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LIMIT BEHAVIOR OF AN OSCILLATING THIN LAYER

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Abstract. We study the limit behavior of a thermal problem, of a containing structure, an oscillating thin layer of thickness and conductivity depending of ε. We use the the epi-convergence method to find the limit problems with interface conditions. The obtained results are tested numerically.

1. Introduction

In mathematical physics, one meets several kinds of boundary problems, the heat conduction, electrostatic, electromagnetic, mechanical of the continuous mediums, where the unknown u satisfies the transmission conditions on the surface of separation between two domains Ω_1 and Ω_2:

$$u_{|\Omega_1} = u_{|\Omega_2} \tag{1.1}$$

$$\sigma_1 |\nabla u|^{p-2} \frac{\partial u}{\partial n}|_{\Omega_1} = \sigma_2 |\nabla u|^{p-2} \frac{\partial u}{\partial n}|_{\Omega_2} \tag{1.2}$$

where $p > 1$ and n represents the outward normal vector to the surface of separation, σ_1 and σ_2 are the associated constants to each domain Ω_1 and Ω_2 respectively. The boundary conditions of type (1.1) and (1.2) are met in thermal conductivity problems, where σ_1 and σ_2 designate the conductivities of two bodies. In the electrostatic or magnetostatic problems σ_1 and σ_2 are the dielectric or permeability constants respectively. A transmission problem with the conditions of type (1.1) and (1.2) and $p = 2$, was studied by Sanchez-Palencia in [8].

Our aim in this work is to study the limit behavior of solutions of a thermal conductivity problem, this last is in a structure containing an oscillating layer of thickness and conductivity depending of ε, ε being a parameter intended to tend towards 0.

A similar problems are found in Brillard and al in [4]. The vectorial case one finds it in Ait Moussa and al, and Brillard and al in [1, 6].

This paper is organized in the following way. In section 2, one expresses the problem to study, and one defines functional spaces for this study in the section 3. In the section 4, one studies the problem (2.1). The section 5 is reserved to the determination of the limits problems. Finally in the section 6, one will give a numerical test illustrating the obtained theoretical results.

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2. Position of the problem

One considers a problem of nonlinear thermal conduction in a body which occupies a bonded domain $\Omega \subset \mathbb{R}^3$, with a Lipschitz border $\partial\Omega$, composed of a layer B_ε, with oscillating border $\Sigma_\varepsilon^{\pm}$, of average interface σ, of very high conductivity, and a remaining region Ω_ε with a constant conductivity (see figure 1). The body occupying the domain Ω, is subject to an outside temperature f, $f\colon \Omega \to \mathbb{R}$, and cooled at the boundary $\partial\Omega$. The equations of the problem are:

$$
\begin{gathered}
\operatorname{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + f = 0 \quad \text{in } \Omega_\varepsilon, \\
\frac{1}{\varepsilon^\alpha}\operatorname{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + f = 0 \quad \text{in } B_\varepsilon, \\
[u^\varepsilon] = 0 \quad \text{on } \Sigma_\varepsilon^{\pm}, \\
|\nabla u^\varepsilon|^{p-2}\frac{\partial u^\varepsilon}{\partial n}\big|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}|\nabla u^\varepsilon|^{p-2}\frac{\partial u^\varepsilon}{\partial n}\big|_{B_\varepsilon} \quad \text{on } \Sigma_\varepsilon^{\pm}, \\
u^\varepsilon = 0 \quad \text{on } \partial\Omega,
\end{gathered}
\tag{2.1}
$$

where n the outward normal to $\partial\Omega$, $p > 1$ and $\alpha \geq 0$.

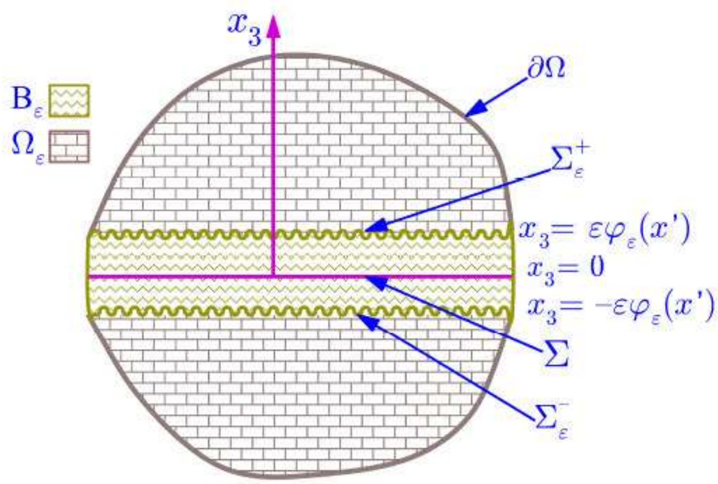

Figure 1. Domain Ω.

Where ε being a positive parameter intended to tend towards zero and φ_ε is a bounded real function, $]0, \varepsilon[^2$-periodic.

3. Notation and functional setting

Here is the notation that will be used in the sequel:
$x = (x', x_3)$ where $x' = (x_1, x_2)$, $\lambda = 1, 2$, $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $Y =]0,1[\times]0,1[$,
$\varphi\colon \mathbb{R}^2 \to [a_1, a_2]$ where φ is Y-periodic and $a_2 \geq a_1 > 0$, $\varphi_\varepsilon(x') = \varphi(\frac{x'}{\varepsilon})$,
$\frac{\partial \varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma) \cap L^\infty(\Sigma)$, $m(\varphi) = (\frac{1}{\int_Y dx'}) \int_Y \varphi(x')dx'$, $\eta(t) = \lim_{\varepsilon\to 0} \varepsilon^{1-t}$ with $t \geq 0$.

