2005-Oujda International Conference on Nonlinear Analysis.
Electronic Journal of Differential Equations, Conference 14, 2006, pp. 135–147.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

OPTIMAL CONTROLS FOR A CLASS OF NONLINEAR EVOLUTION SYSTEMS

ABDELHAQ BENBRIK, MOHAMMED BERRAJAA, SAMIR LAHRECH

ABSTRACT. We consider the abstract nonlinear evolution equation $\dot{z} + Az = uBz + f$. Viewing u as control, we seek to minimize $J(u) = \int_0^T L(z(t), u(t))\,dt$. Under suitable hypotheses, it is shown that there exists an optimal control $\overline{u}$ and that it satisfies the appropriate optimality system. An example involving the p-Laplacian operator demonstrates the applicability of our results.

1. Introduction

In this paper, we investigate the optimal control problem governed by the abstract non linear evolution equation

$$\dot{z} + Az = uBz + f \tag{1.1}$$

These systems with linear operators A and B are called bilinear systems (see. [2, 3, 11]). They appear in many mathematical models from physical processes, for example, models involving the p Laplacian operator (see [4]). Our aim is to investigate the case where A is not linear.

We organize this work as follows: After formulating the problem, we address the question of existence and uniqueness of solutions to these systems. In section 3, we prove the existence theorem of optimal control and we give the necessary conditions of optimality. Finally, we present an example involving the p-Laplacian operator which illustrates the applications of the abstract framework and the results of the theory developed in the previous sections.

2. Setting of the problem

Throughout the paper, H denotes a separable Hilbert space and V a subspace of H having the structure of a reflexive Banach space which is continuously and densely embedded in H.

Identifying H with its dual H', we have the Gelfand triplet $V \hookrightarrow H \hookrightarrow V'$ where V' is the dual of V. We suppose that these embeddings are compact. Let $\langle ., . \rangle$ be the duality pairing between V and V' as well as the inner product on H. Let $\| \cdot \|$,

2000 *Mathematics Subject Classification.* 49J20, 49K20.
Key words and phrases. Optimal control; monotone operator; compact embedding; p-Laplacian; bilinear system.

Published September 20, 2006.

$|\cdot|$ and $\|\cdot\|_{V'}$ denote the norms on V, H and V' respectively. Given a fixed real number $T > 0$ and $2 < p < +\infty$ we introduce the spaces:

$$L^p(V) = L^p(0,T;V), \quad L^p(H) = L^p(0,T;H), \quad L^{p'}(V') = L^{p'}(0,T;V'),$$

where $(\frac{1}{p} + \frac{1}{p'} = 1)$ and $W = \{w \in L^p(V) : \dot{w} \in L^{p'}(V')\}$. Here the derivative is understood in the sense of vector valued distributions.

It is well known that every $w \in W$ is after eventual modification on a set of measure zero, continuous from $[0,T]$ in H and the embedding $W \hookrightarrow \mathcal{C}([0,T];H)$ is continuous [6, 7]. Furthermore, if $V \hookrightarrow H$ compactly, then also $W \hookrightarrow L^p(H)$ compactly.

We study the control problem

$$\inf_u J(u) \tag{2.1}$$

subject to the state equation

$$\begin{gathered} \dot{z} + Az(t) = u(t)Bz(t) + f(t) \\ z(0) = z_0, \end{gathered}$$

where the cost functional is

$$J(u) = \int_0^T L(z(t), u(t))\, dt.$$

Our aim is to provide conditions under which the optimal solutions (2.1) exist. By an optimal solution we mean a control $\overline{u}$ on which the infimum is attained.

For problem (2.1) we need the following hypotheses:

- (H1) $A\colon V \to V'$ is such that:
  - (i) $\|A\varphi\|_{V'} \le \alpha_1 \|\varphi\|^{p-1}$ with $\alpha_1 > 0$.
  - (ii) $\langle A\varphi, \varphi\rangle \ge \alpha_2 \|\varphi\|^p$ with $\alpha_2 > 0$.
  - (iii) $\langle A\varphi_1 - A\varphi_2, \varphi_1 - \varphi_2\rangle \ge \alpha_3 \|\varphi\|^2$ with $\beta > 0$.
  - (iv) $\varphi \to A(\varphi)$ is continuously Frechet differentiable.
- (H2) $B\colon H \to H$ is linear and continuous with $|B\varphi| \le b|\varphi|$, $b > 0$.
- (H3) $u \in L^r(0,T)$ with $r = p/(p-2)$.
- (H4) $f \in L^{p'}(V')$.
- (H5) $z_0 \in H$.
- (H6) $L\colon H \times \mathbb{R} \to \mathbb{R}$ is a integrand convex such that:
  - (i) L is coercive: $\lim_{\|u\|_{L^r(0,T)} \to \infty} \int_0^T L(z(t), u(t))\, dt = +\infty$
  - (ii) $(x,u) \to L(x,u)$ is continuously Frechet differentiable.
  - (iii) for every $x \in \mathcal{C}([0,T], H)$ and every $u \in L^r(0,T)$, $J(u)$ is finite.

