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COMPACTNESS FOR A SCHRÖDINGER OPERATOR IN THE GROUND-STATE SPACE OVER $\mathbb{R}^N$

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. We investigate the compactness of the resolvent $(\mathcal{A}-\lambda I)^{-1}$ of the Schrödinger operator $\mathcal{A} = -\Delta + q(x)\bullet$ acting on the Banach space X,

$$X = \{f \in L^2(\mathbb{R}^N) : f/\varphi \in L^\infty(\mathbb{R}^N)\}, \quad \|f\|_X = \operatorname*{ess\,sup}_{\mathbb{R}^N}(|f|/\varphi),$$

$X \hookrightarrow L^2(\mathbb{R}^N)$, where φ denotes the ground state for $\mathcal{A}$ acting on $L^2(\mathbb{R}^N)$. The potential $q : \mathbb{R}^N \to [q_0, \infty)$, bounded from below, is a "relatively small" perturbation of a radially symmetric potential which is assumed to be monotone increasing (in the radial variable) and growing somewhat faster than $|x|^2$ as $|x| \to \infty$. If Λ is the ground state energy for $\mathcal{A}$, i.e. $\mathcal{A}\varphi = \Lambda\varphi$, we show that the operator $(\mathcal{A}-\lambda I)^{-1} : X \to X$ is not only bounded, but also compact for $\lambda \in (-\infty, \Lambda)$. In particular, the spectra of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ and X coincide; each eigenfunction of $\mathcal{A}$ belongs to X, i.e., its absolute value is bounded by $\mathrm{const}\cdot\varphi$.

1. Introduction

We investigate the compactness of the resolvent $(\mathcal{A} - \lambda I)^{-1}$ of the Schrödinger operator

$$\mathcal{A} \equiv \mathcal{A}_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet \tag{1.1}$$

acting not only on the standard Hilbert space $L^2(\mathbb{R}^N)$, but also on the Banach space X,

$$X \equiv X_q \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}^N) : f/\varphi \in L^\infty(\mathbb{R}^N)\}, \tag{1.2}$$

or on its predual space $X^\odot = L^1(\mathbb{R}^N; \varphi\,\mathrm{d}x)$. Here, $\varphi \equiv \varphi_q$ denotes the (normalized) ground state of $\mathcal{A}$. The electric potential $q(x)$ is assumed to be a continuous function $q : \mathbb{R}^N \to \mathbb{R}$ such that

$$q_0 \stackrel{\text{def}}{=} \inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \to +\infty \text{ as } |x| \to \infty. \tag{1.3}$$

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We denote by $\Lambda \equiv \Lambda_q$ the principal eigenvalue of the operator $\mathcal{A}$ (also called the ground state energy); hence, $\mathcal{A}\varphi = \Lambda\varphi$. It follows from (1.3) that $(\mathcal{A} - \lambda I)^{-1}$ is a compact linear operator acting on $L^2(\mathbb{R}^N)$ for every $\lambda \in (-\infty, \Lambda)$. Furthermore, $(\mathcal{A} - \lambda I)^{-1}$ is bounded as an operator acting on X, by a standard argument using the weak maximum principle. This operator, in general, is *not* compact on X; for instance, not for $q(x) = |x|^2$ ($|x| \geq r_0 > 0$).

As a direct consequence of the Riesz-Schauder theory for compact linear operators (Edwards [8, §9.10, pp. 677–682] or Yosida [23, Chapt. X, Sect. 5, pp. 283–286]), if $(\mathcal{A} - \lambda I)^{-1}$ happens to be compact on X then, for instance, all $L^2(\mathbb{R}^N)$-eigenfunctions of $\mathcal{A}$ actually belong to X. Moreover, one can prove an anti-maximum principle as well. We refer to Alziary, Fleckinger, and Takáč [1, Theorem 2.1, p. 128] for $N = 2$, [2, Theorem 2.1, p. 365] for $N \geq 2$, and to [3, Sect. 3] for further details in applications of the compactness of $(\mathcal{A} - \lambda I)^{-1}$ on X. The corresponding results, under different hypotheses, are stated in Section 3 below.

