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# HIGHER ORDER NONLINEAR DEGENERATE ELLIPTIC PROBLEMS WITH WEAK MONOTONICITY

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Abstract. We prove the existence of solutions for nonlinear degenerate elliptic boundary-value problems of higher order. Solutions are obtained using pseudo-monotonicity theory in a suitable weighted Sobolev space.

## 1. Introduction and statement of results

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure and let $m \geq 1$ be an integer and $p > 1$ be a real number. We will consider the degenerated partial differential operators

$$Au(x) = A^m u(x) + A^{m-1} u(x), \tag{1.1}$$

on $\Omega$ where

$$A^m u(x) = \sum_{|\alpha|=m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, \nabla^m u) \tag{1.2}$$

is the top order part of the degenerated quasilinear operator $A$. and where

$$A^{m-1} u(x) = \sum_{|\alpha|\leq m-1} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \dots, \nabla^m u) \tag{1.3}$$

is the lower order part of $A$. The coefficients $\{A_\alpha(x, \eta, \zeta), |\alpha| \leq m\}$ are real valued functions defined on $\Omega \times \mathbb{R}^{N_{m-1}} \times \mathbb{R}^{N_m}$ (with $N_{m-1} = \text{card}\{\alpha \in \mathbb{N}^N, |\alpha| \leq m-1\}$ and $N_m = \text{card}\{\alpha \in \mathbb{N}^N, \ |\alpha| = m\}$) which satisfy suitable regularity and growth assumptions (see section 2). Let $V$ be a subspace such that

$$W_0^{m,p}(\Omega, w) \subseteq V \subseteq W^{m,p}(\Omega, w), \tag{1.4}$$

where $W^{m,p}(\Omega, w)$ and $W_0^{m,p}(\Omega, w)$ are weighted Sobolev spaces associated to a vector of weights $w = \{w_\alpha \equiv w_\alpha(x), |\alpha| \leq m\}$ on $\Omega$ satisfying some integrability conditions (see sections 2). We deal with the case where $A^{m-1}$ is affine with respect

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to the top order derivatives of $u$, i.e, $A^{m-1}u$ is of the form,

$$\begin{aligned} A^{m-1}u(x) = &\sum_{|\alpha|\le m-1} (-1)^{|\alpha|} D^\alpha L_\alpha(x,u,\dots,\nabla^{m-1}u) \\ &+ \sum_{|\alpha|\le m-1}\sum_{|\beta|=m} (-1)^{|\alpha|} D^\alpha C_{\alpha\beta}(x,u,\dots,\nabla^{m-1}u) D^\beta u \end{aligned} \tag{1.5}$$

where $L_\alpha(x,\eta)$ and $C_{\alpha\beta}(x,\eta)$ are some real valued functions defined on $\Omega\times\mathbb{R}^{N_{m-1}}$. We will assume the following hypotheses:

(H1) For every $u\in V$ and any multi-index $|\beta|\le m-1$, there exists a parameter $q(\beta)\ge 1$ and a weight function $\sigma_\beta=\sigma_\beta(x)$ such that,

$$D^\beta u \in L^{q(\beta)}(\Omega,\sigma_\beta),$$
$$\|D^\beta u(x)\|_{q(\beta),\sigma_\beta} \le \tilde c_\beta \|u\|_{m,p,w}$$

with some constant $\tilde c_\beta>0$ independent of $u$ and moreover, the compact imbedding,

$$V \hookrightarrow\hookrightarrow H^{m-1,q}(\Omega,\sigma) \tag{1.6}$$

holds, where $H^{m-1,q}(\Omega,\sigma)=\{u, D^\beta u\in L^{q(\beta)}(\Omega,\sigma_\beta) \text{ for all } |\beta|\le m-1\}$.

(H2) The functions $\{A_\alpha,|\alpha|=m\}$, $\{L_\alpha,|\alpha|\le m-1\}$ and $\{C_{\alpha\beta},|\alpha|\le m-1 \text{ and } |\beta|=m\}$ are Carathéodory functions and there exists functions $g_\alpha\in L^{p'}(\Omega)$ for all $|\alpha|=m$, $\tilde g_\alpha\in L^{q'(\alpha)}(\Omega)$ for all $|\alpha|\le m-1$, and $\gamma_{\alpha\beta}\in L^{r_\alpha}(\Omega)$ for all $|\alpha|\le m-1$ and all $|\beta|=m$ such that

(i) for all $|\alpha|=m$,

$$\begin{aligned} &|A_\alpha(x,\eta,\zeta)| \\ &\le c_\alpha w_\alpha^{1/p}(x)\Big(g_\alpha(x)+\tilde c_\alpha\sum_{|\beta|=m} w_\beta^{\frac{1}{p'}}|\zeta_\beta|^{p-1}+\tilde c_\alpha\sum_{|\beta|\le m-1}\sigma_\beta^{\frac{1}{p'}}|\eta_\beta|^{\frac{q(\beta)}{p'}}\Big)\end{aligned}$$

(ii) for all $|\alpha|\le m-1$,

$$|L_\alpha(x,\eta)|\le c_\alpha\sigma_\alpha^{\frac{1}{q(\alpha)}}\Big(\tilde g_\alpha(x)+\tilde c_\alpha\sum_{|\beta|\le m-1}\sigma_\beta^{\frac{1}{q'(\alpha)}}|\eta_\beta|^{\frac{q(\beta)}{q'(\alpha)}}\Big)$$

(iii) for all $|\alpha|\le m-1$ and all $|\beta|=m$,

$$\begin{aligned} &|C_{\alpha\beta}(x,\eta)| \\ &\le c_{\alpha\beta}\sigma_\alpha^{\frac{1}{q(\alpha)}}(x) w_\beta^{1/p}(x)\Big(\gamma_{\alpha\beta}(x)+\tilde c_{\alpha\beta}\sum_{|\lambda|\le m-1}\sigma_\lambda^{\frac{1}{r_\alpha}}(x)|\eta_\lambda|^{\frac{q(\lambda)}{r_\alpha}}\Big)\end{aligned}$$

for a.e. $x\in\Omega$, some positive constants $c_\alpha$, $\tilde c_\alpha$ and $\tilde c_{\alpha\beta}$, every $(\eta,\zeta)\in\mathbb{R}^{N_{m-1}}\times\mathbb{R}^{N_m}=\mathbb{R}^d$ and some exponent $r_\alpha$ such that

$$\frac{1}{r_\alpha}+\frac{1}{p}+\frac{1}{q(\alpha)}<1 \quad \text{for all } |\alpha|\le m-1. \tag{1.7}$$

For the existence of $r_\alpha$ see Remark 2.1 below.

Let us consider the degenerated boundary value problem (DBVP) associated to the equation,

$$Au=f\in V^*, \tag{1.8}$$

where $V^*$ is the dual space of $V$ from (1.4). Recently, Drabeck, Kufner and Mustonen proved in [4] the existence result for Dirichlet degenerated problem of second order associated to the operator $A$ of the form,

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) \tag{1.9}$$

where the Carathéodory functions $a_i(x, \eta, \zeta)$ satisfy some simple growth conditions, that is,

$$|a_i(x, \eta, \zeta)| \le c_1 w_i^{1/p}(x)\Big(g(x) + \bar{w}^{\frac{1}{p'}}(x)|\eta|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}|\zeta|^{p-1}\Big) \tag{1.10}$$

where the exponent $q$ and the weight function $\bar{w}(x)$ verify the so called Hardy-type inequality; i.e,

$$\int_\Omega |u(x)|^q \bar{w}(x)\, dx \le c \sum_{i=1}^{N} \int_\Omega |D_i u|^p w_i(x)\, dx \tag{1.11}$$

and the compact imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \bar{w}). \tag{1.12}$$

The authors have proved that the mapping $T$ associated to $A$ from (1.9) is pseudo-monotone in $W_0^{1,p}(\Omega, w)$, by assuming only the so-called weak Leray-Lions condition

$$\sum_{i=1}^{N} (a_i(x, \eta, \zeta) - a_i(x, \eta, \bar{\zeta}))(\zeta_i - \bar{\zeta}_i) \ge 0. \tag{1.13}$$

Our first objective of this paper is to extend the previous result of [4] in the general class of operators $A$ from (1.1), where the lower order part $A^{m-1}$ is of the form (1.5) and where the growth conditions are of the most general form (H2). More precisely, we prove the following result.

