{aligned}&\int_Q \frac{\partial u(x,t)}{\partial t}\varphi\,dx\,dt+\int_Q d^{\lambda}(x)\sum_{i=1}^{N}|\frac{\partial u(x,t)}{\partial x_i}|^{p-1}\operatorname{sgn}(\frac{\partial u}{\partial x_i})\frac{\partial\varphi(x,t)}{\partial x_i}\,dx\,dt\\&+\int_Q \rho d^{\mu}(x)u(x,t)|u(x,t)|^{p-2}\varphi(x,t)\\&=\int_Q f\varphi\,dx\,dt\quad\forall\varphi\in D(Q)\end{aligned}$$

*admits at least one solution* $u$ *in* $L^p(0,T,W_0^{1,p}(\Omega,d^{\lambda}))$*, for any function* $f$ *in* $L^{p'}(0,T,W^{-1,p'}(\Omega,d^{\lambda'}))$ *where* $\lambda'=\lambda(1-p')$ *and* $u_0\in W_0^{1,p}(\Omega,d^{\lambda})$*.*

## References


[1] R. Adams, *Sobolev spaces*, AC, Press, New York, (1975)

[2] Y. Akdim, E. Azroul and A. Benkirane, *Existence Results for Quasilinear Degenerated Equations Via Strong Convergence of Truncations*, Revista Matematica Complutense 17, , N.2, (2004) pp 359-379.

[3] J. Berkovits, V. Mustonen, *Topological degree for perturbation of linear maximal monotone mappings and applications to a class of parabolic problems*, Rend. Mat. Roma, Ser, VII, 12 (1992), pp. 597-621.

[4] L. Boccardo, F. Murat and J. P. Puel *Existence of bunded solutions for nonlinear elliptic unilateral probems*, Ann. Matt. Pura Appl. (4) 152 (1988), pp. 183-196. (English, with French and Italian summaries.

[5] L. Boccardo, F. Murat, *Strongly nonlinear Cauchy problems with gradient dependt lower order nonlinearity*, Pitman Research Notes in Mathematics, 208 (1988), pp. 347-364.

[6] L. Boccardo, F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations,* Nonlinear analysis, T.M.A., 19 (1992), no. 6, pp. 581-597.

[7] H. Brezis, and F.E. Browder, *Strongly nonlinear parabolic initial-boundary value problems*, Proc. Nat. Acad. Sci. U. S. A. 76 (1976). pp. 38-40.

[8] A. Dallaglio A. Orsina , *Non linear parabolic equations with natural growth condition and $L^1$ data.* Nolinear Anal., T.M.A., 27 no. 1 (1996). pp. 59-73.

[9] P. Drabek, A. Kufner and L. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators*, Bull. Austral. Math. Soc. Vol. 58 (1998), 213-221.

[10] P. Drabek, A. Kufner and F. Nicolosi, *Non linear elliptic equations, singular and degenerated cases*, University of West Bohemia, (1996).

[11] A. Kufner, *Weighted Sobolev Spaces*, John Wiley and Sons, (1985).

[12] R. Landes, *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems,* Proc. Roy. Soc. Edinburgh sect. A. 89 (1981), 217-137.

[13] R. Landes, V. Mustonen, *A strongly nonlinear parabolic initial-boundary value problems*, Ark. f. Math. 25. (1987).

[14] R. Landes, V. Mustonen, *On parabolic initial-boundary value problems with critical growth for the gradient*, Ann. Inst. H. Poincaré11(2)(1994)135-158.

[15] J. Leray, J. L. Lions, *Quelques resultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France 93 (1995), 97-107.

[16] J. L. Lions, *quelques methodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, 1969.

[17] Zh. Liu, *nonlinear degenerate parabolic equations*, Acta Math Hungar,77 (1-2), 1997, 147-157.

[18] E. Zeidler, *nonlinear functional analysis and its applications, II A and II B*, Springer-Verlag (New York-Heidlberg, 1990).

Lahsen Aharouch
Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B.P 1796 Atlas Fès, Maroc
*E-mail address*: l_aharouch@yahoo.fr

Elhoussine Azroul
Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B.P 1796 Atlas Fès, Maroc
*E-mail address*: azroul_elhoussine@yahoo.fr

Mohamed Rhoudaf
Département de Mathématiques et Informatique, Faculté des Sciences Dhar-Mahraz, B.P 1796 Atlas Fès, Maroc
*E-mail address*: rhoudaf_mohamed@yahoo.fr