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LERAY LIONS DEGENERATED PROBLEM WITH GENERAL GROWTH CONDITION

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Abstract. In this paper, we study the existence of solutions for the nonlinear degenerated elliptic problem

$$-\mathrm{div}(a(x,u,\nabla u)) = F \quad \text{in } \Omega,$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the coercivity condition, but they verify the general growth condition and only the large monotonicity. The second term F belongs to $W^{-1,p'}(\Omega, w^*)$.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions (i.e., every component $w_i(x)$ is a measurable function which is positive a.e. in Ω) satisfying some integrability conditions. The Objective of this paper is to study the following problem, in the weighted Sobolev space,

$$\begin{aligned} Au &= F \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ to its dual $W^{-1,p'}(\Omega, w^*)$. The principal part A is a differential operator of second order in divergence form defined as,

$$Au = -\operatorname{div}(a(x,u,\nabla u))$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, measurable with respect to x in Ω for every (x,ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) satisfying the coercivity condition. But, on the one hand, they verify the general growth condition in this form

$$|a_i(x,s,\xi)| \leq \beta w_i^{1/p}(x)[k(x) + |s|^{p-1} + \sum_{j=1}^{N} w_j^{1/p'}(x)[\gamma(s)|\xi_j|]^{p-1}]$$

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instead the classical growth condition, where we introduce some continuous function $\gamma(s)$. This type of the growth condition can not guaranteed the existence of the weak solution (See Remark 4.6), for that we overcame this difficulty by introduce an other type of solution so-called T-solution. On the other hand, they verify only the large monotonicity, that is

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)\geq 0 \quad \text{for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N.$$

We overcome this difficulty of the not strict monotonicity thanks to a technique (the L^1-version of Minty's lemma) similar to the one used in [5]. Recently in [6] Boccardo has studied the problem (1.1) in the classical Sobolev space $W_0^{1,p}(\Omega)$. For that the author has proved the existence of the T-solution. Other works in this direction can be found in [5] (where the right hand side $f\in L^1$ and $F\in L^{p'}(\Omega)$) and in [1] (where the existence and nonexistence results for some quasilinear elliptic equations involving the P-Laplaces have proved).

2. Preliminaries

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1<p<\infty$ and $w=\{w_i(x),\, 0\leq i\leq N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that (for each $w_i\neq 0$.)

$$w_i\in L^1_{\rm loc}(\Omega), \tag{2.1}$$

$$w_i^{\frac{-1}{p-1}}\in L^1_{\rm loc}(\Omega), \tag{2.2}$$

for any $0\leq i\leq N$.

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u\in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions fulfill

$$\frac{\partial u}{\partial x_i}\in L^p(\Omega,w_i)\quad \text{for } i=1,\dots,N.$$

Which is a Banach space under the norm

$$\|u\|_{1,p,w}=\Big[\int_\Omega |u(x)|^p w_0(x)\,dx+\sum_{i=1}^N\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big]^{1/p}. \tag{2.3}$$

The condition (2.1) implies that $C_0^\infty(\Omega)$ is a space of $W^{1,p}(\Omega,w)$ and consequently, we can introduce the subspace $W_0^{1,p}(\Omega,w)$ of $W^{1,p}(\Omega,w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Moreover, condition (2.2) implies that $W^{1,p}(\Omega,w)$ as well as $W_0^{1,p}(\Omega,w)$ are reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^*=\{w_i^*=w_i^{1-p'},\ i=0,\dots,N\}$ and where p' is the conjugate of p i.e. $p'=\frac{p}{p-1}$.

3. Basic assumptions and statement of results

Assumption (H1). The expression

$$|||u|||_X=\Big(\sum_{i=1}^N\int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p} \tag{3.1}$$

is a norm defined on X and is equivalent to the norm (2.3).
There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that the Hardy inequality,

$$\Big(\int_\Omega |u(x)|^q \sigma\, dx\Big)^{1/q} \leq c\Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx\Big)^{1/p}, \tag{3.2}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{3.3}$$

expressed by the inequality (3.2) is compact.