**Theorem 1.1.** *Assume that (H1), (H2) and that*

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \zeta) - A_\alpha(x, \eta, \bar{\zeta}))(\zeta_\alpha - \bar{\zeta}_\alpha) \ge 0 \tag{1.14}$$

*for a.e.* $x \in \Omega$*, all* $\eta \in \mathbb{R}^{N_{m-1}}$ *and all* $(\zeta, \bar{\zeta}) \in \mathbb{R}^{N_m} \times \mathbb{R}^{N_m}$ *hold. Then the mapping* $T$ *associated to the operator* $A$ *from* (1.1) *and* (1.5) *is pseudo-monotone in* $V$*.*

*If in addition the degeneracy satisfies*

$$\sum_{|\alpha|\le m} A_\alpha(x, \xi)\xi_\alpha \ge c \sum_{|\alpha|\le m} w_\alpha(x)|\xi_\alpha|^p, \tag{1.15}$$

*for a.e.* $x \in \Omega$*, some* $c > 0$ *and all* $\xi \in \mathbb{R}^{N_{m-1}} \times \mathbb{R}^{N_m}$*, then the DBVP associated to the equation* (1.8) *has at least one solution* $u \in V$*.*

**Remark 1.2.** The statement of Theorem 1.1, is obviously contained in Theorem 3.1 below (it suffices to take $J = \emptyset$) where some general situation is considered.

On the other hand, Drabeck, Kufner and Nikolosi in [6] have studied the existence result for the DBVP from the equation (1.8) with $A$ of the form (1.1) and with more general hypotheses (H1'), (H2'), (H3) (in section 2) and with the so-called Leray-Lions condition

$$\sum_{|\alpha|=m}(A_\alpha(x,\eta,\zeta)-A_\alpha(x,\eta,\bar\zeta))(\zeta_\alpha-\bar\zeta_\alpha)>0. \tag{1.16}$$

The authors have assumed in addition to the previous hypotheses the compact imbedding,

$$V\hookrightarrow\hookrightarrow W^{m-1,p}(\Omega,w) \tag{1.17}$$

and then, have proved that the mapping $T$ satisfies the condition $\alpha(V)$ (see definition 2.5) and hence used the degree theory of general mappings of monotone type. The hypotheses (1.17) play an important role in the work [6], because it is related to some strong converges appearing in the $\alpha(V)$ condition.

Our second objective of this paper, is to prove the same result as in [6] without assuming the compact imbedding (1.17). This is possible by proving the pseudo-monotonicity of the mapping $T$ induced by the operator $A$ from (1.1). More precisely, we have the following result.

**Theorem 1.3.** *Assume that (H1'), (H2'), (H3) and* (1.16). *Then the mapping $T$ associated to operator $A$ from* (1.1) *is pseudo-monotone in $V$. If in addition the degeneracy* (1.15) *is satisfied, then, the DBVP from the equation* (1.8) *has at least one solution $u\in V$.*

**Remark 1.4.** Theorem 1.3 is obviously a consequence of the more general Theorem 3.1 it suffices to take $J^c=\emptyset$).

Hence, this paper can be seen as an extension of the preceding papers [4, 5, 6] (where the second order case without lower order part is considered in the first paper. The degree theory is used in the two last papers) and as a continuation of the papers [2] and [3] (where the second order case with lower order part not equal to zero, is studied in the first paper and where the higher order case with $A^{m-1}\equiv 0$ or with $A^{m-1}\not\equiv 0$ but under restrictions $w_\alpha\equiv 1$ for all $|\alpha|\le m-1$, is considered in the last paper). Finally, note that our approach (based on the theory of pseudo-monotone mappings) can be applied in the case of non reflexive Banach spaces. For example in the general settings of weighted Orlicz-Sobolev spaces (see [1] for this direction). This work is divided into five sections. We start with the introduction of a basic assumptions in section 2. Next, we give our main general result in section 3, which is proved in section 4. Finally, we study in section 5, some particular case (where our basic assumption are satisfied). In our work, we shall adopt many ideas introduced in [7], but the results are generalized and improved.

## 2. Preliminaries and basic assumptions

2.1. **Weighted Sobolev spaces.** Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure. In the sequel we suppose that the vector of weights, on $\Omega$, $w=\{w_\alpha(x):|\alpha|\le m\}$ satisfies the integrability conditions:

$$w_\alpha\in L^1_{\rm loc}(\Omega),$$
$$w_\alpha^{-\frac{1}{p-1}}\in L^1_{\rm loc}(\Omega)$$

for any $|\alpha|\le m$. We denote by $W^{m,p}(\Omega,w)$ $(1<p<\infty)$ the space of all real-valued functions $u$ such that the derivatives in the sense of distributions fulfil

$$D^\alpha u\in L^p(\Omega,w_\alpha)\quad \text{for all } |\alpha|\le m.$$

The weighted Sobolev space $W^{m,p}(\Omega,w)$ is normed when equipped by the norm

$$\|u\|_{m,p,w} = \Big(\sum_{|\alpha|\le m}\int_\Omega |D^\alpha u|^p w_\alpha\ dx\Big)^{1/p}. \tag{2.1}$$

The space $W_0^{m,p}(\Omega,w)$ is defined as the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (2.1). Note that the conditions (2.1) and (2.1) imply that the spaces $W^{m,p}(\Omega,w)$ and $W_0^{m,p}(\Omega,w)$ are reasonably defined and are reflexive Banach spaces (for more details see [6]). We recall that the dual space of $W_0^{m,p}(\Omega,w)$ is equivalent to $W^{-m,p'}(\Omega,w^*)$ where $w^* = \{w^*_\alpha = w_\alpha^{1-p'} : |\alpha|\le m\}$, with $p' = \frac{p}{p-1}$ is the Hölder's conjugate of $p$.

2.2. **Basic assumptions.** Let $J$ be a subset of $\{\alpha\in\mathbb{N}^N,\ |\alpha| = m\}$ and $J^c$ its complement. We will suppose that the coefficients $A_\alpha$ of the operator $A$ from (1.1) are such that

$$\begin{gathered} A_\alpha(x,\eta,\zeta) = B_\alpha(x,\eta,\zeta_J)\quad \forall\alpha\in J,\\ A_\alpha(x,\eta,\zeta) = B_\alpha(x,\eta,\zeta_{J^c})\quad \forall\alpha\in J^c,\\ A_\alpha(x,\eta,\zeta) = L_\alpha(x,\eta,\zeta_J) + \sum_{\beta\in J^c} C_{\alpha\beta}(x,\eta,\zeta_J)\zeta_\beta\quad \forall|\alpha|\le m-1, \end{gathered} \tag{2.2}$$

for a.e. $x\in\Omega$ and where $\{B_\alpha, |\alpha| = m\}$, $\{L_\alpha, |\alpha|\le m-1\}$ and $\{C_{\alpha\beta}, |\alpha|\le m-1$ and $\beta\in J^c\}$ are some Carathéodory functions and where $\zeta_I$ denoted $\zeta_I = \{\zeta_\alpha,\ \alpha\in I\}$. We denote by $N_I = \text{card}\{\alpha\in\mathbb{N}^N,\quad \alpha\in I\}$. Let us introduce the following modified versions of (1.16) and (1.14),

$$\sum_{\alpha\in J}(B_\alpha(x,\eta,\zeta_J) - B_\alpha(x,\eta,\bar\zeta_J))(\zeta_\alpha - \bar\zeta_\alpha) > 0, \tag{2.3}$$

for a.e $x\in\Omega$, all $\eta\in\mathbb{R}^{N_{m-1}}$ and all $\zeta_J\ne\bar\zeta_J\in\mathbb{R}^{N_J}$ and

$$\sum_{\alpha\in J^c}(B_\alpha(x,\eta,\zeta_{J^c}) - B_\alpha(x,\eta,\bar\zeta_{J^c}))(\zeta_\alpha - \bar\zeta_\alpha) \ge 0, \tag{2.4}$$

for a.e $x\in\Omega$ and all $(\eta,\zeta_{J^c},\bar\zeta_{J^c})\in\mathbb{R}^{N_{m-1}}\times\mathbb{R}^{N_{J^c}}\times\mathbb{R}^{N_{J^c}}$.

Let us denote by $m_1 = m - \frac{N}{p}$ and suppose that $m_1 > 0$ i.e, $mp > N$. We denote by $C(\Omega,w_\alpha)$ the weighted spaces of continuous functions, more precisely $C(\Omega,w_\alpha) = \{u = u(x)$ continuous on $\Omega, \|u\|_{C(\Omega,w_\alpha)} = \sup_{x\in\Omega}|u(x)w_\alpha(x)| < \infty\}$.

(H1') Let $u\in V$.