Remark 2.1. $A(\varphi) \in L^{p'}(V')$ if $\varphi \in L^p(V)$ and then

$$\|A\varphi\|_{L^{p'}(V')} \le \alpha_1 \|\varphi\|^{p-1}_{L^p(V)}.$$

For $u \in L^r(0,T)$ and $\varphi \in L^p(V)$ we have $uB\varphi \in L^{p'}(V')$ and

$$\|uB\varphi\|_{L^{p'}(V')} \le \beta_1 \|u\|_{L^r(0,T)} \|\varphi\|_{L^p(V)},$$

where $\beta_1 > 0$. Therefore, the choice of control space is compatible with the equation.

3. Results on the evolution problem

We consider the evolution problem

$$\begin{gathered}\dot{z}(t)+Az(t)=u(t)Bz(t)+f\\ z(0)=z_0\end{gathered}\tag{3.1}$$

We recall that by a solution to the above problem, we mean a function $z\in W$ that satisfies (3.1).

Theorem 3.1. *Under hypothesis (H1)(i), (H1)(ii), (H1)(iii), (H2), (H3), (H4) and (H5), equation* (3.1) *admits a unique solution z such that $z\in L^\infty(H)$ and $z\in W$.*

Proof. **Uniqueness:** if z_1 and z_2 are solutions of (3.1), then $z=z_1-z_2$ satisfies

$$\begin{gathered}\dot{z}+Az_1-Az_2=uBz\\ z(0)=0\end{gathered}\tag{3.2}$$

and for $t\in[0,T]$,

$$\frac{1}{2}|z(t)|^2\le b\int_0^t|u(\tau)||z(\tau)|^2\,d\tau\,.$$

Using the Gronwall lemma, we obtain $z_1=z_2$.

The existence follows from a standard application of the Galerkin method [6] and the a priori estimates given in Lemma 3.2. We remark that by Theorem 3.1, $z\in\mathcal{C}([0,T];H)$. □

Lemma 3.2. *Under the hypothesis of Theorem 3.1, if z is a solution of* (3.1) *then*

$$\|z\|_{L^p(V)}\le K_1\big[|z_0|^2+\|u\|^r_{L^r(0,T)}+\|f\|^{p'}_{L^{p'}(V')}\big]^{1/p}\tag{3.3}$$

$$\|z\|_{L^\infty(H)}\le K_2\big[|z_0|^2+\|u\|^r_{L^r(0,T)}+\|f\|^{p'}_{L^{p'}(V')}\big]^{1/2}\tag{3.4}$$

$$\|\dot{z}\|_{L^{p'}(V')}\le K_3\big[\|z\|^{p-1}_{L^p(V)}+\|u\|^{p-1/p-2}_{L^r(0,T)}+\|f\|_{L^{p'}(V')}\big]\tag{3.5}$$

Proof. Let z be a solution of (3.1), then

$$\int_0^T\langle\dot{z}(t),z(t)\rangle\,dt+\int_0^T\langle Az(t),z(t)\rangle\,dt=\int_0^T\langle u(t)Bz(t),z(t)\rangle\,dt+\int_0^T\langle f(t),z(t)\rangle\,dt.$$

Using (H1)(ii), (H2) and the continuity of the embedding $V\hookrightarrow H$, we have

$$\begin{aligned}&\frac{1}{2}|z(T)|^2-\frac{1}{2}|z_0|^2+\alpha_2\int_0^T\|z(\tau)\|_V^p\,d\tau\\ &\le K_1'\int_0^T|u(\tau)|\|z(\tau)\|^2\,d\tau+\int_0^T\|f(\tau)\|_{V'}\|z(\tau)\|\,d\tau\,.\end{aligned}$$

By the Young inequality [10], for $\frac{1}{r}+\frac{2}{p}=1$, we have

$$\begin{gathered}K_1'\int_0^T|u(\tau)|\|z(\tau)\|^2\,d\tau\le\frac{\alpha_2}{4}\|z\|^p_{L^p(V)}+K_2'\|u\|^r_{L^r(0,T)},\\ \int_0^T\|f(\tau)\|_V\|z(\tau)\|\,d\tau\le\frac{\alpha_2}{4}\|z\|^p_{L^p(V)}+K_3'\|f\|^{p'}_{L^{p'}(V')}.\end{gathered}$$

Hence

$$\frac{\alpha_2}{2}\|z\|^p_{L^p(V)}\le\frac{1}{2}|z_0|^2+K_2'\|u\|^r_{L^r(0,T)}+K_3'\|f\|^{p'}_{L^{p'}(V')},$$

from which, we deduce then (3.3).

Multiplying (3.1) by z and integrating on $[0,t]$ we obtain

$$\frac{1}{2}|z(t)|^2 - \frac{1}{2}|z_0|^2 + \frac{\alpha_2}{2}\int_0^t \|z(\tau)\|^p \, d\tau \le K_2''\|u\|^r_{L^r(0,T)} + K''\|f\|^{p'}_{L^{p'}(V')}$$

and then

$$|z(t)|_H \le K_2\big[|z_0|^2 + \|u\|^r_{L^r(0,T)} + \|f\|^{p'}_{L^{p'}(V')}\big]^{1/2}$$

which implies 3.4.