Note that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark 3.1. If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition: There exists $\nu \in]\frac{N}{P}, +\infty[\cap[\frac{1}{P-1}, +\infty[$ such that

$$w_i^{-\nu} \in L^1(\Omega) \quad \text{for all } i = 1, \dots, N. \tag{3.4}$$

Note that the assumptions (2.1) and (3.4) imply that,

$$|||u||| = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx\Big)^{1/p}, \tag{3.5}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and its equivalent to (2.3) and that, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega), \tag{3.6}$$

is compact for all $1 \leq q \leq p_1^*$ if $p.\nu < N(\nu + 1)$ and for all $q \geq 1$ if $p.\nu \geq N(\nu + 1)$ where $p_1 = \frac{p\nu}{\nu+1}$ and p_1^* is the Sobolev conjugate of p_1 [see [9], pp 30-31].

Assumption (H2).

$$|a_i(x, s, \xi)| \leq \beta w_i^{1/p}(x)[k(x) + |s|^{p-1} + \sum_{j=1}^N w_j^{1/p'}(x)[\gamma(s)|\xi_j|]^{p-1}], \tag{3.7}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) \geq 0 \quad \text{for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.8}$$

$$a(x, s, \xi).\xi \geq \alpha \sum_{i=1}^N w_i |\xi_i|^p, \tag{3.9}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$, $\gamma(s)$ is a continuous function and α, β are strictly positive constants.

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as,

$$T_k(s) = \begin{cases} s & \text{if } s \leq k \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

4. Existence results

Consider the problem

$$\begin{gathered} u \in W_0^{1,p}(\Omega, w), \quad F \in W^{-1,p'}(\Omega, w^*) \\ -\operatorname{div}(a(x, u, \nabla u)) = F \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.1}$$

Definition 4.1. A function u in $W_0^{1,p}(\Omega, w)$ is a T-solution of (4.1) if

$$\int_\Omega a(x, u, \nabla u)\nabla T_k[u-\varphi]\,dx = \langle F, T_k[u-\varphi]\rangle \quad \forall \varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega).$$

Theorem 4.2. *Assume that (H1) and(H2). Then the problem* (4.1) *has at least one T-solution u.*

Remark 4.3. Recall that an existence result for the problem (4.1) can be found in [8] by using the approach of pseudo monotonicity with some particular growths condition, that is $\gamma(s) = 1$.

Remark 4.4. In [9] the authors study the problem (4.1) under the strong hypotheses

$$\begin{gathered}[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \quad \text{for all } \xi \neq \eta \in \mathbb{R}^N,\\ |a_i(x,s,\xi)| \le \beta w_i^{1/p}(x)[k(x) + |s|^{p-1} + \sum_{j=1}^N w_j^{1/p'}(x)|\xi_j|^{p-1}],\end{gathered}$$

instead of (3.8) and (3.7) (respectively). Then the operator A associated to the problem (4.1) verifies the (S^+) condition and is coercive. Hence A is surjective from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$.

Proof of Theorem 4.2. Consider the approximate problem

$$\begin{gathered} u_n \in W_0^{1,p}(\Omega, w) \\ -\operatorname{div}(a(x, T_n(u_n), \nabla u_n)) = F. \end{gathered} \tag{4.2}$$

under the following assumptions:
Assertion (a): A priori estimates The problem (4.2) has a solution by a classical result in [8]. Moreover, by using u_n as test function in (4.2) we have,

$$\int_\Omega a(x, T_n(u_n), \nabla u_n).\nabla u_n\,dx = \int_\Omega F u_n\,dx.$$

Thanks to assumption (3.9), we have

$$\int_\Omega a(x, T_n(u_n), \nabla u_n).\nabla u_n\,dx \ge \alpha \sum_{i=1}^N \int_\Omega |\frac{\partial u_n}{\partial x_i}|^p w_i(x)\,dx = \alpha |||u_n|||^p$$

i.e.,

$$\alpha |||u_n|||^p \le \langle F, u_n\rangle \le \|F\|_{-1,p',w^*} |||u_n|||,$$

which implies $\alpha|||u_n|||^p \le C_1|||u_n|||$ for $p > 1$, with C_1 is a constant positive, then the sequence u_n is bounded in $W_0^{1,p}(\Omega, w)$, thus, there exists a function $u \in W_0^{1,p}(\Omega, w)$ and a subsequence u_{n_j} such that u_{n_j} converges weakly to u in $W_0^{1,p}(\Omega, w)$.
Assertion (b) We shall prove that for φ in $W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x, u_{n_j}, \nabla\varphi)\nabla T_k[u_{n_j} - \varphi]\,dx \le \langle F, T_k[u_{n_j} - \varphi]\rangle. \tag{4.3}$$