In the following C will denote any constant with respect to ε. Also we use the convention $0.+\infty = 0$.

3.1. **Functional setting.** First, we introduce the Banach space $V^\varepsilon = W_0^{1,p}(\Omega)$. Let

$$V^p(\Sigma) = \Big\{u \in W_0^{1,p}(\Omega) : u\big|_\Sigma \in W^{1,p}(\Sigma)\Big\},$$
$$V^C(\Sigma) = \Big\{u \in W_0^{1,p}(\Omega) : u\big|_\Sigma = C\Big\}.$$

The set $V^C(\Sigma)$ is a Banach space with the norm of $W_0^{1,p}(\Omega)$. we show easily that $V^p(\Sigma)$ is a Banach space with the norm

$$u \mapsto \|\nabla u\|_{L^p(\Omega)^3} + \left\|\nabla' u_{|\Sigma}\right\|_{L^p(\Sigma)^2}.$$

Let

$$\mathbb{G}^\alpha = \begin{cases} \Big\{u \in W_0^{1,p}(\Omega) : \eta(\alpha) u\big|_\Sigma \in W^{1,p}(\Sigma)\Big\} & \text{if } \alpha \le 1, \\ V^C & \text{if } \alpha > 1. \end{cases}$$

$$\mathbb{D}^\alpha = \begin{cases} \mathcal{D}(\Omega) & \text{if } \alpha \le 1, \\ \Big\{u \in \mathcal{D}(\Omega) : u\big|_\Sigma = C\Big\} & \text{if } \alpha > 1. \end{cases}$$

It is known that $\overline{\mathbb{D}^\alpha} = \mathbb{G}^\alpha$.

Our goal in this work, is to study the problem (2.1) and its limit behavior.

4. Study of the problem (2.1)

The problem (2.1) is equivalent to the minimization problem

$$\inf_{v \in V^\varepsilon} \Big\{\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla v|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla v|^p - \int_\Omega f.v\Big\}. \tag{4.1}$$

Proposition 4.1. *For $f \in L^{p'}(\Omega)$, the problem (4.1) admits an unique solution u^ε in V^ε.*

The proof of this proposition is based on classical convexity arguments see for example [3].

Lemma 4.1. *For every $f \in L^{p'}(\Omega)$, the family $(u^\varepsilon)_{\varepsilon>0}$ satisfies:*

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} \le C, \tag{4.2}$$

$$\frac{1}{\varepsilon^\alpha}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)} \le C. \tag{4.3}$$

Moreover u^ε is bounded in $W_0^{1,p}(\Omega)$.

Proof. Since u^ε is the solution of the problem (4.1), we have

$$\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^{p-2}\,\nabla u^\varepsilon \nabla v + \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u^\varepsilon|^{p-2}\,\nabla u^\varepsilon \nabla v = \int_\Omega f v, \quad \forall v \in V^\varepsilon.$$

In particular, by taking $v = u^\varepsilon$, one obtains

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^\alpha}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)} = \int_\Omega f u^\varepsilon.$$

According to the inequalities of Hölder and Young, one has

$$\begin{aligned}\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^\alpha}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)} &\le C\|\nabla u^\varepsilon\|_{L^p(\Omega)}\\ &\le C(\|\nabla u^\varepsilon\|_{L^p(\Omega_\varepsilon)} + \|\nabla u^\varepsilon\|_{L^p(B_\varepsilon)})\\ &\le C + \frac{1}{p}\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{1}{p}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)}\\ &\le C + \frac{1}{p}\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^\alpha}\frac{1}{p}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)}.\end{aligned}$$

So that

$$\|\nabla u^\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^\alpha}\|\nabla u^\varepsilon\|^p_{L^p(B_\varepsilon)} \le C.$$

Therefore, one will have the assertions (4.2) and (4.3). It is clair that for a small enough ε, the solution (u^ε) is bounded in $W_0^{1,p}(\Omega)$. □

Let us define the operator "m^ε" which transforms the definite functions u on B_ε into functions definite on Σ by

$$m^\varepsilon u(x_1,x_2) = \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u(x_1,x_2,x_3)dx_3. \tag{4.4}$$

Lemma 4.2. *The operator m^ε definite by* (4.4) *is linear and bounded of $L^p(B_\varepsilon)$ (respectively $W^{1,p}(B_\varepsilon)$) in $L^p(\Sigma)$ (respectively $W^{1,p}(\Sigma)$), with norm $\le C\varepsilon^{-\frac{1}{p}}$, moreover, for all $u \in W^{1,p}(B_\varepsilon)$, one has*

$$\left\|m^\varepsilon u - u_{|\Sigma}\right\|^p_{|_{L^p(\Sigma)}} \le C\varepsilon^{p-1}\int_{B_\varepsilon}|\nabla u|^p. \tag{4.5}$$