Multiplying 3.2 by $\xi \in L^p(V)$, we have

$$\begin{aligned}&|\int_0^T \langle \dot z(t), \xi(t)\rangle \, dt| \\ &\le |\int_0^T \langle Az(t), \xi(t)\rangle \, dt| + |\int_0^T \langle u(t)Bz(t), \xi(t)\rangle| + |\int_0^T \langle f(t), \xi(t)\rangle|.\end{aligned}$$

Hence

$$\Big|\int_0^T \langle \dot z(t), \xi(t)\rangle \, dt\Big| \le \big[\alpha_1\|z\|^{p-1}_{L^p(V)} + \beta_1\|u\|_{L^r(0,T)}\|z\|_{L^p(V)} + \|f\|_{L^{p'}(V')}\big]\|\xi\|_{L^p(V)},$$

which by Young inequality implies (3.5). □

4. Optimal controls

The aim of this section is to prove the existence of optimal controls for problem (2.1). The differentiability of the mapping $u \mapsto z$ permits the characterization of the optimal control $\overline{u}$ by necessary conditions corresponding to $J'(\overline{u}) = 0$.

Existence theorem for the control problem.

Theorem 4.1. *If (H1), (H2), (H3), (H4), (H5) and (H6) hold, then* (2.1) *admits an optimal solution.*

Proof. Let $(u_n)_n$ be a minimizing sequence for (2.1), i.e. the pairs (z_n, u_n) are admissible for (2.1) and

$$\lim_n J(u_n) = \overline{J}.$$

From (H6) we have $\|u_n\|_{L^r(0,T)} \le M$.

And from Lemma 3.2, we know that $(z_n)_n$ belongs to a bounded subset of $L^\infty(H) \cap W$. By passing to a subsequence if necessary, we may assume that

$$\begin{gathered} u_n \rightharpoonup \overline{u} \quad w - L^r(0,T) \\ z_n \rightharpoonup \overline{z} \quad w* - L^\infty(H) \\ z_n \rightharpoonup \overline{z} \quad w - L^p(V) \\ Az_n \rightharpoonup \chi \quad w - L^{p'}(V') \\ u_nBz_n \rightharpoonup \Psi \quad w - L^{p'}(V') \\ \dot z_n \rightharpoonup \Lambda \quad w - L^{p'}(V') \end{gathered}$$

1. Using the convergence $\sigma(\mathcal{D}(0,T;V);\mathcal{D}'(0,T;V'))$ we obtain $\Lambda = \dot{\overline{z}}$.

2. $V \hookrightarrow H$ compactly implies that

$$\begin{gathered} z_n \to \overline{z} \quad s - L^p(H) \\ z_n(t) \to \overline{z}(t) \quad s - H \quad \text{for all } t \in [0,T]. \end{gathered}$$

For $\varphi \in L^p(V)$, we have

$$\begin{aligned}&\int_0^T \langle u_n(t)Bz_n(t), \varphi(t)\rangle\,dt\\&= \int_0^T \langle u_n(t)B(z_n(t)-\overline{z}(t);\varphi(t)\rangle\,dt + \int_0^T \langle u_n(t)B\overline{z}(t),\varphi(t)\rangle\,dt.\end{aligned}$$

Note that

$$\int_0^T \langle u_n(t)B(z_n(t)-\overline{z}(t);\varphi(t)\rangle\,dt \le K_1\|u_n\|_{L^r(0,T)}\|z_n-\overline{z}\|_{L^p(H)}\|\varphi\|_{L^p(H)}$$

and

$$\int_0^T u_n(t)\langle B\overline{z}(t),\varphi(t)\rangle\,dt \to \int_0^T \overline{u}(t)\langle B\overline{z}(t),\varphi(t)\rangle\,dt$$

because $\langle B\overline{z},\varphi\rangle \in L^{r'}(0,T)$. We deduce that $\Psi = \overline{u}B\overline{z}$.

3. For $y \in L^p(V)$, we set

$$X_m = \int_0^T \langle Az_m(t) - Ay(t); z_m(t)-y(t)\rangle\,dt\,.$$

We have

$$X_m = \int_0^T \langle Az_m(t); z_m(t)\rangle\,dt - \int_0^T \langle Az_m(t); y(t)\rangle\,dt - \int_0^T \langle Ay(t); z_m(t)-y(t)\rangle\,dt$$

and

$$\begin{aligned}&\int_0^T \langle Az_m(t), z_m(t)\rangle\,dt\\&= \frac{1}{2}|z_{m,0}|^2 - \frac{1}{2}|z_m(T)|^2 + \int_0^T \langle u_mBz_m(t), z_m(t)\rangle\,dt + \int_0^T \langle f(t), z_m(t)\rangle\,dt.\end{aligned}$$

But

$$\begin{aligned}&\int_0^T \big(\langle u_m(t)Bz_m(t), z_m(t)\rangle - \langle \overline{u}(t)B\overline{z}(t),\overline{z}(t)\rangle\big)\,dt\\&= \int_0^T \langle u_m(t)Bz_m(t), z_m(t)-\overline{z}(t)\rangle\,dt + \int_0^T \langle u_m(t)Bz_m(t) - \overline{u}(t)B\overline{z}(t),\overline{z}(t)\rangle\,dt\end{aligned}$$

The first integral in the right-hand side approaches zero because $z_m \to \overline{z}$ $(s - L^p(H))$. The second integral approaches zero because $u_mBz_m \rightharpoonup \overline{u}B\overline{z}$ $(w - L^{p'}(V'))$. We deduce that

$$\begin{aligned}&\limsup_m \int_0^T \langle Az_m(t), z_m(t)\rangle\,dt\\&\le \frac{1}{2}|z_0|^2 - \frac{1}{2}|\overline{z}(T)|^2 + \int_0^T \langle \overline{u}(t)B\overline{z}(t),\overline{z}(t)\rangle\,dt + \int_0^T \langle f(t),\overline{z}(t)\rangle\,dt\,.\end{aligned}$$