In the work reported in the present article we persue the search (started in Alziary, Fleckinger, and Takáč [3]) for finding "reasonable" *sufficient conditions* on the potential $q(x)$ that guarantee the compactness of $(\mathcal{A} - \lambda I)^{-1}$ on X. Different sufficient conditions on q, which also force $(\mathcal{A} - \lambda I)^{-1}$ compact on X, are formulated in [3, Theorem 3.2(a)]. We will take advantage of some of the recent results obtained in [3] to treat potentials q satisfying

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \quad \text{for all } x \in \mathbb{R}^N, \tag{1.4}$$

where $Q_1, Q_2 : \mathbb{R}_+ \to (0, \infty)$ are some functions ($\mathbb{R}_+ = [0, \infty)$), monotone increasing and continuous, such that $\int_1^\infty Q_1(r)^{-1/2}\, \mathrm{d}r < \infty$ and the difference $Q_2(r)^{1/2} - Q_1(r)^{1/2}$ is in some weighted Lebesgue space $L^1([1, \infty); w(r)\, \mathrm{d}r)$ with a suitable weight function $w : [1, \infty) \to [1, \infty)$ which depends on the growth of $Q_2(r)$ as $r \to \infty$. For instance, one may take $w(r) \equiv 1$ ($r \geq 1$) if $Q_2(r)$ has at most power growth near infinity, i.e., $Q_2(r) \leq \text{const} \cdot r^\alpha$ for $r \geq 1$, with some $\alpha \in (2, \infty)$; see Lemma 8.2 in §8.2 (the Appendix).

Remark 1.1. If $q : \mathbb{R}^N \to \mathbb{R}$ is continuous and satisfies (1.3), then it is easy to see that both functions

$$\tilde{Q}_1(r) \stackrel{\text{def}}{=} \min_{|y| \geq r} q(y) \quad \text{and} \quad \tilde{Q}_2(r) \stackrel{\text{def}}{=} \max_{|y| \leq r} q(y) \quad \text{of } r \in \mathbb{R}_+ \tag{1.5}$$

(where $y \in \mathbb{R}^N$) are monotone increasing and continuous, together with $\tilde{Q}_1(|x|) \leq q(x) \leq \tilde{Q}_2(|x|)$ for all $x \in \mathbb{R}^N$. In particular, one may use the pair of functions $(\tilde{Q}_1, \tilde{Q}_2)$ in place of (Q_1, Q_2) in order to impose sufficient conditions on q that force $(\mathcal{A} - \lambda I)^{-1}$ compact on X.

To be more specific, let us begin with the radially symmetric eigenvalue problem

$$\mathcal{A}v \equiv -\Delta v + q(|x|)v = \lambda v \quad \text{in } L^2(\mathbb{R}^N), \quad 0 \neq v \in L^2(\mathbb{R}^N), \tag{1.6}$$

i.e., let $q(x) \equiv q(r)$ be radially symmetric, where $r = |x| \geq 0$. First, consider the harmonic oscillator, that is, $q(r) = r^2$ for $r \geq 0$. One finds immediately that, except for the ground state φ itself, *no other* eigenfunction v of $\mathcal{A}$ (associated with an eigenvalue $\lambda \neq \Lambda$) can satisfy $v/\varphi \in L^\infty(\mathbb{R}^N)$. We refer to Davies [6, Sect. 4.3, pp. 113–117] for greater details when $N = 1$. On the other hand, if $q(r) = r^{2+\varepsilon}$ for $r \geq 0$ ($\varepsilon > 0$ – a constant), then $v/\varphi \in L^\infty(\mathbb{R}^N)$ holds for *every* eigenfunction v of $\mathcal{A}$, again by results from Davies [6], Corollary 4.5.5 (p. 122) combined with

Lemma 4.2.2 (p. 110) and Theorem 4.2.3 (p. 111). We refer to Davies and Simon [7, Theorem 6.3, p. 359] and M. Hoffmann-Ostenhof [15, Theorem 1.4(i), p. 67] for the same result under much weaker restrictions on $q(x)$. In our present article we impose similar restrictions.