Proof. One has

$$\int_\Sigma |m^\varepsilon u|^p\,dx_1dx_2 = \int_\Sigma (\frac{1}{2\varepsilon\varphi_\varepsilon})^p\left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u dx_3\right|^p dx_1dx_2, \tag{4.6}$$

since $0 < a_1 \le \varphi_\varepsilon \le a_2$, and according to the inequality of Hölder, (4.6) becomes

$$\begin{aligned}\int_\Sigma |m^\varepsilon u|^p\,dx_1dx_2 &\le \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|u|^p\,dx_3\Big)dx_1dx_2\\ &\le \frac{1}{2\varepsilon a_1}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|u|^p\,dx_3\Big)dx_1dx_2,\end{aligned} \tag{4.7}$$

since $u \in L^p(B_\varepsilon)$ and (4.7), it follows that $m^\varepsilon u \in L^p(\Sigma)$. Let $u \in \mathcal{D}(\overline{B_\varepsilon})$. One has

$$\begin{aligned}\frac{\partial}{\partial x_\lambda}(m^\varepsilon u)(x_1,x_2) &= \frac{1}{2}\frac{\partial}{\partial x_\lambda}\Big(\int_{-1}^1 u(x_1,x_2,x_3\varepsilon\varphi_\varepsilon)dx_3\Big)\\ &= \frac{1}{2}\Big(\int_{-1}^1 \frac{\partial u}{\partial x_\lambda}(x_1,x_2,x_3\varepsilon\varphi_\varepsilon)\\ &\quad + \varepsilon x_3\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\frac{\partial u}{\partial x_3}(x_1,x_2,x_3\varepsilon\varphi_\varepsilon)dx_3\Big)\\ &= \frac{1}{2\varepsilon\varphi_\varepsilon}\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\frac{\partial u}{\partial x_\lambda} + (\frac{x_3}{\varepsilon\varphi_\varepsilon})(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda})\frac{\partial u}{\partial x_3}dx_3\Big).\end{aligned}$$

So that

$$\begin{aligned}\int_\Sigma \big|\frac{\partial}{\partial x_\lambda}(m^\varepsilon u)\big|^p &= \int_\Sigma \big|\frac{1}{2\varepsilon\varphi_\varepsilon}\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\frac{\partial u}{\partial x_\lambda}+(\frac{x_3}{\varepsilon\varphi_\varepsilon})(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda})\frac{\partial u}{\partial x_3}dx_3\Big)\big|^p\\ &\le \frac{1}{2\varepsilon a_1}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\big|\frac{\partial u}{\partial x_\lambda}+(\frac{x_3}{\varepsilon\varphi_\varepsilon})(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda})\frac{\partial u}{\partial x_3}\big|^p dx_3\Big).\end{aligned}$$

However, $\frac{\partial\varphi}{\partial x_\lambda}\in\mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$, then $\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}$ is bounded, and therefore

$$\int_\Sigma \big|\frac{\partial}{\partial x_\lambda}(m^\varepsilon u)\big|^p \le \frac{C}{\varepsilon}\int_{B_\varepsilon}\big(\big|\frac{\partial u}{\partial x_\lambda}\big|^p+\big|\frac{\partial u}{\partial x_3}\big|^p\big)dx_3 \le \frac{C}{\varepsilon}\int_{B_\varepsilon}|\nabla u|^p\,,$$

by density arguments, for all $u\in W^{1,p}(B_\varepsilon)$, one has

$$\int_\Sigma \big|\frac{\partial}{\partial x_\lambda}(m^\varepsilon u)\big|^p \quad\le\quad \frac{C}{\varepsilon}\int_{B_\varepsilon}|\nabla u|^p\,.$$

Let $u\in\mathcal{D}(\overline{B_\varepsilon})$, so that

$$\big\|m^\varepsilon u-u_{|\Sigma}\big\|^p_{L^p(\Sigma)}=\int_\Sigma\Big|\Big(\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}u(x_1,x_2,x_3)dx_3\Big)-u(x_1,x_2,0)\Big|^p dx_1dx_2, \tag{4.8}$$

using the inequality of Hölder, (4.8) becomes

$$\begin{aligned}\big\|m^\varepsilon u-u_{|\Sigma}\big\|^p_{L^p(\Sigma)} &\le \frac{1}{2\varepsilon a_1}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|u(x_1,x_2,x_3)-u(x_1,x_2,0)|^p\,dx_3\Big)dx_1dx_2\\ &\le \frac{C}{\varepsilon}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\big|\int_0^{x_3}\frac{\partial u}{\partial x_3}(x_1,x_2,t)dt\big|^p dx_3\Big)dx_1dx_2\\ &\le \frac{C}{\varepsilon}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}|x_3|^{p-1}\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\big|\frac{\partial u}{\partial x_3}(x_1,x_2,t)\big|^p dt\Big)dx_3\Big)dx_1dx_2\\ &\le C\varepsilon^{p-1}\int_\Sigma\Big(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon}\left|\frac{\partial u}{\partial x_3}\right|^p dx_3\Big)dx_1dx_2\\ &\le C\varepsilon^{p-1}\int_{B_\varepsilon}|\nabla u|^p\,dx,\end{aligned}$$

by density arguments, one has for all $u\in W^{1,p}(B_\varepsilon)$

$$\big\|m^\varepsilon u-u_{|\Sigma}\big\|^p_{L^p(\Sigma)}\le C\varepsilon^{p-1}\int_{B_\varepsilon}|\nabla u|^p\,dx. \tag{4.9}$$

Hence the proof of lemma 4.2 is complete. □

Lemma 4.3. *Let $(u^\varepsilon)_{\varepsilon>0}\subset V^\varepsilon$ which satisfies (4.2) and (4.3). Then*

$$\Big\|\nabla'(m^\varepsilon u^\varepsilon)\Big\|^p_{(L^p(\Sigma))^2}\le C\varepsilon^{\alpha-1}. \tag{4.10}$$

Moreover $m^\varepsilon u^\varepsilon$ possess a bounded subsequence in $L^p(\Sigma)$.