Since $\overline{z}$ satisfies

$$\begin{gathered}\dot{\overline{z}} + \chi = \overline{u}B\overline{z} + f\\ \overline{z}(0) = z_0\end{gathered}$$

it follows that

$$\frac{1}{2}|z_0|^2 - \frac{1}{2}|\overline{z}(T)|^2 + \int_0^T \langle \overline{u}(t)B\overline{z}(t),\overline{z}(t)\rangle\,dt + \int_0^T \langle f(t),\overline{z}(t)\rangle\,dt = \int_0^T \langle \chi(t),\overline{z}(t)\rangle\,dt.$$

Hence

$$0 \leq \limsup_m X_m \leq \int_0^T \langle \chi(t) - Ay(t), \overline{z}(t) - y(t)\rangle \, dt \text{ for all } y \in L^p(V)$$

Using the continuity of the operator A we obtain $\chi = A\overline{z}$. We deduce that $(\overline{z}, \overline{u})$ is admissible for (2.1). From (H6) we have

$$\int_0^T L(\overline{z}(t), \overline{u}(t)) \, dt \leq \liminf_m \int_0^T L(z_m(t), u_m(t)) \, dt = \overline{J}$$

Hence $\overline{u}$ is an optimal control. □

Optimality conditions. Before proceeding with investigation of the mapping $\Theta\colon u \mapsto z$, where z is defined by (3.1), we introduce a technical lemma generalizing the Gronwall inequality.

Lemma 4.2. *Let $T > 0$ and $c \geq 0$. Assume that λ and m are integrable in $[0, T]$ with positive values. Let $\varphi\colon [0, T] \to \mathbb{R}^+$ be such that:*

(a) *$\lambda\varphi$ and $\lambda\varphi^2$ are integrable on $[0, T]$.*
(b) *$\frac{1}{2}\varphi^2(t) \leq \frac{1}{2}c^2 + \int_0^t \lambda(s)\varphi(s) \, ds + \int_0^t m(s)\varphi^2(s) \, ds$ for $t \geq 0$.*

Then

$$\varphi(t) \leq \Big[c + \int_0^t \lambda(s) \, ds\Big] \exp\Big(\int_0^t m(s) \, ds\Big).$$

Proof. Set

$$\Psi(t) = \Big[c^2 + 2\int_0^t \lambda(s)\varphi(s) \, ds + 2\int_0^t m(s)\varphi^2(s) \, ds\Big]^{1/2}.$$

We have that $\varphi(t) \leq \Psi(t)$ and $\dot{\Psi} \leq \lambda(t) + m(t)\Psi(t)$. Then

$$\frac{d}{dt}\Big[\Psi(t) \exp\Big(-\int_0^t m(s) \, ds\Big) - \int_0^t \lambda(s) \exp\Big(-\int_0^s m(\tau) \, d\tau\Big)\Big] \leq 0.$$

Hence

$$\Psi(t) \leq \exp\Big(\int_0^t m(\tau) \, d\tau\Big)\Big[c + \int_0^t \lambda(\tau) \exp\Big(-\int_0^\tau m(s) \, ds\Big) d\tau\Big],$$

which completes the proof. □

Lemma 4.3. *Suppose the hypothesis (H1), (H2), (H3), (H4) and (H5) hold, then the mapping $\Theta\colon L^r(0, T) \to L^\infty(H) \cap L^2(V)$, $u \mapsto z$ is locally Lipschitz.*

Proof. Let $\overline{u}$ and h be in $L^r(0, T)$ with $\|h\|_{L^r(0,T)} \leq 1$. Set $\overline{z} = \Theta(\overline{u})$, $z_h = \Theta(\overline{u} + h)$ and $z = z_h - \overline{z}$. Then z satisfies

$$\dot{z} + Az_h - A\overline{z} = \overline{u}Bz + hBz_h$$
$$z(0) = 0$$

Multiplying by z and integrating on $[0, t]$ we have

$$\frac{1}{2}|z(t)|^2 + \beta\int_0^t \|z(\tau)\|_V^2 \, d\tau \leq b\int_0^t |\overline{u}(\tau)||z(\tau)|^2 \, d\tau + b\int_0^t |h(\tau)||z_h(\tau)||z(\tau)| \, d\tau \,.$$

Invoking the Lemma 4.2, we have

$$|z(t)|_H \leq \exp\Big(b\int_0^t |\overline{u}(\tau)| \, d\tau\Big)\Big[b\int_0^t |h(\tau)||z_h(\tau)| \, d\tau\Big]$$

but
$$\int_0^t |h(\tau)||z_h(\tau)|\,d\tau \le K\|h\|_{L^r(0,T)}\|z_h\|_{L^\infty(H)}$$
and
$$\|z_h\|_{L^\infty(H)}\| \le K_1\Big[|z_0|^2 + \|\overline{u}+h\|^r_{L^r(0,T)} + \|f\|^{p'}_{L^{p'}(V')}\Big]^{1/2} \le K'$$
where K' is a positive constant depending on z_0, $\overline{u}$ and f (because $\|h\| \le 1$). We obtain
$$\|z\|_{L^\infty(H)} \le K'_1\|h\|_{L^r(0,T)},$$
$$\|z\|_{L^2(V)} \le K'_2\|h\|_{L^r(0,T)}$$
□

Theorem 4.4. *Suppose that:*

(i) *The hypothesis of Lemma 4.3 are satisfied with $f = 0$.*

(ii) *For φ and Ψ in $\mathcal{C}([0,T];H)$ with $\|\Psi\|_{\mathcal{C}([0,T];H)} \le 1$ we have*
$$\|A'(\varphi(t)+\Psi(t)) - A'(\varphi(t))\|_{\mathcal{L}(H)} \le \gamma(t)|\Psi(t)|_H$$
where $\gamma \in L^1(0,T)$.