Now assume $\lambda \in \mathbb{C}$, $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$, and let $u \in L^2(\mathbb{R}^N)$ be a solution of the equation

$$-\Delta u + q(x)u = \lambda u + f(x) \quad \text{in } L^2(\mathbb{R}^N) \tag{1.7}$$

in the sense of distributions on $\mathbb{R}^N$. A related sufficient condition on q is imposed in Alziary and Takáč [4, Theorem 2.1, p. 284] to obtain

$$u \geq c\varphi \quad \text{in } \mathbb{R}^N \ (\varphi\text{-}\textit{positivity}\,) \tag{1.8}$$

for $\lambda < \Lambda$, with some constant $c > 0$. Somewhat stronger sufficient conditions on q and f guarantee also

$$u \leq -c\varphi \quad \text{in } \mathbb{R}^N \ (\varphi\text{-}\textit{negativity}\,) \tag{1.9}$$

provided $\Lambda < \lambda < \Lambda + \delta$, where $\delta \equiv \delta(f) > 0$ is sufficiently small, $c > 0$; see Alziary, Fleckinger, and Takáč [1, Theorem 2.1, p. 128] for $N = 2$ and [2, Theorem 2.1, p. 365] for $N \geq 2$. Moreover, in both inequalities (1.8) and (1.9) we have $c \equiv c(\lambda) \to +\infty$ as $\lambda \to \Lambda$. Again, the harmonic oscillator $q(x) \equiv |x|^2$ and a suitably chosen positive function f provide easy counterexamples to both, (1.8) and (1.9).

From the proof of Theorem 3.2, Part (a), we will derive (1.8) whenever $f \in X^{\odot}$, $0 \leq f \not\equiv 0$ in $\mathbb{R}^N$, and $\lambda < \Lambda$. This result is stated as Theorem 3.1. Directly from Theorem 3.2, Part (c), we will derive also (1.9) whenever $f \in X$, $\int_{\mathbb{R}^N} f\varphi \, dx > 0$, and $\Lambda < \lambda < \Lambda + \delta$ ($\delta > 0$ – small enough). This is the anti-maximum principle in Theorem 3.4.

The proof of Theorem 3.2 is based on the asymptotic equivalence of $\varphi_{Q_1}(|x|)$ and $\varphi_{Q_2}(|x|)$ as $|x| \to \infty$, for Q_1 and Q_2 as described above. An important ingredient here is Lemma 4.1 which generalizes Titchmarsh' lemma [22, Sect. 8.2, p. 165] applied in Alziary and Takáč [4, Lemma 3.2, p. 286], and in Alziary, Fleckinger, and Takáč [1, p. 132] and [2, p. 366], with a slightly different class of potentials $Q_1(r)$ and $Q_2(r)$. In Alziary, Fleckinger, and Takáč [3, Lemma 4.1] the authors use a WKB-type asymptotic formula due to Hartman and Wintner [13, eq. (xxv), p. 49].

Asymptotic estimates for radially symmetric solutions of the Schrödinger equation with $q(x) = Q_j(|x|)$ ($j = 1, 2$) are combined with standard comparison results for solutions with different, but pointwise ordered (nonradial) potentials in order to control the asymptotic behavior of these solutions at infinity, and thus retain the compactness of the resolvent from the radially symmetric case (Proposition 5.2). In our approach it is crucial that the ground states $\varphi_j(x) \equiv \varphi_{Q_j}(|x|)$ corresponding to the potentials $Q_j(|x|)$ ($j = 1, 2$) are *comparable* , that is, $\varphi_1/\varphi_2 \in L^\infty(\mathbb{R}^N)$ (by Proposition 7.1) and $\varphi_2/\varphi_1 \in L^\infty(\mathbb{R}^N)$ (by Corollary 5.3).

This article is organized as follows. In the next section (Section 2) we describe the type of potentials $q(x)$ we are concerned with, together with some basic notations. Section 3 contains our main results. These results are proved in Sections 4 through 7.