Proof. Thanks to lemma 4.2, one has

$$\big\|\frac{\partial(m^\varepsilon u^\varepsilon)}{\partial x_\lambda}\big\|^p_{L^p(\Sigma)^2}\le C\varepsilon^{-1}\int_{B_\varepsilon}|\nabla u^\varepsilon|^p\,dx\,.$$

According to (4.3), one has

$$\big\|\frac{\partial(m^\varepsilon u^\varepsilon)}{\partial x_\lambda}\big\|^p_{L^p(\Sigma)^2}\le C\varepsilon^{\alpha-1}.$$

Let us show that $(m^\varepsilon u^\varepsilon)$ is a bounded sequence in $L^p(\Sigma)$. From (4.5), (see, lemma 4.2), one has

$$\left\|m^\varepsilon u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right\|^p_{L^p(\Sigma)} \le C\varepsilon^{p-1}\int_{B_\varepsilon} |\nabla u^\varepsilon|^p \,dx.$$

According to (4.3), one obtains

$$\left\|m^\varepsilon u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right\|^p_{L^p(\Sigma)} \le C\varepsilon^{\alpha+p-1}.$$

As u^ε satisfies (4.2) and (4.3), so u^ε is bounded in $W^{1,p}_0(\Omega)$, it follows that there exists $u^* \in W^{1,p}_0(\Omega)$ and a subsequence of u^ε, still denoted by u^ε, such that $u^\varepsilon \rightharpoonup u^*$ in $W^{1,p}_0(\Omega)$, so ${u^\varepsilon}_{|\Sigma}$ is a bounded sequence in $L^p(\Sigma)$. Since

$$\|m^\varepsilon u^\varepsilon\|_{L^p(\Sigma)} \le \left\|m^\varepsilon u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right\|_{L^p(\Sigma)} + \left\|{u^\varepsilon}_{|\Sigma}\right\|_{L^p(\Sigma)}, \tag{4.11}$$

from (4.11), there exists a constant $C > 0$ such that $\|m^\varepsilon u^\varepsilon\|^p_{L^p(\Sigma)} \le C$. □

Proposition 4.2. *The solution of the problem* (4.1), $(u^\varepsilon)_\varepsilon$, *possess a subsequence weakly convergent toward an element* u^* *in* $W^{1,p}_0(\Omega)$ *satisfying*

1. *If* $\alpha = 1$*:* $u^*\big|_\Sigma \in W^{1,p}(\Sigma)$.
2. *If* $\alpha > 1$*:* $u^*\big|_\Sigma = C$.

Proof. According to lemma 4.1, the sequence u^ε is bounded in $W^{1,p}_0(\Omega)$, it follows that there exists an element $u^* \in W^{1,p}_0(\Omega)$ and a subsequence of u^ε, still denoted by u^ε such that $u^\varepsilon \rightharpoonup u^*$ in $W^{1,p}_0(\Omega)$. One has

$$\left\|m^\varepsilon u^\varepsilon - {u^\varepsilon}_{|\Sigma}\right\|_{L^p(\Sigma)} \le C\varepsilon^{\frac{\alpha+p-1}{p}} \quad \text{and } {u^\varepsilon}_{|\Sigma} \rightharpoonup {u^*}_{|\Sigma} \text{ in } L^p(\Sigma).$$

For $\alpha = 1$, according to the evaluation (4.10), the sequence $\nabla' m^\varepsilon u^\varepsilon$ possess a subsequence, still denoted by $\nabla' m^\varepsilon u^\varepsilon$ weakly convergent to an element u^2 in $L^p(\Sigma)^2$, as $m^\varepsilon u^\varepsilon \rightharpoonup {u^*}_{|\Sigma}$ in $L^p(\Sigma)$, so one concludes that $m^\varepsilon u^\varepsilon \rightharpoonup {u^*}_{|\Sigma}$ in $W^{1,p}(\Sigma)$ and $\nabla' {u^*}_{|\Sigma} = u^2$. Hence ${u^*}_{|\Sigma} \in W^{1,p}(\Sigma)$.

For $\alpha > 1$, one shows, as in the case $\alpha = 1$ and taking $u^2 = 0$, that ${u^*}_{|\Sigma} = C$. Hence the proof of proposition 4.2 is complete. □

The limit behavior of the problem (4.1), will be derived with the epi-convergence method, (see definition 7.1).

5. Limit behavior

Let

$$F^\varepsilon(u) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u|^p, \quad \forall u \in W^{1,p}_0(\Omega), \tag{5.1}$$

$$G(u) = -\int_\Omega fu, \quad \forall u \in W^{1,p}_0(\Omega). \tag{5.2}$$

One denotes by τ_f the weak topology on $W^{1,p}_0(\Omega)$.

Theorem 5.1. *According to the values of* α, *there exists a functional* F^α *defined on* $W^{1,p}_0(\Omega)$ *with value in* $\mathbb{R} \cup \{+\infty\}$ *such that* $\tau_f - \lim_e F^\varepsilon = F^\alpha$ *in* $W^{1,p}_0(\Omega)$, *where the functional* F^α *is given by*

(1) *If* $0 \le \alpha < 1$*:*

$$F^\alpha(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,, \quad \forall u \in W_0^{1,p}(\Omega).$$

(2) *If* $\alpha \ge 1$*:*

$$F^\alpha(u) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{2m(\varphi)\,\eta(\alpha)}{p}\int_\Sigma \left|\nabla' u_{|\Sigma}\right|^p & \text{if } u \in \mathbb{G}^\alpha, \\ +\infty & \text{if } u \in W_0^{1,p}(\Omega)\setminus \mathbb{G}^\alpha. \end{cases}$$