Let n_j large enough ($n_j > k + \|\varphi\|_{L^\infty(\Omega)}$), we have by choosing $T_k[u_{n_j} - \varphi]$ as test function in (4.2)

$$\int_\Omega a(x, u_{n_j}, \nabla u_{n_j})\nabla T_k[u_{n_j} - \varphi]\,dx = \langle F, T_k[u_{n_j} - \varphi]\rangle,$$

i.e.,

$$\begin{aligned}&\int_\Omega a(x,u_{n_j},\nabla u_{n_j})\nabla T_k[u_{n_j}-\varphi]\,dx+\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi]\,dx\\&-\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi]\,dx\\&=\langle F,T_k[u_{n_j}-\varphi]\rangle,\end{aligned}$$

which implies

$$\begin{aligned}\int_\Omega&[a(x,u_{n_j},\nabla u_{n_j})-a(x,u_{n_j},\nabla\varphi)]\nabla T_k[u_{n_j}-\varphi]\,dx\\&+\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi]\,dx=\langle F,T_k[u_{n_j}-\varphi]\rangle.\end{aligned}\tag{4.4}$$

Thanks to assumption (3.8) and the definition of truncating function, we have,

$$\int_\Omega[a(x,u_{n_j},\nabla u_{n_j})-a(x,u_{n_j},\nabla\varphi)]\nabla T_k[u_{n_j}-\varphi]\,dx\ge 0.\tag{4.5}$$

Combining (4.4) and (4.5), we obtain (4.3).
Assertion (c) We claim that,

$$\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi]\,dx\to\int_\Omega a(x,u,\nabla\varphi)\nabla T_k[u-\varphi]\,dx$$

and that

$$\langle F,T_k[u_{n_j}-\varphi]\rangle\to\langle F,T_k[u-\varphi]\rangle.$$

Indeed, first, by virtue of $u_{n_j}\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$, and [3, Lemma 2.4], we have

$$T_k(u_{n_j}-\varphi)\rightharpoonup T_k(u-\varphi)\quad\text{in } W_0^{1,p}(\Omega,w).\tag{4.6}$$

Which gives

$$\frac{\partial T_k}{\partial x_i}(u_{n_j}-\varphi)\rightharpoonup\frac{\partial T_k}{\partial x_i}(u_{n_j}-\varphi)\quad\text{in } L^p(\Omega,w_i).\tag{4.7}$$

Note that $\nabla T_k(u_{n_j}-\varphi)$ is not zero on the subset $\{x\in\Omega:|u_{n_j}-\varphi(x)|\le k\}$ (subset of $\{x\in\Omega:|u_{n_j}(x)|\le k+\|\varphi\|_{L^\infty(\Omega)}\}$,). Thus thanks to assumption (3.7), we have

$$\begin{aligned}|a_i(x,u_{n_j},\nabla\varphi)|^{p'}w_i^{-p'/p}&\le[k(x)+|u_{n_j}|^{p-1}+\gamma_0^{p-1}\sum_{k=1}^N|\frac{\partial\varphi}{\partial x_k}|^{p-1}w_k^{1/p'}]^{p'}\\&\le\beta[k(x)^{p'}+|u_{n_j}|^p+\gamma_0^p\sum_{k=1}^N|\frac{\partial\varphi}{\partial x_k}|^p w_k].\end{aligned}\tag{4.8}$$

where $\{\gamma_0=\sup|\gamma(s)|,|s|\le k+\|\varphi\|_\infty\}$. Since $u_{n_j}\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and $W_0^{1,p}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,\sigma)$, it follows that $u_{n_j}\to u$ strongly in $L^q(\Omega,\sigma)$ and $u_{n_j}\to u$ a.e. in Ω. Combining (4.7), (4.8) and By Vitali's theorem we obtain,

$$\int_\Omega a(x,u_{n_j},\nabla\varphi)\nabla T_k[u_{n_j}-\varphi]\,dx\to\int_\Omega a(x,u,\nabla\varphi)\nabla T_k[u-\varphi]\,dx.\tag{4.9}$$

Secondly, we show that

$$\langle F,T_k[u_{n_j}-\varphi]\rangle\to\langle F,T_k[u-\varphi]\rangle.\tag{4.10}$$

In view of (4.6) and since $F \in W^{-1,p'}(\Omega, w^*)$, we get
$$\langle F, T_k[u_{n_j} - \varphi]\rangle \to \langle F, T_k[u - \varphi]\rangle. \tag{4.11}$$
The convergence (4.9) and (4.11) allow to pass to the limit in the inequality (4.3), and to obtain
$$\int_\Omega a(x, u, \nabla\varphi)\nabla T_k[u - \varphi]\, dx \le \langle F, T_k[u - \varphi]\rangle. \tag{4.12}$$
Now we introduce the following Lemma which will be proved later and which is considered as an L^1 version of Minty's lemma (in weighted Sbolev spaces).