Then the mapping $\Theta\colon L^r(0,T) \to L^\infty(H)\cap L^2(V)$ is Fréchet differentiable and the derivative $\Theta'_{\overline{u}}.h$ is a solution of
$$\begin{gathered}\dot y(t) + A'_{\overline{z}(t)}y(t) = \overline{u}(t)By(t) + h(t)B\overline{z}(t)\\ y(0)=0\end{gathered}\tag{4.1}$$
where $\overline{z} = \Theta(\overline{u})$.

Proof. **1.** Since A is strongly monotone. For $\lambda > 0$, φ and Ψ in V, we have
$$\Big\langle \frac{1}{\lambda}(A(\varphi+\lambda\Psi) - A(\varphi)), \Psi\Big\rangle \ge \beta\|\Psi\|_V^2.$$
Hence $\langle A'_\varphi\Psi, \Psi\rangle \ge \beta\|\Psi\|_V^2$

2. For $\overline{u} \in L^r(0,T)$, the mapping $h \mapsto y$ defined by (4.1) is linear. Multiplying (4.1) by y and integrating on $[0,t]$ we obtain
$$\frac12|y(t)|^2 + \beta\int_0^t \|y(\tau)\|_V^2\,d\tau \le b\int_0^t |\overline{u}(\tau)||y(\tau)|^2\,d\tau + |h(\tau)||\overline{z}(\tau)||y(\tau)|\,d\tau$$
By Lemma 4.3,
$$|y(t)| \le b\int_0^t |h(\tau)||\overline{z}(\tau)|\,d\tau \exp\Big[b\int_0^t |\overline{u}(\tau)|\Big],$$
but
$$\|\overline{z}\|_{L^\infty(H)} \le K_1\big[|z_0|^2 + \|\overline{u}\|^r_{L^r(0,T)}\big]^{1/2}.$$
Then
$$\|y\|_{L^\infty(H)} \le K'_1\|h\|_{L^r(0,T)},$$
$$\|y\|_{L^2(V)} \le K'_2\|h\|_{L^r(0,T)},$$
where K'_i are positive constants depending on z_0 and $\overline{u}$. Hence the mapping $h \mapsto y$ is continuous.

3. Set $\overline{z} = \Theta(\overline{u})$, $z_h = \Theta(\overline{u}+h)$, $z = z_h - \overline{z}$ and $w = z - y$ where y is a solution of (4.1). We have

$$\begin{gathered} \dot{w}(t) + A'_{\overline{z}(t)} w(t) = \overline{u}(t)Bw(t) + h(t)Bz(t) + g(t) \\ w(0) = 0 \end{gathered} \tag{4.2}$$

where

$$\begin{aligned} g(t) &= A'_{\overline{z}(t)} z(t) - \big(Az_h(t) - A\overline{z}(t)\big) \\ &= \int_0^1 [A'\overline{z}(t) - A'(\overline{z}(t) + sz(t))] z(t)\, ds. \end{aligned}$$

Then

$$|g(t)|_H \le \int_0^1 \gamma(t)|sz(t)||z(t)|\, ds = \frac{\gamma(t)}{2}|z(t)|^2$$

On the other hand, multiplying (4.2) by w and integrating on $[0,t]$ we obtain

$$\begin{aligned} &\frac{1}{2}|w(t)|^2 + \beta \int_0^t \|w(\tau)\|_V^2\, d\tau \\ &\le b\int_0^t |\overline{u}(\tau)||w(\tau)|^2\, d\tau + b\int_0^t |h(\tau)||z(\tau)||w(\tau)|\, d\tau + \frac{1}{2}\int_0^t \gamma(\tau)|z(\tau)|^2|w(\tau)|\, d\tau \end{aligned}$$

Then Lemma 4.3 gives

$$|w(t)| \le \exp\Big(b\int_0^t |\overline{u}(\tau)|\, d\tau\Big)\Big[b\int_0^t |h(\tau)||z(\tau)|\, d\tau + \frac{1}{2}\int_0^t \gamma(\tau)|z(\tau)|^2\, d\tau\Big],$$

but $\|z\|_{L^\infty(H)} \le K\|h\|_{L^r(0,T)}$ then

$$\begin{gathered} \|w\|_{L^\infty(H)} \le K_1\|h\|^2_{L^r(0,T)}, \\ \|w\|_{L^2(V)} \le K_2\|h\|^2_{L^r(0,T)}. \end{gathered}$$

It follows that Θ is fréchet differentiable from $L^r(0,T)$ on $L^\infty(H)\cap L^2(V)$ and $\Theta'_{\overline{u}}.h$ is a solution of (4.1). ☐