Proof. (a) One is going to determine the upper epi-limit: Let $u \in \mathbb{G}^\alpha \subset W_0^{1,p}(\Omega)$, there exists a sequence (u^n) in $\mathbb{D}^\alpha$ such that

$$u^n \to u \text{ in } \mathbb{G}^\alpha, \text{ when } n \to +\infty.$$

So that $u^n \to u$ in $W_0^{1,p}(\Omega)$. Let θ be a smooth function satisfying

$$\theta(x_3) = 1 \text{ if } |x_3| \le 1,\ \theta(x_3) = 0 \text{ if } |x_3| \ge 2 \text{ and } |\theta'(x_3)| \le 2\ \forall x \in \mathbb{R},$$

and set

$$\theta_\varepsilon(x) = \theta(\frac{x_3}{\varepsilon\varphi_\varepsilon});$$

we define

$$u^{\varepsilon,n} = \theta_\varepsilon(x) u^n{}_{|\Sigma} + (1-\theta_\varepsilon(x))u^n,$$

One shows easily that $u^{\varepsilon,n} \in V^\varepsilon$ and $u^{\varepsilon,n} \to u^n$ in $\mathbb{G}^\alpha$, when $\varepsilon \to 0$. Since

$$F^\varepsilon(u^{\varepsilon,n}) = \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u^{\varepsilon,n}|^p\,,$$

so that

$$\begin{aligned} F^\varepsilon(u^{\varepsilon,n}) &= \frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{1}{p}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u^{\varepsilon,n}|^p \\ &= \frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} |\nabla u^n|^p + \frac{1}{p}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' u^n{}_{|\Sigma}\right|^p. \end{aligned} \tag{5.3}$$

Since φ_ε is bounded, one verifies easily that

$$\lim_{\varepsilon\to 0}\Big\{\frac{1}{p}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} |\nabla u^{\varepsilon,n}|^p\Big\} = 0. \tag{5.4}$$

(1) If $\alpha \le 1$: Since $\varphi_\varepsilon \rightharpoonup^* m(\varphi)$ in $L^\infty(\Sigma)$ and $\varepsilon^{1-\alpha} \to \eta(\alpha)$, it follows that

$$\lim_{\varepsilon\to 0} \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' u^n{}_{|\Sigma}\right|^p = \frac{2m(\varphi)\eta(\alpha)}{p}\int_\Sigma \left|\nabla' u^n{}_{|\Sigma}\right|^p.$$

By passage to the upper limit, one has

$$\begin{aligned} \limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0}\Big(\frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} |\nabla u^n|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' u^n{}_{|\Sigma}\right|^p\Big) \\ &= \frac{1}{p}\int_\Omega |\nabla u^n|^p + \frac{2m(\varphi)\,\eta(\alpha)}{p}\int_\Sigma \left|\nabla' u^n{}_{|\Sigma}\right|^p. \end{aligned}$$

(2) If $\alpha > 1$: By passage to the upper limit, one has

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0}\Big(\frac{1}{p}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} |\nabla u^n|^p\Big)\\ &= \frac{1}{p}\int_\Omega |\nabla u^n|^p .\end{aligned}$$

Since $u^n \to u$ in $\mathbb{G}^\alpha$, when $n \to +\infty$. According to the classical result, diagonalization's lemma [2, Lemma 1.15], there exists a function $n(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$, such that $u^{\varepsilon,n(\varepsilon)} \to u$ in $\mathbb{G}^\alpha$ when $\varepsilon \to 0$. While n approaches $+\infty$, one will have

(1) If $\alpha \neq 1$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) &\le \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n})\\ &\le \frac{1}{p}\int_\Omega |\nabla u|^p .\end{aligned}$$

(2) If $\alpha = 1$:

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) &\le \limsup_{n\to+\infty}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n})\\ &\le \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_\Sigma \big|\nabla' u_{|\Sigma}\big|^p .\end{aligned}$$

If $u \in W_0^{1,p}(\Omega)\setminus \mathbb{G}^\alpha$, it is clear that, for every $u^\varepsilon \in W_0^{1,p}(\Omega)$, $u^\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, one has

$$\limsup_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \le +\infty.$$

(b) One is going to determine the lower epi-limit. Let $u \in \mathbb{G}^\alpha$ and (u^ε) be a sequence in $W_0^{1,p}(\Omega)$ such that $u^\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, so that

$$\chi_{\Omega_\varepsilon}\nabla u^\varepsilon \rightharpoonup \nabla u \quad \text{in } L^p(\Omega)^3. \tag{5.5}$$

(1) If $\alpha \neq 1$: Since

$$F^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^p .$$

According to (5.5) and by passage to the lower limit, one obtains

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \ge \frac{1}{p}\int_\Omega |\nabla u|^p .$$

(2) If $\alpha = 1$: If $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty$, there is nothing to prove, because

$$\frac{1}{p}\int_\Omega |\nabla u|^p + \frac{2m(\varphi)\eta(\alpha)}{p}\int_\Sigma \big|\nabla' u_{|\Sigma}\big|^p \le +\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $F^\varepsilon(u^\varepsilon)$ still denoted by $F^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that $F^\varepsilon(u^\varepsilon) \le C$, which implies that

$$\frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u^\varepsilon|^p \le C. \tag{5.6}$$