(i) For $|\beta| < m_1$, there is a weight function $\sigma_\beta = \sigma_\beta(x)$ such that, $D^\beta u\in C(\Omega,\sigma_\beta)$ and moreover,

$$\sup_{x\in\Omega}|D^\beta u(x)\sigma_\beta(x)| \le \tilde c_\beta\|u\|_{m,p,w} \tag{2.5}$$

with some constant $\tilde c_\beta > 0$ independent of $u$. When we denote by $k(x,u(x))$ the expression $\sum_{|\beta|<m_1}|\sigma_\beta(x)D^\beta u(x)|$, then, in view of (2.5),

$$|k(x,u(x))| \le c\|u\|_{m,p,w}\quad \text{for all } u\in V. \tag{2.6}$$

(ii) For $m_1\le|\beta|\le m-1$, there is a parameter $q(\beta)\ge 1$ and a weight function $\sigma_\beta = \sigma_\beta(x)$ such that $D^\beta u\in L^{q(\beta)}(\Omega,\sigma_\beta)$ and moreover,

$$\|D^\beta u(x)\|_{q(\beta),\sigma_\beta} \le \tilde c_\beta\|u\|_{m,p,w} \tag{2.7}$$

for some constant $\tilde c_\beta > 0$ independent of $u$.

(iii) The imbedding $V \hookrightarrow\hookrightarrow H^{m-1,q}(\Omega,\sigma)$ compact, where $H^{m-1,q}(\Omega,\sigma) = \{u, D^\beta u \in X_\beta, \text{ for all } |\beta| \le m-1\}$ with $X_\beta = L^{q(\beta)}(\Omega,\sigma_\beta)$ for $m_1 \le |\beta| \le m-1$ and $X_\beta = C(\Omega,\sigma_\beta)$ for $|\beta| < m_1$.

(H2') There exists functions $g_\alpha \in L^{p'}(\Omega)$ for $|\alpha| = m$, $\tilde{g}_\alpha \in L^{q'(\alpha)}(\Omega)$ for $m_1 \le |\alpha| \le m-1$, $\hat{g}_\alpha \in L^1(\Omega)$ for $|\alpha| < m_1$, $\gamma_{\alpha\beta} \in L^{r_\alpha}(\Omega)$ for all $|\alpha| \le m-1$ and $\beta \in J^c$ and some positive constants $\tilde{c}_\alpha$ and $\tilde{c}_{\alpha\beta}$, moreover there exists a positive continuous, non decreasing function $G(t)$, $t \ge 0$, such that the following estimates hold:

(i) For $\alpha \in J$,
$$
\begin{aligned}
&|B_\alpha(x,\eta,\zeta_J)| \\
&\le G(k(x,\kappa)) w_\alpha^{1/p}\Big(g_\alpha(x) + \tilde{c}_\alpha \sum_{\beta\in J} w_\beta^{\frac{1}{p'}} |\zeta_\beta|^{p-1} + \tilde{c}_\alpha \sum_{m_1\le|\beta|\le m-1} \sigma_\beta^{\frac{1}{p'}} |\eta_\beta|^{\frac{q(\beta)}{p'}}\Big)
\end{aligned}
$$
(ii) for $\alpha \in J^c$,
$$
\begin{aligned}
&|B_\alpha(x,\eta,\zeta_{J^c})| \\
&\le G(k(x,\kappa)) w_\alpha^{1/p}\Big(g_\alpha(x) + \tilde{c}_\alpha \sum_{\beta\in J^c} w_\beta^{\frac{1}{p'}} |\zeta_\beta|^{p-1} + \tilde{c}_\alpha \sum_{m_1\le|\beta|\le m-1} \sigma_\beta^{\frac{1}{p'}} |\eta_\beta|^{\frac{q(\beta)}{p'}}\Big)
\end{aligned}
$$
(iii) for $m_1 \le |\alpha| \le m-1$,
$$
\begin{aligned}
&|L_\alpha(x,\eta,\zeta_J)| \\
&\le G(k(x,\kappa)) \sigma_\alpha^{\frac{1}{q(\alpha)}} \Big(\tilde{g}_\alpha(x) + \tilde{c}_\alpha \sum_{\beta\in J} w_\beta^{\frac{1}{q'(\alpha)}} |\zeta_\beta|^{\frac{p}{q'(\alpha)}} + \tilde{c}_\alpha \sum_{m_1\le|\beta|\le m-1} \sigma_\beta^{\frac{1}{q'(\alpha)}} |\eta_\beta|^{\frac{q(\beta)}{q'(\alpha)}}\Big)
\end{aligned}
$$
(iv) for $|\alpha| < m_1$,
$$
\begin{aligned}
&|L_\alpha(x,\eta,\zeta_J)| \\
&\le G(k(x,\kappa)) \sigma_\alpha \Big(\hat{g}_\alpha(x) + \tilde{c}_\alpha \sum_{\beta\in J} w_\beta |\zeta_\beta|^{p} + \tilde{c}_\alpha \sum_{m_1\le|\beta|\le m-1} \sigma_\beta |\eta_\beta|^{q(\beta)}\Big)
\end{aligned}
$$
(v) for $m_1 \le |\alpha| \le m-1$ and $\beta \in J^c$,
$$
\begin{aligned}
&|C_{\alpha\beta}(x,\eta,\zeta_J)| \\
&\le G(k(x,\kappa)) \sigma_\alpha^{\frac{1}{q(\alpha)}} w_\beta^{1/p} \Big(\gamma_{\alpha\beta}(x) + \tilde{c}_{\alpha\beta} \sum_{\lambda\in J} w_\lambda^{\frac{1}{r_\alpha}} |\zeta_\lambda|^{\frac{p}{r_\alpha}} + \tilde{c}_{\alpha\beta} \sum_{m_1\le|\lambda|\le m-1} \sigma_\lambda^{\frac{1}{r_\alpha}} |\eta_\lambda|^{\frac{q(\lambda)}{r_\alpha}}\Big)
\end{aligned}
$$
(vi) for $|\alpha| < m_1$ and $\beta \in J^c$,
$$
\begin{aligned}
&|C_{\alpha\beta}(x,\eta,\zeta_J)| \\
&\le G(k(x,\kappa)) \sigma_\alpha w_\beta^{1/p} \Big(\gamma_{\alpha\beta}(x) + \tilde{c}_{\alpha\beta} \sum_{\lambda\in J} w_\lambda^{\frac{1}{r_\alpha}} |\zeta_\lambda|^{\frac{p}{r_\alpha}} + \tilde{c}_{\alpha\beta} \sum_{m_1\le|\lambda|\le m-1} \sigma_\lambda^{\frac{1}{r_\alpha}} |\eta_\lambda|^{\frac{q(\lambda)}{r_\alpha}}\Big)
\end{aligned}
$$
for a.e. $x \in \Omega$, every $\eta \in \mathbb{R}^{N_{m-1}}$ and every $\zeta_I \in \mathbb{R}^{N_I}$ where $\kappa = \{\eta_\beta, |\beta| < m_1\}$ and
$$
\frac{1}{r_\alpha} + \frac{1}{p} + \frac{1}{q(\alpha)} < 1
$$
for any $m_1 \le |\alpha| \le m-1$ and any $\beta \in J^c$ and with
$$
\frac{1}{r_\alpha} + \frac{1}{p} < 1
$$

for any $|\alpha| < m_1$ and any $\beta \in J^c$. Note that the exponent $q'(\alpha)$ denotes the Hölder's conjugate of $q(\alpha)$.

**Remark 2.1.** For all $m_1 \le |\alpha| \le m-1$, the such $r_\alpha$ satisfying $\frac{1}{r_\alpha} + \frac{1}{p} + \frac{1}{q(\alpha)} < 1$ exists when $q(\alpha) > p'$. And we can choose $r_\alpha > p'$ when $|\alpha| < m_1$.

**Remark 2.2.** If $m_1 \le 0$, then the set of multi-indices $\xi_\beta$ with $|\beta| < m_1$ is empty. Then we set $G(t) \equiv 1$ and since the cases iv)and vi) in $(H_2')$ are irrelevant, we obtain the growth condition of type C [6]. Further if we do not differ between $|\alpha| = m$ and $|\alpha| \le m-1$ i.e, if we take $\tilde{g}_\alpha = g_\alpha \in L^{p'}(\Omega)$ we immediately obtain the growth conditions of type (B) [6]. Finally if we choose $q(\beta) = p$ and $\sigma_\beta = w_\beta$, we obtain the growth condition of type A [6].

(H3) Let $G_1$ be a continuous positive, nonincreasing function on $[0,\infty)$, and let $G_2$ be a continuous positive, nondecreasing function on $[0,\infty)$, we will suppose that for every $\xi = (\kappa, \eta, \zeta) \in \mathbb{R}^d$ and for a.e. $x \in \Omega$ the ellipticity condition holds

$$\begin{aligned}&\sum_{|\alpha|=m} A_\alpha(x,\kappa,\eta,\zeta)\zeta_\alpha \\ &\ge G_1(h(x,\kappa)) \sum_{|\beta|=m} w_\beta |\zeta_\beta|^p - G_2(h(x,\kappa)) \sum_{m_1 \le |\beta| \le m-1} \sigma_\beta |\eta_\beta|^{q(\beta)},\end{aligned}$$

where $\kappa = \{\xi_\beta,\ |\beta| < m_1\} \in \mathbb{R}^{d_1}$, $\eta = \{\xi_\beta,\ m_1 \le |\beta| \le m-1\} \in \mathbb{R}^{d_2}$, $\zeta = \{\xi_\beta,\ |\beta| = m\} \in \mathbb{R}^{N_m}$ and $d_1 + d_2 = N_{m-1}$.