Theorem 4.5. *Assume the hypotheses of Theorem 4.4 and (H6) hold. Then an optimal control* $\overline{u}$*, its corresponding state* $\overline{z}$*, and its adjoint state* p *are necessarily tied by the optimality system:*

(1) $\dot{\overline{z}} + A\overline{z} = \overline{u}B\overline{z}$ $\overline{z}(0) = z_0$
(2) $-\dot{p} + A'^{*}_{\overline{z}}p = \overline{u}B^*p + \partial_1 L(\overline{z}(t), \overline{u}(t))$ $p(T) = 0$
(3) $\langle B\overline{z}(t), p(t)\rangle + \partial_2 L(\overline{z}(t), \overline{u}(t)) = 0$ *a.e. in* $[0,T]$

Proof. Since L is Fréchet differentiable, we deduce that the functional

$$J(u) = \int_0^T L(z(t), u(t))\, dt$$

is Fréchet differentiable on $L^r(0,T)$. Since $\overline{u}$ is a minimum point for J,

$$J'_{\overline{u}}.h = 0, \quad \forall h \in L^r(0,T)$$

but

$$J'_{\overline{u}}.h = \int_0^T \langle \partial_1 L(\overline{z}(t), \overline{u}(t), y(t)\rangle\, dt + \int_0^T \langle \partial_2 L(\overline{z}(t), \overline{u}(t), h(t)\rangle\, dt$$

where $y = \Theta'_{\overline{u}}.h$. We define p by (2), then

$$\begin{aligned} J'(u).h &= \int_0^T \langle -\dot{p}(t) + A'^{*}_{\overline{z}(t)}p(t) - \overline{u}(t)B^*p(t), y(t)\rangle\,dt + \int_0^T h(t)\partial_2 L(\overline{z}(t), \overline{u}(t))\,dt \\ &= \int_0^T \langle p(t), \dot{y}(t) + A'_{\overline{z}(t)}y(t) - \overline{u}(t)By(t)\rangle\,dt + \int_0^T h(t)\partial_2 L(\overline{z}(t), \overline{u}(t))\,dt \\ &= \int_0^T \big[\langle p(t), B\overline{z}(t)\rangle + \partial_2 L(\overline{z}(t), \overline{u}(t))\big]h(t)\,dt \end{aligned}$$

Hence part (3) of the theorem is consequence of the above equality. □

5. Example

In this section, we present an example which illustrates the application of the results of the theory developed in the previous sections. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial\Omega$. We consider the control problem (2.1) with

$$J(u) = \int_Q |z(x,t)|^4\,dx\,dt + \int_0^T |u(t)|^2_{\mathbb{R}^N}\,dt$$

Where z satisfies the nonlinear evolution equation

$$\begin{gathered} \frac{\partial z}{\partial t} - \operatorname{div}(|\nabla z|^2\nabla z) = \sum_{i=1}^N u_i(t)\frac{\partial z}{\partial x_i} \quad \text{in } Q = \Omega\times]0,T[\\ z = 0 \quad \text{in } \Sigma = \Gamma\times]0,T[\\ z(x,0) = z_0(x) \end{gathered} \tag{5.1}$$

Setting $V = W_0^{1,4}(\Omega)$, $H = L^2(\Omega)$ and $V' = W^{-1,4/3}(\Omega)$ we have $V \hookrightarrow H \hookrightarrow V'$ continuously and densely. Furthermore $V \hookrightarrow H$ compactly.

The equation (5.1) can be written in the form

$$\begin{gathered} \dot{z}(t) + Az(t) = u(t)Bz(t) \\ z(0) = z_0 \end{gathered}$$

where

(1) $A\colon V \to V'$, $\varphi \mapsto -\operatorname{div}(|\nabla\varphi|^2\nabla\varphi)$ which satisfies (H1) (see [6]).
(2) $B = (B_1, \dots, B_N)$ with $B_i\colon V \to H$, $\varphi \mapsto B_i\varphi = \varphi_{x_i}$. Hence $\|B_i\varphi\|_H \le b_i\|\varphi\|_V$, $b_i > 0$ and $\|B\varphi\|_{H^N} \le b\|\varphi\|_V$.
(3) $u = (u_1, \dots, u_N) \in \mathcal{U} = L^2(0,T;\mathbb{R}^N)$. Here

$$u(t)Bz(t) = \sum_{i=1}^N u_i(t)B_iz(t) = \sum_{i=1}^N u_i(t)\frac{\partial z(t)}{\partial x_i}.$$

The cost function becomes

$$J(u) = \|u\|^2_{L^2(0,T;\mathbb{R}^N)} + \|z\|^4_{L^4(0,T;Q)}$$

Since $\int_\Omega u(t)Bz(x,t)z(x,t)\,dx = 0$, the a priori estimates given by Lemma 3.2 become

$$\begin{gathered}\|z\|_{L^4(V)} \le K_1|z_0|^{1/2},\\ \|z\|_{L^\infty(H)} \le K_2|z_0|,\\ \|\dot z\|_{L^{4/3}(V')} \le K_3\big[\|z\|^{3/2}_{L^4(V)} + \|u\|^{3/2}_{L^2(0,T;\mathbb{R}^N)}\big]\end{gathered}$$

Corollary 5.1. *For z_0 in $L^2(\Omega)$ and u in $L^2(0,T;\mathbb{R}^N)$, the equation (3.1) with $f=0$ admits a unique solution which satisfies*

$$\begin{gathered}z \in L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;W^{1,4}_0(\Omega)),\\ \dot z \in L^{4/3}(0,T;W^{-1,4/3}(\Omega))\end{gathered}$$