2. Hypotheses and notation

We consider the Schrödinger equation (1.7), i.e.,

$$-\Delta u + q(x)u = \lambda u + f(x) \quad \text{in } L^2(\mathbb{R}^N).$$

Here, $f \in L^2(\mathbb{R}^N)$ is a given function, $\lambda \in \mathbb{C}$ is a complex parameter, and the potential $q : \mathbb{R}^N \to \mathbb{R}$ is a continuous function; we always assume that q satisfies (1.3), i.e.,

$$q_0 \stackrel{\text{def}}{=} \inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \to +\infty \quad \text{as } |x| \to \infty.$$

We interpret equation (1.7) as the operator equation $\mathcal{A}u = \lambda u + f$ in $L^2(\mathbb{R}^N)$, where the Schrödinger operator (1.1),

$$\mathcal{A} \equiv \mathcal{A}_q \stackrel{\text{def}}{=} -\Delta + q(x)\bullet \quad \text{on } L^2(\mathbb{R}^N),$$

is defined formally as follows: We first define the quadratic (Hermitian) form

$$\mathcal{Q}_q(v,w) \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \bar{w} + q(x) v \bar{w})\, \mathrm{d}x \tag{2.1}$$

for every pair $v, w \in \mathcal{V}_q$ where

$$\mathcal{V}_q \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}^N) : \mathcal{Q}_q(f,f) < \infty\}. \tag{2.2}$$

Then $\mathcal{A}$ is defined to be the Friedrichs representation of the quadratic form $\mathcal{Q}_q$ in $L^2(\Omega)$; $L^2(\Omega)$ is endowed with the natural inner product

$$(v,w)_{L^2(\mathbb{R}^N)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} v\bar{w}\, \mathrm{d}x, \quad v, w \in L^2(\Omega).$$

This means that $\mathcal{A}$ is a positive definite, selfadjoint linear operator on $L^2(\Omega)$ with domain $\mathrm{dom}(\mathcal{A})$ dense in $\mathcal{V}_q$ and

$$\int_{\mathbb{R}^N} (\mathcal{A}v)\bar{w}\, \mathrm{d}x = \mathcal{Q}_q(v,w) \quad \text{for all } v, w \in \mathrm{dom}(\mathcal{A});$$

see Kato [16, Theorem VI.2.1, p. 322]. Notice that $\mathcal{V}_q$ is a Hilbert space with the inner product $(v,w)_q = \mathcal{Q}_q(v,w)$ and the norm $\|v\|_{\mathcal{V}_q} = ((v,v)_q)^{1/2}$. The embedding $\mathcal{V}_q \hookrightarrow L^2(\mathbb{R}^N)$ is compact, by (1.3).

The principal eigenvalue $\Lambda \equiv \Lambda_q$ of the operator $\mathcal{A} \equiv \mathcal{A}_q$ can be obtained from the Rayleigh quotient

$$\Lambda \equiv \Lambda_q = \inf\left\{\mathcal{Q}_q(f,f) : f \in \mathcal{V}_q \ \text{ with } \ \|f\|_{L^2(\mathbb{R}^N)} = 1\right\}, \quad \Lambda > 0. \tag{2.3}$$

This eigenvalue is simple with the associated eigenfunction $\varphi \equiv \varphi_q$ normalized by $\varphi > 0$ throughout $\mathbb{R}^N$ and $\|\varphi\|_{L^2(\mathbb{R}^N)} = 1$; φ is a minimizer for the Rayleigh quotient above. The reader is referred to Edmunds and Evans [9] or Reed and Simon [19, Chapt. XIII] for these and other basic facts about Schrödinger operators.

We set $r = |x|$ for $x \in \mathbb{R}^N$, so $r \in \mathbb{R}_+$, where $\mathbb{R}_+ \stackrel{\text{def}}{=} [0,\infty)$. If q is a radially symmetric potential, $q(x) = q(r)$ for $x \in \mathbb{R}^N$, then also the eigenfunction φ must be radially symmetric. This follows directly from Λ being a simple eigenvalue.

Since our technique is based on a perturbation argument for a relatively small perturbation of a radially symmetric potential, which is assumed to satisfy certain differentiability and growth conditions in the radial variable $r = |x|$, $r \in \mathbb{R}_+$, we bound the potential $q : \mathbb{R}^N \to \mathbb{R}$ by such radially symmetric potentials from below and above.