Under these assumptions, the differential operator (1.1) generates a mapping $T$ from $V$ to its dual $V^*$ through the formula

$$\begin{aligned}\langle Tu, v\rangle &= \sum_{\alpha\in J} \int_\Omega B_\alpha(x,\eta(u),\zeta_J(\nabla^m u)) D^\alpha v\,dx \\ &\quad + \sum_{\alpha\in J^c} \int_\Omega B_\alpha(x,\eta(u),\zeta_{J^c}(\nabla^m u)) D^\alpha v\,dx \\ &\quad + \sum_{|\alpha|\le m-1} \int_\Omega L_\alpha(x,\eta(u),\zeta_J(\nabla^m u)) D^\alpha v\,dx \\ &\quad + \sum_{|\alpha|\le m-1} \sum_{\beta\in J^c} \int_\Omega C_{\alpha\beta}(x,\eta(u),\zeta_J(\nabla^m u)) D^\beta u D^\alpha v\,dx,\end{aligned} \tag{2.8}$$

for all $u, v \in V$ and where $\langle .,. \rangle$ denotes the duality pairing between $V^*$ and $V$. The mapping $T$ is well defined and bounded, this can be easily seen by Hölder's inequality and the following lemma.

**Lemma 2.3.** *Let $\Omega$ be a subset of $\mathbb{R}^N$ with finite measure and let $f \in L^p(\Omega,\sigma_1)$, $g \in L^q(\Omega,\sigma_2)$ where $\sigma_1$ and $\sigma_2$ are weight functions in $\Omega$ and let $h \in L^r(\Omega, \sigma_1^{-\frac{r}{p}} \sigma_2^{-\frac{r}{q}})$ with*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le 1.$$

*Then $fgh \in L^1(\Omega)$.*

Indeed. Let $\frac{1}{s}=\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leq 1$. By Hölder's inequality we have,

$$\int_\Omega |fgh|^s\,dx \leq \Big(\int_\Omega f^p\sigma_1\,dx\Big)^{\frac{s}{p}}\Big(\int_\Omega g^q\sigma_2\,dx\Big)^{\frac{s}{q}}\Big(\int_\Omega h^r\sigma_1^{-\frac{r}{p}}\sigma_2^{-\frac{r}{q}}\,dx\Big)^{\frac{s}{r}}<\infty.$$

Then $fgh\in L^s(\Omega)$, which implies that, $fgh\in L^1(\Omega)$.

Let us recall the following definitions.

**Definition 2.4.** A mapping $T$ from $X$ to its dual $X^*$, is called pseudo-monotone, if for every sequence $\{u_n\}\subset X$ with $u_n\rightharpoonup u$ in $X$ and $\limsup_{n\to\infty}\langle Tu_n,u_n-u\rangle\leq 0$, one has

$$\liminf_{n\to\infty}\langle Tu_n,u_n-v\rangle\geq\langle Tu,u-v\rangle\quad\text{for all } v\in X.$$

**Definition 2.5.** Let $X$ be a reflexive Banach space. The mapping $T$ from $X$ to $X^*$ is said to satisfy condition $\alpha(X)$ if the assumptions

$$u_n\rightharpoonup u\quad\text{in } X\quad\text{and}\quad\limsup_{n\to\infty}\langle T\,u_n,u_n-u\rangle\leq 0,$$

imply $u_n\to u$ in $X$.

Obviously, the class $\alpha(X)$ of operators is contained in the class of pseudo-monotone operators.

## 3. Main general result

The aim of this section, is to prove the following result.

**Theorem 3.1.** *Assume that (H1'), (H2'), (H3),* (2.3) *and* (2.4) *hold. Then, the mapping $T$ defined by* (2.8) *is pseudo-monotone in $V$.*

**Remark 3.2.**
(1) When $J=\emptyset$, the previous theorem applies in particular to operators like (1.1) with $A_\alpha$, $|\alpha|\leq m-1$ affine with respect to $\nabla^m u$. This gives from (1.14) a sufficient condition (see Theorem 1.1).
(2) When $J=\emptyset$, $m=1$ and $A_0\equiv 0$, we immediately obtain [4, proposition 1].
(3) When $A_\alpha\equiv 0$ for all $|\alpha|\leq m-1$ and $J=\emptyset$ (resp. $J^c=\emptyset$) we obtain Theorem 8.1 (resp. Theorem 8.3) of [1] with some simple the growth conditions.

**Remark 3.3.** Since the hypothesis (H3) concerns only the terms $L_\alpha$ with $|\alpha|<m_1$ (see Remark 4.5 below), then the statement of Theorem 3.1 remains true without assuming (H3), when $m_1\leq 0$.

**Remark 3.4.** If we take $m_1\leq 0$, $q(\beta)=p$ and $\sigma_\beta=w_\beta$, then $X_\beta=L^p(\Omega,w_\beta)$ for all $|\beta|\leq m-1$, hence the growth condition (H2') is of the type $A$ (see [6]) and the statement of Theorem 3.1 remains true without assuming (H3).

Applying the previous theorem, we obtain the following existence results, which generalize the corresponding (cf. [1, 4]) and extend the corresponding in [5, 6].

**corollary 3.5.** *Assume the hyptheses in Theorem 3.1 and the condtion on the degeneracy* (1.15)*. Then the DBVP from the equation* (1.8) *has at least one solution $u\in V$.*

**Remark 3.6.** If the expression,

$$|||u|||_V=\Big(\sum_{|\alpha|=m}\int_\Omega w_\alpha(x)|D^\alpha u|^p\,dx\Big)^{1/p}$$

is a norm in $V$ equivalent to the usual norm (2.1) (see section 5 where this fact is verified for $V = W_0^{m,p}(\Omega, w)$), then we can replace in Corollary 3.5, the degeneracy (1.15) by the weaker condition

$$\sum_{|\alpha|\le m} A_\alpha(x,\xi)\xi_\alpha \ge c \sum_{|\alpha|=m} w_\alpha |\xi_\alpha|^p. \tag{3.1}$$

## 4. Proof of Theorem 3.1

For this goal, we need the following lemmas.

**Lemma 4.1.** *Let $(g_n)_n$ be a sequence of $L^p(\Omega, \tilde\sigma)$ and let $g \in L^p(\Omega,\tilde\sigma)$ $(1 < p < \infty)$, where $\tilde\sigma$ is a weight function in $\Omega$. If $g_n \to g$ in measure (in particular a.e in $\Omega$) and it is bounded in $L^p(\Omega,\tilde\sigma)$, then $g_n \to g$ in $L^q(\Omega, \tilde\sigma^{\frac{q}{p}})$ for all $q < p$.*

*Proof.* Let $\varepsilon > 0$ and set $A_n = \{x \in \Omega / |g_n(x) - g(x)|\tilde\sigma^{1/p}(x) \le (\frac{\varepsilon}{2\,\mathrm{meas}(\Omega)})^{1/q}\}$. We have

$$\begin{aligned}\int_\Omega |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx &= \int_{A_n} |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx + \int_{A_n^c} |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx\\ &\le \frac{\varepsilon}{2} + \int_{A_n^c} |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx.\end{aligned}$$

By Hölder inequality, one can see that

$$\begin{aligned}\int_{A_n^c} |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx &\le \Big(\int_\Omega |g_n - g|^p \tilde\sigma\,dx\Big)^{\frac{q}{p}} \Big(\mathrm{meas}(A_n^c)\Big)^{1-\frac{q}{p}}\\ &\le M\Big(\mathrm{meas}(A_n^c)\Big)^{1-\frac{q}{p}},\end{aligned}$$

where $M$ is a constant does not depend on $n$. On the other hand, since $g_n \to g$ in measure, $\mathrm{meas}(A_n^c) \to 0$ as $n \to \infty$. Then there exists some $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\int_{A_n^c} |g_n - g|^q \tilde\sigma^{\frac{q}{p}}\,dx \le \frac{\varepsilon}{2}.$$

□

The following lemma is a generalization of [9, Lemma 3.2] in weighted spaces.