Proposition 5.2. *The mapping $\Theta\colon \mathcal{U} \to \mathcal{C}([0,T];H)$, $u \mapsto z$, with z the solution of (3.1) with $f=0$. is differentiable in the sense of Fréchet, and $\Theta'_{\overline{u}}.h$ satisfies*

$$\begin{gathered}\dot y + A'_{\overline{z}(t)}y(t) = \overline{u}(t)By(t) + h(t)B\overline{z}(t)\\ y(0)=0,\end{gathered}\tag{5.2}$$

where $z = (\Theta(u))(t)$ and

$$A'_\varphi.h = -\sum_{i=1}^N \big[|\nabla\varphi|^2 h_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1\varphi_{x_i}\big]_{x_i}$$

with $\langle\nabla\varphi,\nabla h\rangle_1 = \sum_{i=1}^N \varphi_{x_i}h_{x_i}$ (φ and $h \in V$).

Proof. **1.** For $\varphi \in V$, the mapping, $A'_\varphi\colon V \to V'$,

$$h \mapsto A'_\varphi h = -\sum_{i=1}^N \big[|\nabla\varphi|^2 h_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1\varphi_{x_i}\big]_{x_i}$$

is linear and for $v \in V$ we have

$$\langle A'_\varphi h, v\rangle_{V',V} = \sum_{i=1}^N \int_\Omega f_i v_{x_i}\,dx$$

with $f_i = |\nabla\varphi|^2 h_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1\varphi_{x_i}$. Furthermore,

$$\begin{aligned}\|f_i\|^{4/3}_{L^{4/3}(\Omega)} &\le K_1\Big[\int_\Omega |\nabla\varphi|^{8/3}|h_{x_i}|^{4/3}\,dx + \int_\Omega |\langle\nabla\varphi,\nabla h\rangle_1|^{4/3}|\varphi_{x_i}|^{4/3}\,dx\Big]\\ &\le 2K_1\Big[\Big(\int_\Omega|\nabla\varphi|^4\Big)^{1/4}\Big(\int_\Omega|\nabla h|^4\Big)^{1/3}\Big].\end{aligned}$$

Then

$$\|f_i\|_{L^{4/3}(\Omega)} \le K\|\varphi\|^2_{W^{1,4}_0(\Omega)}\|h\|_{W^{1,4}_0(\Omega)}.$$

Using the norm in V' it follows that $A'_\varphi \in \mathcal{L}(V,V')$.

For φ and h in V, we have

$$A(\varphi+h) - A(\varphi) - A'_\varphi(h) = F(\varphi,h)$$

where

$$F(\varphi,h) = -\sum_{i=1}^N [|\nabla h|^2 h_{x_i} + |\nabla h|^2\varphi_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1 h_{x_i}]_{x_i}\,.$$

For $v \in V$,

$$\langle F, v\rangle_{V',V} = \sum_{i=1}^{N} \int_\Omega f_i v_{x_i}\, dx,$$

where

$$f_i = |\nabla h|^2 \varphi_{x_i} + |\nabla h|^2 h_{x_i} + 2\langle \nabla\varphi, \nabla h\rangle_1 h_{x_i}.$$

Then

$$\begin{aligned}\|f_i\|^{4/3}_{L^{4/3}(\Omega)} &\le K'\Big[\int_\Omega |\nabla\varphi|^{4/3}|\nabla h|^{8/3}\, dx + \int_\Omega |\nabla\varphi|^{4/3}|\nabla h|^{8/3}\, dx + \int_\Omega |\nabla h|^4\, dx\Big]\\ &\le K''\Big[\|\varphi\|^{4/3}_{W_0^{1,4}(\Omega)}\|h\|^{8/3}_{W_0^{1,4}(\Omega)} + \|h\|^4_{W_0^{1,4}(\Omega)}\Big].\end{aligned}$$

We deduce that

$$\|A(\varphi+h) - A(\varphi) - A'_\varphi(h)\|_{V'} \le K[\|\varphi\|_V\|h\|_V^2 + \|h\|_V^3\|]$$

Hence A is differentiable in the sense of Frechet.

2. The equation (5.2) admits a unique solution satisfying

$$y \in L^2(V)\cap L^\infty(H), \quad \dot y \in L^2(V').$$

The existence follows from a standard application of the Galerkin method and the a priori estimates obtained for (4.1). We remark that $y \in \mathcal{C}([0,T];H)$.