So u^ε satisfies the hypothesis of the lemma 4.3, and according to this last, $\nabla' m^\varepsilon u^\varepsilon$ is bounded in $L^p(\Sigma)^2$, so there exists an element $u_1 \in L^p(\Sigma)^2$ and a subsequence of $\nabla' m^\varepsilon u^\varepsilon$, still denoted by $\nabla' m^\varepsilon u^\varepsilon$, such that $\nabla' m^\varepsilon u^\varepsilon \rightharpoonup u_1$

in $L^p(\Sigma)^2$, since ${u^\varepsilon}_{|\Sigma} \rightharpoonup u_{|\Sigma}$ in $L^p(\Sigma)$, and thanks to (4.5) and (5.6), one has $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\Sigma}$ in $L^p(\Sigma)$, then $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\Sigma}$ in $W^{1,p}(\Sigma)$, therefore $u_1 = \nabla' u_{|\Sigma}$, so that $\nabla' m^\varepsilon u^\varepsilon \rightharpoonup \nabla' u_{|\Sigma}$ in $L^p(\Sigma)^2$. One has

$$\begin{aligned} F^\varepsilon(u^\varepsilon) &\geq \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^p + \frac{1}{p\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla u^\varepsilon|^p \\ &\geq \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^p + \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' m^\varepsilon u^\varepsilon\right|^p . \end{aligned}$$

Using the sub-differential inequality of

$$v \to \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|v\right|^p, \forall v \in L^p(\Sigma)^2,$$

one has

$$\begin{aligned} F^\varepsilon(u^\varepsilon) \geq \frac{1}{p}\int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^p &+ \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' u_{|\Sigma}\right|^p \\ &+ \frac{2\varepsilon^{1-\alpha}}{p}\int_\Sigma \varphi_\varepsilon \left|\nabla' u_{|\Sigma}\right|^{p-2} \nabla' u_{|\Sigma}(\nabla' m^\varepsilon u^\varepsilon - \nabla' u_{|\Sigma}). \end{aligned}$$

Thanks to lemma 7.1, one has $\varphi_\varepsilon \to m(\varphi)$ in $L^{p'}(\Sigma)$, so according to (5.5) and by passage to the lower limit, one obtains

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) \geq \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{2m(\varphi)\,\eta(\alpha)}{p}\int_\Sigma \left|\nabla' u_{|\Sigma}\right|^p .$$

If $u \in W_0^{1,p}(\Omega) \setminus \mathbb{G}^\alpha$ and $u^\varepsilon \in W_0^{1,p}(\Omega)$, such that $u^\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.
Assume that

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty.$$

So there exists a constant $C > 0$ and a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$, such that

$$F^\varepsilon(u^\varepsilon) < C. \tag{5.7}$$

For $0 \leq \alpha < 1$, there is nothing to prove.
Otherwise, One takes the same way as in the case $u \in \mathbb{G}^{\alpha=1}$, one has $\nabla' m^\varepsilon u^\varepsilon$ is bounded in $L^p(\Sigma)^2$, so there exists an element $u_1 \in L^p(\Sigma)^2$ and a subsequence of $\nabla' m^\varepsilon u^\varepsilon$, still denoted by $\nabla' m^\varepsilon u^\varepsilon$, such that $\nabla' m^\varepsilon u^\varepsilon \rightharpoonup u_1$ in $L^p(\Sigma)^2$, since ${u^\varepsilon}_{|\Sigma} \rightharpoonup u_{|\Sigma}$ in $L^p(\Sigma)$, and thanks to (4.5) and (5.7), one has $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\Sigma}$ in $L^p(\Sigma)$, then $m^\varepsilon u^\varepsilon \rightharpoonup u_{|\Sigma}$ in $W^{1,p}(\Sigma)$, so that $u \in \mathbb{G}^\alpha$ what contradicts the fact that $u \notin \mathbb{G}^\alpha$, so that

$$\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty.$$

Hence the proof of theorem 5.1 is complete. □

In the sequel, one is interested to limit problem determination partner to the problem (4.1), when ε approaches zero. Thanks to the epi-convergence results, (see theorem 7.3, Proposition 7.2) and Theorem 5.1, and according to τ_f-continuity of G in $W_0^{1,p}(\Omega)$, one has $F^\varepsilon + G$ τ_f-epi-converges toward $F^\alpha + G$ in $W_0^{1,p}(\Omega)$.

Proposition 5.2. *For every $f \in L^{p'}(\Omega)$ and according to the parameter values of α, there exists $u^* \in W_0^{1,p}(\Omega)$ satisfying*

$$u^\varepsilon \rightharpoonup u^* \, in \; W_0^{1,p}(\Omega),$$
$$F^\alpha(u^*) + G(u^*) = \inf_{v \in \mathbb{G}^\alpha} \Big\{ F^\alpha(v) + G(v) \Big\}.$$

Proof. Thanks to lemma 4.1, the family (u^ε) is bounded in $W_0^{1,p}(\Omega)$, therefore it possess a τ_f- cluster point u^* in $W_0^{1,p}(\Omega)$. And thanks to a classical epi-convergence result (see theorem 7.3), one has u^* is a solution of the problem Find

$$\inf_{v \in W_0^{1,p}(\Omega)} \Big\{ F^\alpha(v) + G(v) \Big\}. \tag{5.8}$$

Since F^α equals $+\infty$ on $W_0^{1,p}(\Omega) \setminus \mathbb{G}^\alpha$, (5.8) becomes

$$\inf_{v \in \mathbb{G}^\alpha} \Big\{ F^\alpha(v) + G(v) \Big\}. \tag{5.9}$$

According to the uniqueness of solutions of the problem (5.8), so that u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon \rightharpoonup u^*$ in $W_0^{1,p}(\Omega)$. □

Remark 5.3. One shows that the limit behavior of a constituted structure of two mediums of constant conductivity united by an oscillating non linear thin layer of thickness ε, which the conductivity depends on the negative powers of ε, is describes by a problem with interface Σ, (Σ the middle interface of the thin layer). Following the powers of ε, to the interface Σ, one has, on the interface Σ, the heat continuity, a bidimensional problem or the constant heat.