**Lemma 4.2.** *Let $g \in L^q(\Omega, \tilde\sigma)$ and let $g_n \in L^q(\Omega,\tilde\sigma)$, with $\|g_n\|_{q,\tilde\sigma} \le c$ $(1 < q < \infty)$. If $g_n(x) \to g(x)$ a.e. in $\Omega$, then $g_n \rightharpoonup g$ in $L^q(\Omega,\tilde\sigma)$, where $\rightharpoonup$ denotes weak convergence.*

*Proof.* Since $g_n\tilde\sigma^{\frac{1}{q}}$ is bounded in $L^q(\Omega)$ and $g_n(x)\tilde\sigma^{\frac{1}{q}}(x) \to g(x)\tilde\sigma^{\frac{1}{q}}(x)$, a.e. in $\Omega$, then by [9, lemma 3.2],

$$g_n\tilde\sigma^{\frac{1}{q}} \rightharpoonup g\tilde\sigma^{\frac{1}{q}} \quad \text{in } L^q(\Omega).$$

Moreover, for all $\varphi \in L^{q'}(\Omega, \tilde\sigma^{1-q'})$, we have $\varphi\tilde\sigma^{-\frac{1}{q}} \in L^{q'}(\Omega)$. Then

$$\int_\Omega g_n\varphi\,dx \to \int_\Omega g\varphi\,dx; \quad \text{i.e. } g_n \rightharpoonup g \quad \text{in } L^q(\Omega,\tilde\sigma).$$

□

**Proof of Theorem 3.1.** Let $(u_n)_n$ be a sequence in $V$ such that: $u_n \rightharpoonup u$ in $V$ and

$$\limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0, \tag{4.1}$$

i.e.,

$$\begin{aligned}
&\limsup_{n\to\infty} \Big\{ \int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \\
&+ \int_\Omega \sum_{\alpha\in J^c} B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \\
&+ \int_\Omega \sum_{|\alpha|\le m-1} L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \\
&+ \int_\Omega \sum_{|\alpha|\le m-1} \sum_{\beta\in J^c} C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)\,dx \Big\} \le 0.
\end{aligned}$$

**(a)** We shall prove that

$$\langle Tu_n, v\rangle \to \langle Tu, v\rangle \quad asn\to\infty\ \forall v\in V. \tag{4.2}$$

By (H1')(iii), the compact imbedding implies that for a subsequence

$$\begin{gathered}
D^\alpha u_n \to D^\alpha u \quad \text{in } X_\alpha \\
D^\alpha u_n \to D^\alpha u \quad \text{a.e. in } \Omega \forall\ |\alpha|\le m-1.
\end{gathered} \tag{4.3}$$

**Step (1)** We shall prove that

$$\lim_{n\to\infty} \sum_{|\alpha|\le m-1} \int_\Omega L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u) = 0. \tag{4.4}$$

(i) We show that

$$\lim_{n\to\infty} \sum_{m_1\le|\alpha|\le m-1} \int_\Omega L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx = 0. \tag{4.5}$$

Let $m_1 \le |\alpha| \le m-1$ be fixed. Thanks to (H2'), we have

$$\begin{aligned}
&\int_\Omega |L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)|\,dx \\
&\le \int_\Omega G(k(x,u_n(x)))\sigma_\alpha^{\frac{1}{q(\alpha)}} |(D^\alpha u_n - D^\alpha u)||\tilde g_\alpha|\,dx \\
&\quad + \tilde c_\alpha \sum_{\beta\in J} \int_\Omega G(k(x,u_n(x)))\sigma_\alpha^{\frac{1}{q(\alpha)}} |(D^\alpha u_n - D^\alpha u)| w_\beta^{\frac{1}{q'(\alpha)}} |D^\beta u_n(x)|^{\frac{p}{q'(\alpha)}}\,dx \\
&\quad + \tilde c_\alpha \sum_{m_1\le|\beta|\le m-1} \int_\Omega G(k(x,u_n(x)))\sigma_\alpha^{\frac{1}{q(\alpha)}} |(D^\alpha u_n - D^\alpha u)| \sigma_\beta^{\frac{1}{q'(\alpha)}} |D^\beta u_n(x)|^{\frac{q(\beta)}{q'(\alpha)}}\,dx.
\end{aligned}$$

By (2.6) we have,

$$G(k(x,u_n(x)) \le G(c\|u_n\|_{m,p,w}).$$

Applying the Hölder's inequality with exponents $q(\alpha)$ and $q'(\alpha)$ we obtain

$$\begin{aligned}&\int_\Omega |L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)|\,dx\\&\le G(c\|u_n\|_{m,p,w})\|D^\alpha u_n - D^\alpha u\|_{q(\alpha),\sigma_\alpha}\Big(\|\tilde g_\alpha\|_{q'(\alpha)}\\&\quad + \tilde c_\alpha\sum_{\beta\in J}\|D^\beta u_n\|_{p,w_\beta}^{\frac{p}{q'(\alpha)}} + \tilde c_\alpha\sum_{m_1\le|\beta|\le m-1}\|D^\beta u_n\|_{q(\beta),\sigma_\beta}^{\frac{q(\beta)}{q'(\alpha)}}\Big).\end{aligned}$$

Thanks to (2.7), we have $\|D^\beta u_n\|_{q(\beta),\sigma_\beta} \le \tilde c_\beta\|u_n\|_{m,p,w}$ for all $m_1 \le |\beta| \le m-1$. Since $\|D^\beta u_n\|_{p,w_\beta} \le \|u_n\|_{m,p,w}$ for all $\beta\in J$, we conclude that

$$\begin{aligned}&\int_\Omega |L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)|\,dx\\&\le \|D^\alpha u_n - D^\alpha u\|_{q(\alpha),\sigma_\alpha} R_\alpha(\|u_n\|_{m,p,w})\end{aligned}$$

with,

$$R_\alpha(t) = G(c_1 t)\Big(\|\tilde g_\alpha\|_{q'(\alpha)} + c_2 t^{\frac{p}{q'(\alpha)}} + c_3\sum_{m_1\le|\beta|\le m-1} t^{\frac{q(\beta)}{q'(\alpha)}}\Big)$$

which is a positive continuous function, hence $R_\alpha(\|u_n\|_{m,p,w})$ is bounded. Moreover, by (4.3) we have,

$$\|D^\alpha u_n - D^\alpha u\|_{q(\alpha),\sigma_\alpha} \to 0 \quad \text{as } n\to\infty.$$

then

$$\int_\Omega L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \to 0,$$

which yields (4.5).
(ii) We show that

$$\lim_{n\to\infty}\sum_{|\alpha|<m_1}\int_\Omega L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx = 0. \tag{4.6}$$

Let $|\alpha| < m_1$ be fixed. Similarly by virtue of (H2'),

$$\begin{aligned}&\int_\Omega |L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)|\,dx\\&\le G(c\|u_n\|_{m,p,w})\sup_{x\in\Omega}(|(D^\alpha u_n - D^\alpha u)\sigma_\alpha|)\Big(\|\hat g_\alpha\|_1 + \tilde c_\alpha\sum_{\beta\in J}\|D^\beta u_n\|_{p,w_\beta}^p\\&\quad + \tilde c_\alpha\sum_{m_1\le|\beta|\le m-1}\|D^\beta u_n\|_{q(\beta),\sigma_\beta}^{q(\beta)}\Big).\end{aligned}$$

It follows from (2.7) and $\|D^\beta u_n\|_{p,w_\beta} \le \|u_n\|_{m,p,w}$ for all $\beta\in J$ that

$$\begin{aligned}&\int_\Omega |L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)|\,dx\\&\le \|D^\alpha u_n - D^\alpha u\|_{C(\Omega,\sigma_\alpha)}\tilde R_\alpha(\|u_n\|_{m,p,w}),\end{aligned}$$

where

$$\tilde R_\alpha(t) = G(c_1 t)\Big(\|\hat g_\alpha\|_1 + c_2 t^p + c_3\sum_{m_1\le|\beta|\le m-1} t^{q(\beta)}\Big).$$

This function is also positive and continuous, hence $\tilde{R}_\alpha(\|u_n\|_{m,p,w})$ is bounded. Since, by (4.3) $\|D^\alpha u_n - D^\alpha u\|_{C(\Omega,\sigma_\alpha)} \to 0$ as $n \to \infty$, it follows that

$$\int_\Omega L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \to 0,$$

which yields (4.6). Thus, due to (4.5) and (4.6) we conclude (4.4).
**Step (2)** We shall prove that

$$\lim_{n\to\infty} \sum_{|\alpha|\le m-1} \sum_{\beta\in J^c} \int_\Omega C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)\,dx = 0. \quad (4.7)$$

(i) Let $m_1 \le |\alpha| \le m-1$ and $\beta \in J^c$ be fixed. And let $s_\alpha$ such that,

$$\frac{1}{s_\alpha} = \frac{1}{q(\alpha)} + \frac{1}{p} + \frac{1}{r_\alpha} < 1.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)|^{s_\alpha}\,dx \\
&\le \Big(\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))|^{r_\alpha}\sigma_\alpha^{-\frac{r_\alpha}{q(\alpha)}} w_\beta^{-\frac{r_\alpha}{p}}\,dx\Big)^{\frac{s_\alpha}{r_\alpha}} \\
&\quad\times \Big(\int_\Omega |D^\beta u_n|^p w_\beta\,dx\Big)^{\frac{s_\alpha}{p}} \Big(\int_\Omega |D^\alpha u_n - D^\alpha u|^{q(\alpha)}\sigma_\alpha\,dx\Big)^{\frac{s_\alpha}{q(\alpha)}}.
\end{aligned}$$