In order to formulate our hypotheses on the potential $q(x)$, $x \in \mathbb{R}^N$, we first introduce the following class (Q) of *auxiliary functions* $Q(r)$ of $r = |x| \geq 0$:

(Q) $Q : \mathbb{R}_+ \to (0, \infty)$ is a continuous function that is monotone increasing in some interval $[r_0, \infty)$, $0 < r_0 < \infty$, and satisfies

$$\int_{r_0}^{\infty} Q(r)^{-1/2} \, \mathrm{d}r < \infty. \tag{2.4}$$

In particular, we have $Q'(r) \geq 0$ for a.e. $r \geq r_0$, and $Q(r) \to \infty$ as $r \to \infty$.

Example 8.1 in the Appendix, §8.1, is essential for understanding potentials of class (Q). The potential $q(x) = q(|x|)$ exhibited in this example belongs to class (Q), but does *not* belong to the analogue of this class defined in Alziary, Fleckinger, and Takáč [3]. There, instead of Q monotone increasing in $[r_0, \infty)$, it is required that there be a constant γ, $1 < \gamma \leq 2$, such that

$$\int_{r_0}^{\infty} \Big| \frac{\mathrm{d}}{\mathrm{d}r} \Big(Q(r)^{-1/2} \Big) \Big|^{\gamma} Q(r)^{1/2} \, \mathrm{d}r < \infty.$$

As a consequence, in our present paper we do not need to employ the Hartman--Wintner asymptotic formula from [13], eq. (xxv) on p. 49 or eq. (158) on p. 80. Condition (2) originally appeared in the work of Hartman and Wintner [13], on p. 49, eq. (xxiv), and on p. 80, eq. (157).

We impose the following hypothesis on the growth of q from below:

(H1) There exists a function $Q : \mathbb{R}_+ \to (0, \infty)$ of class (Q) such that

$$q(x) - \Lambda + \frac{(N-1)(N-3)}{4r^2} \geq Q(r) > 0 \quad \text{holds for all } |x| = r > r_0. \tag{2.5}$$

Remark 2.1. The term $-\Lambda + \frac{(N-1)(N-3)}{4r^2}$ has been added for convenience only; it may be left out by replacing $Q(r)$ by $Q(r) + \Lambda + 1$ if $N \geq 1$ and taking also $r_0 > 0$ large enough if $N = 2$.

In several results we need stronger hypotheses than (H1). Writing $x = rx'$ ($x \in \mathbb{R}^N \setminus \{\mathbf{0}\}$) with the radial and azimuthal variables $r = |x|$ and $x' = x/|x|$, respectively, we frequently impose the following stronger restrictions on the growth of $q(x)$ in r and the variation of $q(x)$ in x':

We assume that

(H2) There exist two functions $Q_1, Q_2 : \mathbb{R}_+ \to (0, \infty)$ of class (Q) and two positive constants $C_{12}, r_0 \in (0, \infty)$, such that

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_{12} \, Q_1(|x|) \quad \text{for all } x \in \mathbb{R}^N \,, \tag{2.6}$$

together with

$$\int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{r_0}^{s} \exp\Big(- \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}] \, \mathrm{d}t \Big) \, \mathrm{d}r \, \mathrm{d}s < \infty \,. \tag{2.7}$$

In fact, it suffices to assume inequalities (2.6) only for all $|x| = r > r_0$ with $r_0 > 0$ large enough, provided $q : \mathbb{R}^N \to (0, \infty)$ is a continuous function. Indeed, then one can find some extensions $\tilde{Q}_1, \tilde{Q}_2 : \mathbb{R}_+ \to (0, \infty)$ of class (Q) of (the restrictions of) functions $Q_1, Q_2 : [r_0 + 1, \infty) \to (0, \infty)$, respectively, from $[r_0 + 1, \infty)$ to $\mathbb{R}_+$ such that $\tilde{Q}_j(r) = Q_j(r)$ for $r \geq r_0 + 1$; $j = 1, 2$, and inequalities (2.6) hold for all $x \in \mathbb{R}^N$ with $\tilde{Q}_j$ in place of Q_j.