6. Numerical solutions

We showed that for a small enough ε, the solution u^ε of the problem (4.1), in a certain sense, approaches the solution u^* of the limit problem (5.9). In this section we interest to the numerical treatment of the problem (4.1) and (5.9), to illustrate the obtained theoretical results. Take the problems (4.1) and (5.9), with

$$\Omega =]-1,1[\times]-1,1[,$$
$$f(x,y) = 0.01\exp(-x^2 - y^2),$$
$$\varphi_\varepsilon(x) = 1.2 + \sin(\pi \frac{x}{\varepsilon}).$$

We solve numerically the problems (4.1) and (5.9), using the language FreeFem++ (see,[7]), with the finite elements method and using Newtons method, with $p = 3.5$ and $\varepsilon = 1e - 06$, and one will have the results shown in figures.

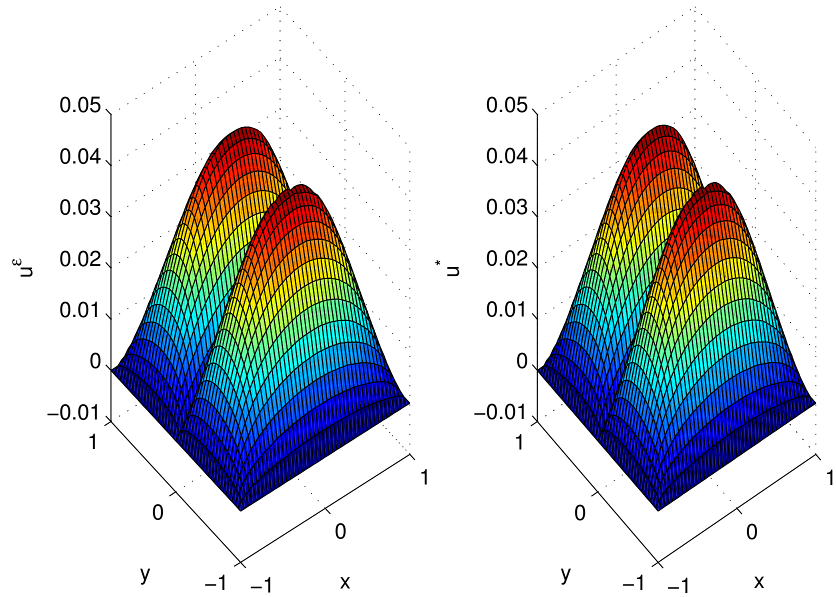

FIGURE 2. Solution to (4.1) (left) and to the limit problem (5.9) for $\alpha = 0.01$ (right)

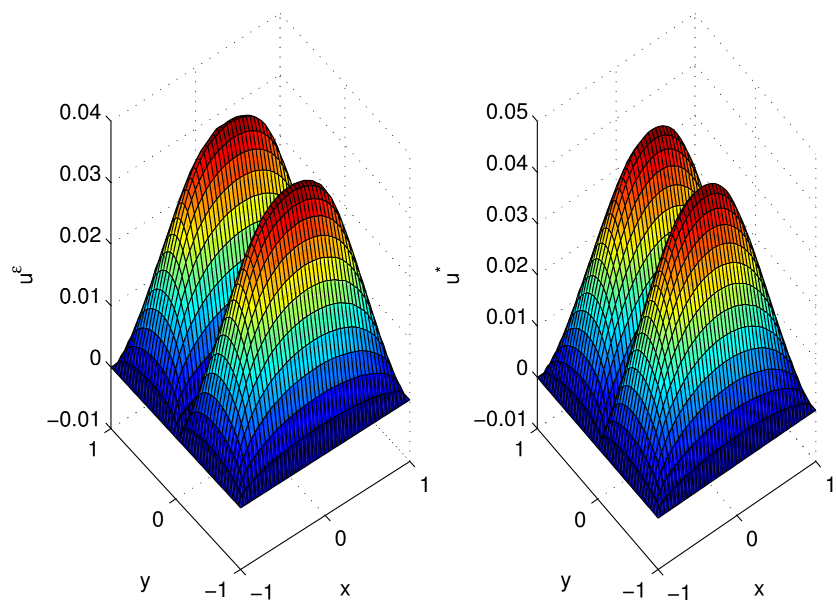

FIGURE 3. Solution to (4.1) (left) and to the limit problem (5.9) for $\alpha = 1$ (right)

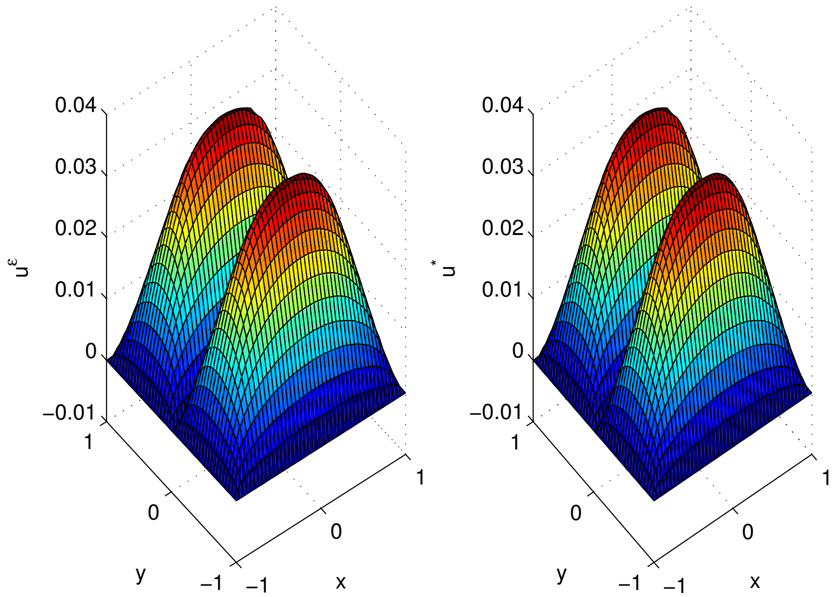

FIGURE 4. Solution to (4.1) (left) and to the limit problem (5.9) for $\alpha = 4.5$ (right)