By (H2') the sequences $\{C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n)),\ m_1 \le |\alpha| \le m-1,\ \beta\in J^c\}$ (resp $\{D^\beta u_n,\ \beta\in J^c\}$) remain bounded in $L^{r_\alpha}(\Omega,\sigma_\alpha^{-\frac{r_\alpha}{q(\alpha)}} w_\beta^{-\frac{r_\alpha}{p}})$ (resp $L^p(\Omega,w_\beta)$). Moreover, $\|D^\alpha u_n - D^\alpha u\|^{s_\alpha}_{q(\alpha),\sigma_\alpha} \to 0$ as $n\to\infty$. Then

$$\lim_{n\to\infty}\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)|^{s_\alpha}\,dx = 0.$$

Consequently,

$$\lim_{n\to\infty}\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)|\,dx = 0,$$

i.e,

$$\lim_{n\to\infty} \sum_{m_1\le|\alpha|\le m-1} \sum_{\beta\in J^c} \int_\Omega C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u) = 0. \quad (4.8)$$

(ii) Let $|\alpha| < m_1$ and $\beta\in J^c$ be fixed. By (H2') the sequences

$$\{\sigma_\alpha^{-1}C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n, |\alpha| < m_1, \beta\in J^c\}$$

remain bounded in $L^{s_\alpha}(\Omega)$ with $\frac{1}{s_\alpha} = \frac{1}{p} + \frac{1}{r_\alpha} < 1$. Indeed,

$$\begin{aligned}
&\int_\Omega |\sigma_\alpha^{-1}C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n|^{s_\alpha}\,dx \\
&\le \Big(\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))|^{r_\alpha}\sigma_\alpha^{-r_\alpha} w_\beta^{-\frac{r_\alpha}{p}}\,dx\Big)^{\frac{s_\alpha}{r_\alpha}} \Big(\int_\Omega |D^\beta u_n|^p w_\beta\,dx\Big)^{\frac{s_\alpha}{p}}.
\end{aligned}$$

The right hand side is bounded because $\{C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))\}$ is bounded in $L^r(\Omega,\sigma_\alpha^{-r_\alpha} w_\beta^{-\frac{r_\alpha}{p}})$ and $\{D^\beta u_n\}$ is bounded in $L^p(\Omega,w_\beta)$.

Thanks to $s_\alpha \geq 1$, the sequences $\{\sigma_\alpha^{-1} C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n, |\alpha| < m_1, \beta \in J^c\}$ remain bounded in $L^1(\Omega)$. Since

$$\begin{aligned}
&\int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)|\,dx \\
&\leq \sup_{x\in\Omega}(|(D^\alpha u_n - D^\alpha u)\sigma_\alpha|) \int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n \sigma_\alpha^{-1}|\,dx
\end{aligned}$$

it follows that

$$\lim_{n\to\infty} \int_\Omega |C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)|\,dx = 0$$

(because $\sup_{x\in\Omega}(|(D^\alpha u_n - D^\alpha u)\sigma_\alpha|) \to 0$). Which gives

$$\lim_{n\to\infty} \sum_{|\alpha|<m_1} \sum_{\beta\in J^c} \int_\Omega C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n(D^\alpha u_n - D^\alpha u)\,dx = 0. \tag{4.9}$$

Combining (4.8) and (4.9) we obtain (4.7).
**Step (3)** We shall prove that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{\alpha\in J} \int_\Omega B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n)) \\
&- B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u)))(D^\alpha u_n - D^\alpha u)\,dx = 0
\end{aligned} \tag{4.10}$$

and that

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \sum_{\alpha\in J^c} (B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n)) \\
&- B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u)))(D^\alpha u_n - D^\alpha u)\,dx = 0.
\end{aligned} \tag{4.11}$$

Combining (4.1), (4.4) and (4.7) one obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{\alpha\in J} \int_\Omega B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \\
&+ \sum_{\alpha\in J^c} \int_\Omega B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))(D^\alpha u_n - D^\alpha u)\,dx \leq 0.
\end{aligned} \tag{4.12}$$

Thanks to (4.3) and (H2') one deduce that

$$\begin{gathered}
B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u)) \to B_\alpha(x,\eta(u),\zeta_J(\nabla^m u)) \quad \text{in } L^{p'}(\Omega,w_\alpha^*),\ \alpha\in J \\
B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u)) \to B_\alpha(x,\eta(u),\zeta_{J^c}(\nabla^m u)) \quad \text{in } L^{p'}(\Omega,w_\alpha^*),\ \alpha\in J^c.
\end{gathered}$$

Since $D^\alpha u_n \rightharpoonup D^\alpha u$ in $L^p(\Omega,w_\alpha)$ for all $|\alpha| = m$, one can write

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u))(D^\alpha u_n - D^\alpha u)\,dx = 0\,, \\
&\lim_{n\to\infty} \int_\Omega \sum_{\alpha\in J^c} B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u))(D^\alpha u_n - D^\alpha u)\,dx = 0\,.
\end{aligned} \tag{4.13}$$

Combining (4.12), (4.13), (2.3) and (2.4) we conclude the assertions (4.10) and (4.11).

**Step (4)** To prove the relation (4.2), it suffices to show the following assertions:
**(i)** For every $v \in V$,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\alpha v\,dx \\ &= \int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u),\zeta_J(\nabla^m u))D^\alpha v\,dx. \end{aligned} \tag{4.14}$$

**(ii)** For every $v \in V$,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega \sum_{m_1\le|\alpha|\le m-1} L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\alpha v\,dx \\ &= \int_\Omega \sum_{m_1\le|\alpha|\le m-1} L_\alpha(x,\eta(u),\zeta_J(\nabla^m u))D^\alpha v\,dx. \end{aligned} \tag{4.15}$$

**(iii)** For every $v \in V$,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega \sum_{|\alpha|<m_1} L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\alpha v\,dx \\ &= \int_\Omega \sum_{|\alpha|<m_1} L_\alpha(x,\eta(u),\zeta_J(\nabla^m u))D^\alpha v\,dx. \end{aligned} \tag{4.16}$$

**(iv)** For every $v \in V$,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega \sum_{|\alpha|\le m-1}\sum_{\beta\in J^c} C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\beta u_n D^\alpha v \\ &= \int_\Omega \sum_{|\alpha|\le m-1}\sum_{\beta\in J^c} C_{\alpha\beta}(x,\eta(u),\zeta_J(\nabla^m u))D^\beta u D^\alpha v. \end{aligned} \tag{4.17}$$

**(v)** For every $v \in V$,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega \sum_{\alpha\in J^c} (B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))D^\alpha v\,dx \\ &= \int_\Omega \sum_{\alpha\in J^c} (B_\alpha(x,\eta(u),\zeta_{J^c}(\nabla^m u))D^\alpha v\,dx. \end{aligned} \tag{4.18}$$

*Proof of assertions (i)and (ii).* Invoking Landes [8, lemma 6], we obtain from (4.10) and the strict monotonicity (2.3) that

$$D^\alpha u_n \to D^\alpha u \quad \text{a.e in } \Omega \text{ for each } \alpha \in J, \tag{4.19}$$

which gives

$$\begin{gathered} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n)) \to B_\alpha(x,\eta(u),\zeta_J(\nabla^m u)) \quad \text{a.e. in } \Omega\ \forall\alpha\in J\,, \\ L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n)) \to L_\alpha(x,\eta(u),\zeta_J(\nabla^m u)) \\ \text{a.e in } \Omega\ \forall m_1 \le |\alpha| \le m-1. \end{gathered}$$

¿From the growth condition (H2'), the sequence $\{B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n)), \alpha \in J\}$ (resp $\{L_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))\quad m_1 \le |\alpha| \le m-1\}$) are bounded in $L^{p'}(\Omega, w_\alpha^*)$ (resp. $L^{q'(\alpha)}(\Omega,\sigma_\alpha^*)$), hence by Lemma 4.2 we have

$$B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n)) \rightharpoonup B_\alpha(x,\eta(u),\zeta_J(\nabla^m u))$$

in $L^{p'}(\Omega, w_\alpha^*)$ for all $\alpha \in J$ and

$$L_\alpha(x, \eta(u_n), \zeta_J(\nabla^m u_n)) \rightharpoonup L_\alpha(x, \eta(u), \zeta_J(\nabla^m u))$$

in $L^{q'(\alpha)}(\Omega, \sigma_\alpha^*)$ for all $m_1 \le |\alpha| \le m-1$, which implies (i) and (ii). □