3. The mapping $A'\colon V \to \mathcal{L}(V,V')$, $\varphi \mapsto A'_\varphi$ is locally Lipschitz. Let φ and ψ be in V with ψ in neighbourhood of 0. For h in V, we have

$$A'_{\varphi+\psi}h = -\sum_{i=1}^{N}\big[|\nabla(\varphi+\psi)|^2 h_{x_i} + 2\langle\nabla(\varphi+\psi), \nabla h\rangle_1((\varphi+\psi))_{x_i}\big]_{x_i}$$

and $(A'_{\varphi+\psi} - A'_\varphi)h = F$, where

$$\begin{aligned}F = -\sum_{i=1}^{N}\Big[&|\nabla\psi|^2 h_{x_i} + 2\langle\nabla\varphi,\nabla\psi\rangle_1 h_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1\psi_{x_i}\\ &+ 2\langle\nabla\psi,\nabla h\rangle_1\varphi_{x_i} + 2\langle\nabla\psi,\nabla h\rangle_1\psi_{x_i}\Big]_{x_i}.\end{aligned}$$

Then for $v \in V$,

$$\langle F, v\rangle_{V',V} = \sum_{i=1}^{N}\int_\Omega f_i v_{x_i}\, dx,$$

where

$$f_i = |\nabla\psi|^2 h_{x_i} + 2\langle\nabla\varphi,\nabla\psi\rangle_1 h_{x_i} + 2\langle\nabla\psi,\nabla h\rangle_1\varphi_{x_i} + 2\langle\nabla\psi,\nabla h\rangle_1\psi_{x_i} + 2\langle\nabla\varphi,\nabla h\rangle_1\psi_{x_i}.$$

Hence

$$\|f_i\|_{L^{p'}(\Omega)} \le K\Big[\|\psi\|_V^2 + \|\psi\|_V\|\varphi\|_V\Big]\|h\|_V.$$

Since ψ is in neighbourhood of 0,

$$\|(A'_{\varphi+\psi} - A'_\varphi)h\|_{V'} \le K'\|\varphi\|_V\|\psi\|_V\|h\|_V.$$

Hence

$$\|A'_{\varphi+\psi} - A'_\varphi\|_{\mathcal{L}(V,V')} \le K''\|\psi\|_V.$$

It follows by theorem 4.4. that Θ is Frechet differentiable and its derivative $\Theta'_u.h$ satisfies (5.2) $\square$

Now the functional J can be written as $J(u) = \int_0^T L(z(t), u(t))\,dt$ with L satisfying (H6).

The differentiability of Θ and the norm ensures the differentiability of J and the expression of derivative is

$$dJ(u).h = 4\int_Q |z(x,t)|^2 z(x,t)y(x,t)\,dx\,dt + 2\int_0^T \langle u(t), h(t)\rangle_{\mathbb{R}^N}\,dt$$

where $y = \Theta'_u h$.

From Theorems 3.1, 4.4 and 4.5, we get the following result.

Corollary 5.3. *An optimal control $\overline{u}$, its corresponding state $\overline{z}$, and its adjoint state p are necessarily tied by the optimality system: For $1 \le i \le N$ and $t \in [0,T]$,*

$$\begin{gathered}
\overline{u}_i(t) = -2\int_\Omega p(x,t)\frac{\partial z}{\partial x_i}(x,t)\,dx \\
\frac{\partial \overline{z}}{\partial t} - \operatorname{div}(|\nabla \overline{z}|^2 \nabla \overline{z} = \sum_{i=1}^N \overline{u}_i(t)\frac{\partial \overline{z}}{\partial x_i} \quad \text{in } Q \\
\overline{z}(x,t) = 0 \quad \text{in } \Sigma \\
\overline{z}(x,0) = z_0(x) \quad \text{in } \Omega \\
-\frac{\partial \overline{p}}{\partial t} + A'_{\overline{z}(t)}\overline{p} = -\sum_{i=1}^N \overline{u}_i(t)\frac{\partial \overline{p}}{\partial x_i} + |\overline{z}(x,t)|^2\overline{z}(x,t) \quad \text{in } Q \\
\overline{p}(x,t) = 0 \quad \text{in } \Sigma \\
\overline{p}(x,T) = 0 \quad \text{in } \Omega
\end{gathered}$$

References

[1] V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach spaces.* Mathematics and its applications, Reidel publishing Company, 19xx.

[2] A. Benbrik, A. Addou *Existence and uniqueness of optimal contrôl for distributed-parameter bilinear system.* Journal of dynamical and control system, vol 8, number 2, pp1 41-152, April 2002.

[3] C. Bruni, G. Dipillo and G. Koch, Bilinear systems. An application class of "nearly linear" systems in theory and applications. *IEEE, transactions on automatic control*, vol. **AC 19**, no. 4, August 1974.

[4] J. Colinge and J. Rappaz, *A strongly non linear problem arising in glaciology.* R.A.I.R.O., **33**, 396–406, 1999.

[5] D. Dan Tiba, *Optimal control of nonsmooth distributed parameter systems.* Lecture notes in Mathematics, vol **1459**, Springer-Verlag.

[6] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[7] J. L. Lions and E. Magénes, *Problèmes aux limites non homgènes et applications.* Volumes **1**, **2** and **3**, Dunod, 1968.

[8] J. L. Lions, *Contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles.* Dunod, 1968.

[9] S. Migorski, *Variational stability analysis of optimal control problems for systems governed by nonlinear second order evolution equations.* J. Math. systems, estimation and control, **6**, no. 4, 1–24, (1996).

[10] D.S. Mitrinović, *Analytic Inequalities.* Springer-Verlag, Berlin,1970.

[11] R.R. Mohler, *Bilinear control processes.* Academic Press, 1975.

Abdelhaq Benbrik
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: benbrik@sciences.univ-oujda.ac.ma

Mohammed Berrajaa
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: berrajaa@sciences.univ-oujda.ac.ma

Samir Lahrech
Département de Mathématiques et Informatique, Faculté des Sciences, Université Mohammed 1er, Oujda, Maroc
E-mail address: lahrech@sciences.univ-oujda.ac.ma