Figures 2, 3, 4 show that the solution of (4.1) approach the one of the limit problem (5.9), for $\alpha = 0.01$, 1 and 4.5 with an error of order $1.64e-010$, $1.1e{-}05$, and $9e-014$, respectively.

7. Appendix

Definition 7.1 ([2, Definition 1.9]). *Let $(\mathbb{X}, \tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted $\tau - lim_e F^\varepsilon = F$, if the following assertions are satisfied*

- *For all $x \in \mathbb{X}$, there exists x^0_ε, $x^0_\varepsilon \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon\to 0} F^\varepsilon(x^0_\varepsilon) \le F(x)$.*
- *For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon\to 0} F^\varepsilon(x_\varepsilon) \ge F(x)$.*

Note the following stability result of the epi-convergence.

Proposition 7.2 ([2, p. 40]). *Suppose that F^ε epi-converges to F in $(\mathbb{X}, \tau)$ and that $G\colon \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, is $\tau -$ continuous. Then $F^\varepsilon + G$ epi-converges to $F + G$ in $(\mathbb{X}, \tau)$*

This epi-convergence is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [5]. It is well suited to the asymptotic analysis of sequences of minimization problems since one has the following fundamental result.

Theorem 7.3 ([2, theorem 1.10]). *Suppose that*

(1) *F^ε admits a minimizer on $\mathbb{X}$,*
(2) *The sequence $(\overline{u}^\varepsilon)$ is τ-relatively compact,*

(3) *The sequence F^ε epi-converges to F in this topology τ.*

Then every cluster point $\overline{u}$ of the sequence $(\overline{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon'\to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

if $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\overline{u}^\varepsilon)_\varepsilon$ which converges to $\overline{u}$.

Lemma 7.1. *Let $\varphi \in L^\infty(\Sigma)$, a Y-periodic, $Y =]0,1[\times]0,1[$. Let*

$$\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon}), \text{ for a small enough } \varepsilon > 0.$$

So that

$$\begin{gathered} \varphi_\varepsilon \to m(\varphi) \quad \text{in } L^s(\Sigma) \text{ for } 1 \le s < \infty, \\ \varphi_\varepsilon \rightharpoonup^* m(\varphi) \quad \text{in } L^\infty(\Sigma). \end{gathered}$$

Proof. Since φ_ε is a εY-periodic, so one has

$$\begin{gathered} \varphi_\varepsilon \rightharpoonup m(\varphi) \quad \text{in } L^s(\Sigma) \text{ for } 1 \le s < \infty, \\ \varphi_\varepsilon \rightharpoonup^* m(\varphi) \quad \text{in } L^\infty(\Sigma). \end{gathered} \tag{7.1}$$

Since φ is bounded a.e. in Σ, so for every $s \ge 1$, there exists a constant $C > 0$, such that

$$\begin{aligned} \int_\Sigma |\varphi_\varepsilon - m(\varphi)|^s &\le C \int_\Sigma |\varphi_\varepsilon - m(\varphi)| \\ &\le C\Big(\int_{\varphi \ge m(\varphi)} (\varphi_\varepsilon - m(\varphi)) - \int_{\varphi \le m(\varphi)} (\varphi_\varepsilon - m(\varphi))\Big). \end{aligned} \tag{7.2}$$

Passing to the limit in (7.2), one has $\varphi_\varepsilon \to m(\varphi)$ in $L^s(\Sigma)$ for $1 \le s < \infty$. □

References

[1] Ait Moussa A. et Licht C., *Comportement asymptotique d'une plaque mince non linaire*, J. Math du Maroc, N^{o}2, 1994, p. 1-16.

[2] Attouch H., *Variational Convergence for Functions and Operators*, Pitman (London) 1984.

[3] Brezis H., *Analyse fonctionnelle, Théorie et Applications*, Masson(1992).

[4] Brillard A., *Asymptotic analysis of nonlinear thin layers*, International Series of Numerical Mathematics, Vol 123, 1997.

[5] De Giorgi, E., *Convergence problems for functions and operators. Proceedings of the International Congress on "Recent Methods in Nonlinear analysis"*, Rome 1978. De Giorgi, Mosco, Spagnolo Eds. Pitagora Editrice (Bologna) 1979.

[6] Ganghoffer J. F., Brillard A. and Schultz J.; *Modelling of mechanical behavior of joints bounded by a nonlinear incompressible elastic adhesive*, European Journal of Mechanics. A/Solids. Vol 16, N^{o}2, 255-276. 1997.

[7] Hecht F., Pironneau O., Le Hyaric A., Ohtsuka K., FreeFem++ Manual, downloadable at http://www.Freefem.org.

[8] Phum Huy H. and Sanchez-Palencia E., *Phénoménes de transmission à travers des couches minces de conductivité élevée*, Journal of Mathematical Analysis and Applications, 47, pp.284-309,1974.

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