*Proof of assertion (iii).* In virtue of the growth condition (H2') we have for all $v \in V$ and all $|\alpha| < m_1$

$$\begin{aligned}|L_\alpha(x, \eta(u_n), \zeta_J(\nabla^m u_n)) D^\alpha v| &\le |D^\alpha v| G(k(x, \eta(u_n)))\sigma_\alpha \Big(\hat{g}_\alpha(x) + \tilde{c}_\alpha \sum_{\beta \in J} w_\beta |D^\beta u_n|^p \\ &\quad + \tilde{c}_\alpha \sum_{m_1 \le |\beta| \le m-1} \sigma_\beta |D^\beta u_n|^{q(\beta)}\Big).\end{aligned}$$

Since $G(k(x, \eta(u_n))) \le c_1$ and $\sup_{x \in \Omega}(|D^\alpha v \sigma_\alpha|) \le c_2$ for all $|\alpha| < m_1$, where $c_i (i = 1, 2)$ are some positive constants, it follows that

$$\begin{aligned}&|L_\alpha(x, \eta(u_n), \zeta_J(\nabla^m u_n)) D^\alpha v| \\ &\le c\Big(\hat{g}_\alpha(x) + c_\alpha \sum_{\beta \in J} w_\beta |D^\beta u_n|^p + c_\alpha \sum_{m_1 \le |\beta| \le m-1} \sigma_\beta |D^\beta u_n|^{q(\beta)}\Big) = g_n.\end{aligned}$$

It follows from (4.3) and (4.19) that

$$L_\alpha(x, \eta(u_n), \zeta_J(\nabla^m u_n)) \to L_\alpha(x, \eta(u), \zeta_J(\nabla^m u)) \quad \text{a.e. in } \Omega \ \forall |\alpha| < m_1$$

and

$$g_n \to g = c\Big(\hat{g}_\alpha(x) + c_\alpha \sum_{\beta \in J} w_\beta |D^\beta u|^p + c_\alpha \sum_{m_1 \le |\beta| \le m-1} \sigma_\beta |D^\beta u|^{q(\beta)}\Big) a.e \text{a.e. in } \Omega.$$

**Lemma 4.3.** *$D^\beta u_n \to D^\beta u$ as $n \to \infty$ in $L^p(\Omega, w_\beta)$ for all $\beta \in J$.*

By (4.3) and Lemma 4.3 we obtain

$$\int_\Omega g_n \, dx \to \int_\Omega g \, dx.$$

By the generalized Lebesgue theorem we have,

$$\int_\Omega L_\alpha(x, \eta(u_n), \zeta_J(\nabla^m u_n)) D^\alpha v \, dx \to \int_\Omega L_\alpha(x, \eta(u), \zeta_J(\nabla^m u)) D^\alpha v \, dx$$

for all $|\alpha| < m_1$ which implies (4.16). □

*Proof of assertion (iv).* By (4.3) and (4.19) we have for each $|\alpha| \le m-1$ and each $\beta \in J^c$,

$$C_{\alpha\beta}(x, \eta(u_n), \zeta_J(\nabla^m u_n)) \to C_{\alpha\beta}(x, \eta(u), \zeta_J(\nabla^m u)) \quad \text{a.e. in } \Omega.$$

So, from (H2') the sequences $\{C_{\alpha\beta}(x, \eta(u_n), \zeta_J(\nabla^m u_n)), m_1 \le |\alpha| \le m-1 \text{ and } \beta \in J^c\}$ (resp. $\{C_{\alpha\beta}(x, \eta(u_n), \zeta_J(\nabla^m u_n)), |\alpha| < m_1 \text{ and } \beta \in J^c\}$) remain bounded in $L^{r_\alpha}(\Omega, \sigma_\alpha^{-\frac{r_\alpha}{q(\alpha)}} w_\beta^{-\frac{r_\alpha}{p}})$ (resp. $L^{r_\alpha}(\Omega, \sigma_\alpha^{-r_\alpha} w_\beta^{-\frac{r_\alpha}{p}})$). Then Lemma 4.1 yields

$$C_{\alpha\beta}(x, \eta(u_n), \zeta_J(\nabla^m u_n)) \to C_{\alpha\beta}(x, \eta(u), \zeta_J(\nabla^m u))$$

in $L^q(\Omega, \sigma_\alpha^{-\frac{q}{q(\alpha)}} w_\beta^{-\frac{q}{p}})$ for all $q < r_\alpha$, all $m_1 \le |\alpha| \le m-1$ and all $\beta \in J^c$. Lemma 4.1 also yields

$$C_{\alpha\beta}(x, \eta(u_n), \zeta_J(\nabla^m u_n)) \to C_{\alpha\beta}(x, \eta(u), \zeta_J(\nabla^m u))$$

in $L^q(\Omega,\sigma_\alpha^{-q}w_\beta^{-\frac{q}{p}})$ for all $q<r_\alpha$, all $|\alpha|<m_1$ and all $\beta\in J^c$.

Let $s_\alpha$ such that $\frac{1}{s_\alpha}=\frac{1}{p}+\frac{1}{q(\alpha)}$. Remark that $r_\alpha>s'_\alpha=\frac{s_\alpha}{s_\alpha-1}$ for $m_1\le|\alpha|\le m-1$ and since $p'<r_\alpha$ for $|\alpha|<m_1$ one has

$$C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))\to C_{\alpha\beta}(x,\eta(u),\zeta_J(\nabla^m u))$$

in $L^{s'_\alpha}(\Omega,\sigma_\alpha^{-\frac{s'_\alpha}{q(\alpha)}}w_\beta^{-\frac{s'_\alpha}{p}})$ for all $m_1\le|\alpha|\le m-1$. Also one has

$$C_{\alpha\beta}(x,\eta(u_n),\zeta_J(\nabla^m u_n))\sigma_\alpha^{-1}\to C_{\alpha\beta}(x,\eta(u),\zeta_J(\nabla^m u))\sigma_\alpha^{-1} \tag{4.20}$$

in $L^{p'}(\Omega,w_\beta^{-\frac{p'}{p}})$ for all $|\alpha|<m_1$.

**Lemma 4.4.** *For all $v\in V$, one has*

1. *$D^\beta u_nD^\alpha v\rightharpoonup D^\beta uD^\alpha v$ in $L^{s_\alpha}(\Omega,\sigma_\alpha^{\frac{s_\alpha}{q(\alpha)}}w_\beta^{\frac{s_\alpha}{p}})$ for each $m_1\le|\alpha|\le m-1$ and each $\beta\in J^c$.*
2. *$D^\beta u_nD^\alpha v\sigma_\alpha\rightharpoonup D^\beta uD^\alpha v\sigma_\alpha$ in $L^p(\Omega,w_\beta)$ for each $|\alpha|<m_1$ and each $\beta\in J^c$.*

In view of (4.20) and Lemma 4.4 we conclude (4.17). □

*Proof of assertion (v).* First we show that

$$\int_\Omega\sum_{\alpha\in J^c}(B_\alpha(x,\eta(u),\zeta_{J^c}(v))-h_\alpha)(v_\alpha-D^\alpha u)\,dx\ge 0 \tag{4.21}$$

for all $v=(v_\alpha)\in\prod_{|\alpha|=m}L^p(\Omega,w_\alpha)$, where $h_\alpha$ stands for the weak limit of $\{B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n)),\alpha\in J^c\}$ in $L^{p'}(\Omega,w_\alpha^*)$. Indeed by (4.11) we have,

$$\limsup_{n\to\infty}\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))(D^\alpha u_n-D^\alpha u)\,dx\le 0,$$

implies

$$\limsup_{n\to\infty}\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))D^\alpha u_n\,dx\le\int_\Omega\sum_{\alpha\in J^c}h_\alpha D^\alpha u\,dx \tag{4.22}$$

and from weak Leray-Lions condition (2.4), for any $v=(v_\alpha)\in\prod_{|\alpha|=m}L^p(\Omega,w_\alpha)$, we obtain

$$\begin{aligned}&\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))D^\alpha u_n\,dx\\ &\ge\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))v_\alpha\,dx\\ &\quad+\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u_n),\zeta_{J^c}(v))(D^\alpha u_n-v_\alpha)\,dx.\end{aligned}$$

Letting $n\to\infty$ we conclude by (4.22) that

$$\int_\Omega\sum_{\alpha\in J^c}h_\alpha D^\alpha u\,dx\ge\int_\Omega\sum_{\alpha\in J^c}h_\alpha v_\alpha\,dx+\int_\Omega\sum_{\alpha\in J^c}B_\alpha(x,\eta(u),\zeta_{J^c}(v))(D^\alpha u-v_\alpha)\,dx$$

and hence (4.21) follows. Choosing $v=D^\alpha u+t\hat{w}$ with $t>0$, $\hat{w}=(\hat{w}_\alpha)\in\prod_{|\alpha|=m}L^p(\Omega,w_\alpha)$ and letting $t\to 0$ we obtain $h_\alpha=B_\alpha(x,\eta(u),\zeta_{J^c}(\nabla^m u))$ a.e. in $\Omega$ which implies (4.18). □