Remark 2.2. (a) Assuming (2.6), we will show in the Appendix, §8.2, that the latter condition, (2.7), is satisfied in the following two cases:

(i) when Q_2 has at most power growth near infinity and the condition

$$\int_{r_0}^{\infty} \Big(Q_2(r)^{1/2} - Q_1(r)^{1/2} \Big) \,\mathrm{d}r < \infty \quad \text{for some } 0 < r_0 < \infty \tag{2.8}$$

is valid; or

(ii) when Q_2 has at most exponential power growth near infinity, i.e., $Q_2(r) \leq \gamma \cdot \exp(\beta r^{\alpha})$ for all $r \geq r_0$, where $\alpha, \beta, \gamma > 0$ and $r_0 > 0$ are some constants, and

$$\int_{r_0}^{\infty} \Big(Q_2(r)^{1/2} - Q_1(r)^{1/2} \Big) \Big[1 + \log^{+}(Q_1(r)^{1/2} + Q_2(r)^{1/2}) \Big] \,\mathrm{d}r < \infty \,. \tag{2.9}$$

Clearly, condition (2.9) is stronger than (2.8).

(b) With regard to Remark 2.1 and from the point of view of spectral theory, the case

$$0 \leq Q_2(r) - Q_1(r) \leq C \quad \text{for all } r \geq r_0 \tag{2.10}$$

is of importance. Here, $C, r_0 \in (0,\infty)$ are some constants. Then the condition (cf. (2.4))

$$\int_{r_0}^{\infty} Q_2(r)^{-1/2} \,\mathrm{d}r < \infty$$

implies also condition (2.7). Indeed, combining (2.10) with the fact that Q_1 and Q_2 are of class (Q), we compute

$$\begin{aligned}
&\int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{r_0}^{s} \exp\Big(- \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}] \,\mathrm{d}t \Big) \,\mathrm{d}r \,\mathrm{d}s \\
&\leq C \int_{r_0}^{\infty} \int_{r_0}^{s} \exp\Big(- \int_{r}^{s} [Q_1(r)^{1/2} + Q_2(r)^{1/2}] \,\mathrm{d}t \Big) \,\mathrm{d}r \,\mathrm{d}s \\
&= C \int_{r_0}^{\infty} \int_{r}^{\infty} \exp\Big(- [Q_1(r)^{1/2} + Q_2(r)^{1/2}](s-r) \Big) \,\mathrm{d}s \,\mathrm{d}r \\
&= C \int_{r_0}^{\infty} [Q_1(r)^{1/2} + Q_2(r)^{1/2}]^{-1} \,\mathrm{d}r < \infty \,.
\end{aligned} \tag{2.11}$$

We will see in §7.2 (the proof of Theorem 3.2, Part (a)) that (H2) guarantees that all ground states φ_q, φ_{Q_1}, and φ_{Q_2} are *comparable* : There exist some constants $0 < \gamma_1 \leq \gamma_2 < \infty$ such that $\gamma_1 \varphi_q \leq \varphi_{Q_j} \leq \gamma_2 \varphi_q$ in $\mathbb{R}^N$; $j = 1, 2$.

Remark 2.3. Define the functions $\tilde{Q}_1, \tilde{Q}_2 : \mathbb{R}_+ \to (0,\infty)$ as in (1.5). Recall that both $\tilde{Q}_1$ and $\tilde{Q}_2$ are monotone increasing and continuous, and satisfy $\tilde{Q}_1(|x|) \leq q(x) \leq \tilde{Q}_2(|x|)$ for all $x \in \mathbb{R}^N$. The functions $Q_1, Q_2 : \mathbb{R}_+ \to (0,\infty)$ in condition (2.6) being monotone increasing and continuous, as well, this condition is equivalent to

$$Q_1(r) \leq \tilde{Q}_1(r) \leq \tilde{Q}_2(r) \leq Q_2(r) \leq C_{12}\, Q_1(r) \quad \text{for all } r \in \mathbb{R}_+ \,. \tag{2.12}$$

As far as condition (2.7) is concerned, it holds equivalently for the pair $(\tilde{Q}_1, \tilde{Q}_2)$ in place of (Q_1, Q_2) provided either of cases (i) or (ii) from Remark 2.2 occurs.