Result (4.12) and the following lemma complete the proof of Theorem 4.2. □

Lemma 4.5. *Let u be a measurable function such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega, w)$ for every $k > 0$. Then the following two statements are equivalent:*

(i) *For every φ in $W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and every $k > 0$,*
$$\int_\Omega a(x, u, \nabla\varphi)\nabla T_k[u - \varphi]\, dx \le \int_\Omega F\nabla T_k(u - \varphi)\, dx\,.$$
(ii) *For every φ in $W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and every $k > 0$,*
$$\int_\Omega a(x, u, \nabla u)\nabla T_k[u - \varphi]\, dx = \int_\Omega F\nabla T_k(u - \varphi)\, dx\,.$$

Proof. Note that (ii) implies (i) is easily proved adding and subtracting
$$\int_\Omega a(x, u, \nabla\varphi)\nabla T_k[u - \varphi]\, dx,$$
and then using assumption (3.8). Thus, it only remains to prove that (i) implies (ii). Let h and k be positive real numbers, let $\lambda \in]-1, 1[$ and $\psi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$. Choosing, $\varphi = T_h(u - \lambda T_k(u - \psi)) \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ as test function in (4.12), we have,
$$I \le J, \tag{4.13}$$
with
$$I = \int_\Omega a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\, dx,$$
$$J = \langle F, T_k(u - T_h(u - \lambda T_k(u - \psi)))\rangle.$$
Put $A_{hk} = \{x \in \Omega : |u - T_h(u - \lambda T_k(u - \psi))| \le k\}$ and $B_h = \{x \in \Omega : |u - \lambda T_k(u - \psi)| \le h\}$. Then, we have
$$\begin{aligned} I = &\int_{A_{kh}\cap B_h} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\, dx \\ &+ \int_{A_{kh}\cap B_h^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\, dx \\ &+ \int_{A_{kh}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi)))\, dx. \end{aligned}$$
Since $\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))$ is zero in A_{kh}^C, we obtain
$$\int_{A_{kh}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\, dx = 0. \tag{4.14}$$

Moreover, if $x \in B_h^C$, we have $\nabla T_h(u - \lambda T_k(u - \psi) = 0$ which implies,

$$\begin{aligned}&\int_{A_{kh} \cap B_h^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\,dx \\ &= \int_{A_{kh} \cap B_h^C} a(x, u, 0)\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\,dx.\end{aligned}$$

Now, thanks to assumption (3.9), we have $a(x, u, 0) = 0$. Then

$$\int_{A_{kh} \cap B_h} a(x, u, 0)\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\,dx = 0. \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$I = \int_{A_{kh} \cap B_h} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\,dx,$$

letting $h \to +\infty$, we have

$$A_{kh} \to \{x, |T_k(u - \psi)| \le k\} = \Omega, \tag{4.16}$$

and $B_h \to \Omega$ which implies

$$A_{kh} \cap B_h \to \Omega. \tag{4.17}$$

Then

$$\begin{aligned}&\lim_{h\to+\infty} \int_{A_{kh} \cap B_h} a(x, u, \nabla T_h(u - \lambda T_k(u - \psi))\nabla T_k(u - T_h(u - \lambda T_k(u - \psi))\,dx \\ &= \lambda \int_\Omega a(x, u, \nabla(u - \lambda T_k(u - \psi)\nabla T_k(u - \psi)\,dx.\end{aligned} \tag{4.18}$$

On the other hand, we have

$$J = \langle F, T_k[u - T_h(u - \lambda T_k(u - \psi)]\rangle.$$

Then

$$\lim_{h\to+\infty} \langle F, T_k(u - T_h(u - \lambda T_k(u - \psi))\rangle = \lambda\langle F, T_k[u - \psi]\rangle. \tag{4.19}$$

Together (4.18), (4.19) and passing to the limit in (4.13), we obtain

$$\lambda \int_\Omega a(x, u, \nabla(u - \lambda T_k(u - \psi)\nabla T_k(u - \psi)\,dx \le \lambda\langle F, T_k[u - \psi]\rangle$$

for every $\psi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$, and for $k > 0$. Choosing $\lambda > 0$ dividing by λ, and then letting λ tend to zero , we obtain