**(b)** We shall prove that
$$\liminf_{n\to\infty}\langle Tu_n,u_n\rangle \geq \langle Tu,u\rangle. \tag{4.23}$$
In view of monotonicity condition (2.3) and (2.4) we have
$$\begin{aligned}
&\int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\alpha u_n + \int_\Omega \sum_{\alpha\in J^c} B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))D^\alpha u_n\\
&\geq \int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u_n))D^\alpha u\\
&\quad + \int_\Omega \sum_{\alpha\in J} B_\alpha(x,\eta(u_n),\zeta_J(\nabla^m u))(D^\alpha u_n - D^\alpha u)\\
&\quad + \int_\Omega \sum_{\alpha\in J^c} B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u_n))D^\alpha u\\
&\quad + \int_\Omega \sum_{\alpha\in J^c} B_\alpha(x,\eta(u_n),\zeta_{J^c}(\nabla^m u))(D^\alpha u_n - D^\alpha u).
\end{aligned}$$
Letting $n\to\infty$ and using (4.4) and (4.7)we obtain (4.23).

*Proof of Lemma 4.3.* Let $E$ be a measurable subset of $\Omega$, in view of steps 1, 2, 3 and 4 in [6, Lemma 2.7], we obtain
$$\lim_{\text{meas}\, E\to 0}\int_E \sum_{\beta\in J} w_\beta |D^\beta u_n(x)|^p\,dx = 0 \tag{4.24}$$
uniformly with respect to $n\in\mathbb{N}$, i.e the sequence $(w_\beta|D^\beta u_n|^p)$ is equi-integrable. And due to (4.19) we have
$$w_\beta|D^\beta u_n|^p \to w_\beta|D^\beta u|^p \quad \text{a.e. } \forall\beta\in J.$$
Since meas$(\Omega)<\infty$, by Vitali's theorem we obtain $D^\beta u_n \to D^\beta u$ in $L^p(\Omega,w_\beta)$ for all $\beta\in J$. □

*Proof of Lemma 4.4.* (1) Let $m_1\leq|\alpha|\leq m-1$ and $\beta\in J^c$ fixed. Let $\varphi\in L^{s'_\alpha}(\Omega,\sigma_\alpha^{-\frac{s'_\alpha}{q(\alpha)}} w_\beta^{-\frac{s'_\alpha}{p}})$.
Since, $\frac{1}{p'}=\frac{1}{s'_\alpha}+\frac{1}{q(\alpha)}$, by Hölder's inequality we obtain
$$\int_\Omega |D^\alpha v\varphi|^{p'} w_\beta^{1-p'} \leq \Big(\int_\Omega |D^\alpha v|^{q(\alpha)}\sigma_\alpha\Big)^{\frac{p'}{q(\alpha)}}\Big(\int_\Omega |\varphi|^{s'_\alpha} w_\beta^{\frac{s'_\alpha(1-p')}{p'}}\sigma_\alpha^{\frac{-s'_\alpha}{q(\alpha)}}\Big)^{\frac{p'}{s'_\alpha}} < \infty$$
(because $\frac{s'_\alpha(1-p')}{p'}=\frac{-s'_\alpha}{p}$). Then $D^\alpha v\varphi\in L^{p'}(\Omega,w_\beta^{1-p'})$ and since $D^\beta u_n \rightharpoonup D^\beta u$ in $L^p(\Omega,w_\alpha)$, we have
$$\int_\Omega D^\beta u_n D^\alpha v\varphi \to \int_\Omega D^\beta u_n D^\alpha v\varphi \quad \text{for all } \varphi\in L^{s'_\alpha}(\Omega,\sigma_\alpha^{-\frac{s'_\alpha}{q(\alpha)}} w_\alpha^{-\frac{s'_\alpha}{p}});$$
i.e.,
$$D^\beta u_n D^\alpha v \rightharpoonup D^\beta u D^\alpha v \in L^{s_\alpha}(\Omega,\sigma_\alpha^{\frac{s_\alpha}{q(\alpha)}} w_\beta^{\frac{s_\alpha}{p}})$$
for all $m_1\leq|\alpha|\leq m-1$ and all $\beta\in J^c$.

(2) Let $|\alpha| < m_1$ and $\beta \in J^c$ and let $\varphi \in L^{p'}(\Omega, w_\beta^*)$. Thanks to $D^\alpha v \in C(\Omega, \sigma_\alpha)$ $\forall v \in V$, we have $D^\alpha v \sigma_\alpha \varphi \in L^{p'}(\Omega, w_\beta^*)$. Since $D^\beta u_n \rightharpoonup D^\beta u$ in $L^p(\Omega, w_\alpha)$, we have

$$\int_\Omega D^\beta u_n D^\alpha v \sigma_\alpha \varphi \, dx \to \int_\Omega D^\beta u D^\alpha v \sigma_\alpha \varphi \, dx \quad \text{for all } \varphi \in L^{p'}(\Omega, w_\beta^*).$$

□

**Remark 4.5.** Note that, the ellepticity condition (H3) is only used to prove (4.24) (see step 3 in [6, lemma 2.7], which concerns only the equality (4.16) corresponding to a terms $L_\alpha$ with $|\alpha| < m_1$).

## 5. Specific case

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ satisfying the cone condition. In the sequel we assume in addition that the collection of weight functions $w = \{w_\alpha(x), |\alpha| \le m\}$ satisfies $w_\alpha(x) = 1$ for all $|\alpha| \le m-1$, and the integrability condition: There exists $\nu \in ]\frac{N}{P}, \infty[\cap[\frac{1}{P-1}, \infty[$ such that

$$w_\alpha^{-\nu} \in L^1(\Omega) \quad \forall |\alpha| = m. \tag{5.1}$$

Note that (5.1) is stronger than (2.1). Assumptions (2.1) and (5.1) imply

$$|||u|||_V = \Big( \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p w_\alpha(x) \, dx \Big)^{1/p}$$

is a norm defined on $V = W_0^{m,p}(\Omega, w)$ and it's equivalent to (2.1). Let

$$m_1 = \frac{mp\nu - N(\nu+1)}{p\nu} = m - \frac{N}{p_1} \quad \text{with} \quad p_1 = \frac{p\nu}{\nu+1}. \tag{5.2}$$

**Remark 5.1** ([5])**.** Under the above assumption the following continuous imbeddings hold: (i) For $k < m_1$,

$$W^{m,p}(\Omega, w) \hookrightarrow C^k(\bar{\Omega}).$$

(ii) For $k = m_1$, with arbitrary $r, 1 < r < \infty$,

$$W^{m,p}(\Omega, w) \hookrightarrow W^{k,r}(\Omega).$$

(iii) For $k > m_1$,

$$W^{m,p}(\Omega, w) \hookrightarrow W^{k,r_k}(\Omega)$$

where $r_k$ satisfies $1 < r_k \le q_k = \frac{p\nu N}{N(\nu+1) - p\nu(m-k)}$.

Moreover the imbedding (i) and (ii) are compact and (iii) is compact if $r_k < q_k$.

Now, we define

$$H^{m-1,q}(\Omega, \sigma) = \prod_{|\beta| \le m-1} X_\beta,$$

where $X_\beta = L^{q(\beta)}(\Omega, \sigma_\beta), q(\beta) > 1$ for $m_1 \le |\beta| \le m-1$ and $X_\beta = C^{|\beta|}(\Omega, \sigma_\beta)$ for $|\beta| < m_1$.

Also we define the assumption

(H4) Let

$$1 < q(\beta) < q_{|\beta|} = \frac{p\nu N}{N(\nu+1) - p\nu(m-|\beta|)}$$

for $m_1 < |\beta| \leq m-1$ and $q(\beta)$ arbitrary if $|\beta| = m_1$ and $\sigma_\beta \equiv 1$ for all $\beta \leq m-1$.

**Remark 5.2.** If (H4) is satisfied, then by Remark 5.1,

$$W^{m,p}(\Omega, w) \hookrightarrow\hookrightarrow H^{m-1,q}(\Omega)$$

which implies immediately that (H2')(iii) with $\sigma \equiv 1$.

**Theorem 5.3.** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^d$. And assume that* (2.1), (5.1), *(H1'), (H2')(,i,ii), (H3), (H4),* (2.3) *and* (2.4) *are satisfied. Then the operator $T$ defined in* (2.8) *is pseudo-monotone in $V = W_0^{m,p}(\Omega, w)$.*

*If in addition the degeneracy* (3.1) *is satisfied, then the degenerate boundary-value problem from* (1.8) *has at least one solution $u \in V$.*

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