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \psi)\,dx \le \langle F, T_k[u - \psi]\rangle. \tag{4.20}$$

For $\lambda < 0$, dividing by λ, and then letting λ tend to zero , we obtain

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \psi)\,dx \ge \langle F, T_k[u - \psi]\rangle, \tag{4.21}$$

Combining (4.20) and (4.21), we conclude that

$$\int_\Omega a(x, u, \nabla u)\nabla T_k(u - \psi)\,dx = \langle F, T_k[u - \psi]\rangle.$$

This completes the proof of Lemma. □

Remark 4.6. (1) The fact that the terms $T_n(u_n)$ is introduced in (4.2) and also $\gamma(s)$ is a continuous function, allow to have a weak solution for the a approximate problem.
(2) Since in the formulation of the problem (4.1), we have $a(x,u,\nabla u)$ instead of $a(x,T_n(u_n),\nabla u_n)$, then the term $a(x,u,\nabla u)$ may not belongs in $L^{p'}(\Omega,w^*)$ and not in $L^1(\Omega)$, thus the problem (4.1) can have a T-solutions but, not a weak solution.

For example if $w_i\equiv 1$, $i=1,\dots,N$ and $a(x,u,\nabla u)=e^{|u|}|\nabla u|^{p-2}\nabla u$, with $\gamma(s)=e^{|s|}$ then

$$
\begin{gathered}
u\in W_0^{1,p}(\Omega,w), F\in W^{-1,p'}(\Omega,w^*)\\
-\operatorname{div}(e^{|u|}|\nabla u|^{p-2}\nabla u)=F \quad \text{in } \Omega\\
u=0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

our simple problem has a T-solutions, but not a weak solution

Example 4.7. Let us consider the special case:

$$
a_i(x,\eta,\xi)=e^{|s|}w_i(x)|\xi_i|^{p-1}\operatorname{sgn}(\xi_i)\quad i=1,\dots,N,
$$

with $w_i(x)$ is a weight function ($i=1,\dots,N$). For simplicity, we shall suppose that $w_i(x)=w(x)$, for $i=1,\dots,N-1$, and $w_N(x)\equiv 0$ it is easy to show that the $a_i(x,s,\xi)$ are Caracthéodory function satisfying the growth condition (3.7) and the coercivity (3.8). On the other hand the monotonicity condition is verified. In fact,

$$
\begin{aligned}
&\sum_{i=1}^{N}(a_i(x,s,\xi)-a_i(x,s,\hat\xi))(\xi_i-\hat\xi_i)\\
&=e^{|s|}w(x)\sum_{i=1}^{N-1}(|\xi_i|^{p-1}\operatorname{sgn}(\xi_i)-|\hat\xi_i|^{p-1}\operatorname{sgn}(\hat\xi_i))(\xi_i-\hat\xi_i)\geq 0
\end{aligned}
$$

for almost all $x\in\Omega$ and for all $\xi,\hat\xi\in\mathbb{R}^N$. This last inequality can not be strict, since for $\xi\neq\hat\xi$ with $\xi_N\neq\hat\xi_N$ and $\xi_i=\hat\xi_i$, $i=1,\dots,N-1$. The corresponding expression is zero. In particular, let us use special weight functions w expressed in terms of the distance to the bounded $\partial\Omega$. Denote $d(x)=\operatorname{dist}(x,\partial\Omega)$ and set $w(x)=d^\lambda(x)$, such that,

$$
\lambda<\min(\frac{p}{N},p-1) \tag{4.22}
$$

Remark 4.8. Condition (4.22) is sufficient for (3.4) to hold [see [10],pp 40-41].

Finally, the hypotheses of Theorem 4.2 are satisfied. Therefore, for all $F\in\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$ the following problem has at last one solution:

$$
\begin{gathered}
T_k(u)\in W_0^{1,p}(\Omega,w),\\
\int_\Omega\sum_{i=1}^{N}w_i(x)e^{|u|}|\frac{\partial u}{\partial x_i}|^{p-1}\operatorname{sgn}\big(\frac{\partial u}{\partial x_i}\big)\frac{\partial T_k(u-\varphi)}{\partial x_i}\,dx=\int_\Omega FT_k(u-\varphi)\,dx\\
\forall\varphi\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)\,.
\end{gathered}
$$

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