3. Main results

For any complex number $\lambda \in \mathbb{C}$ that is not an eigenvalue of the operator $\mathcal{A} = -\Delta + q(x)\bullet$ on $L^2(\mathbb{R}^N)$, we denote by

$$(\mathcal{A} - \lambda I)^{-1} = (-\Delta + q(x) \bullet - \lambda I)^{-1}$$

the resolvent of $\mathcal{A}$ on $L^2(\mathbb{R}^N)$ given by (cf. eq. (1.7))

$$u(x) = \left[(\mathcal{A} - \lambda I)^{-1} f\right](x), \quad x \in \mathbb{R}^N.$$

Now let us fix any real number $\lambda < \Lambda$ and consider the resolvent $(\mathcal{A} - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$. By the weak maximum principle (see the proof of Proposition 7.1), the operator $(\mathcal{A} - \lambda I)^{-1}$ is *positive*, that is, for $f \in L^2(\mathbb{R}^N)$ and $u = (\mathcal{A} - \lambda I)^{-1} f$ we have

$$f \geq 0 \text{ a.e. in } \mathbb{R}^N \quad \Longrightarrow \quad u \geq 0 \text{ a.e. in } \mathbb{R}^N. \tag{3.1}$$

Consequently, given any constant $C > 0$, we have also

$$|f| \leq C\varphi \text{ in } \mathbb{R}^N \quad \Longrightarrow \quad |u| \leq C(\Lambda - \lambda)^{-1}\varphi \text{ in } \mathbb{R}^N, \tag{3.2}$$

by linearity. We denote by $\mathcal{K}|_X$ the restriction of $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ to the Banach space X defined in (1.2). Hence, $\mathcal{K}|_X$ is a bounded linear operator on X with the operator norm $\leq (\Lambda - \lambda)^{-1}$, by (3.2).

Clearly, X is the dual space of the Lebesgue space $X^{\odot} = L^1(\mathbb{R}^N; \varphi\, \mathrm{d}x)$ with respect to the duality induced by the natural inner product on $L^2(\mathbb{R}^N)$. The embeddings

$$X \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow X^{\odot}$$

are dense and continuous. Furthermore, $\mathcal{K}$ possesses a unique extension $\mathcal{K}|_{X^{\odot}}$ to a bounded linear operator on $X^{\odot}$ (by Lemma 8.4 below). Finally, it is obvious that $\mathcal{K}|_X : X \to X$ is the adjoint of $\mathcal{K}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$.

3.1. **Main theorems.** Throughout this paragraph we assume that $q(x)$ is a potential that satisfies (H2). Under this hypothesis we are able to show the following *ground-state positivity* of the weak solution to the Schrödinger equation (1.7) in $X^{\odot}$.

Theorem 3.1. *Let* (H2) *be satisfied and let* $-\infty < \lambda < \Lambda$. *Assume that* $f \in X^{\odot}$ *satisfies* $f \geq 0$ *almost everywhere and* $f \not\equiv 0$ *in* $\mathbb{R}^N$. *Then the (unique) solution* $u \in X^{\odot}$ *to equation* (1.7) *(in the sense of distributions on* $\mathbb{R}^N$*) is given by* $u = (\mathcal{A} - \lambda I)^{-1}|_{X^{\odot}} f$ *and satisfies* $u \geq c\varphi$ *almost everywhere in* $\mathbb{R}^N$*, with some constant* $c \equiv c(f) > 0$.

In the literature, the inequality $u \geq c\varphi$ is often called briefly *φ-positivity.* In Protter and Weinberger [17, Chapt. 2, Theorem 10, p. 73], a similar result is referred to as the *generalized maximum principle.*

This result has been established in Alziary and Takáč [4, Theorem 2.1, p. 284] under slightly different growth hypotheses on the potentials Q_1 and Q_2 in conditions (2.6) and (2.7). In [4, eq. (2), p. 283], a closely related class (Q) is used where $Q(r)$ still satisfies a condition similar to (2.4), namely, $\int_{r_0}^{\infty} Q(r)^{-\beta}\, \mathrm{d}r < \infty$ with some constant $\beta \in (0, 1/2)$. Also the "potential variation" condition (2.7) in our present work is somewhat different from that assumed in [4, eq. (5), p. 283]. Nevertheless, our proof of Theorem 3.1 follows similar steps as does the proof of Theorem 2.1 in [4, pp. 289–290]. Theorem 3.1 will be proved in Section 7, first for $q(x) = Q(|x|)$ of class (Q), as Proposition 7.1 in §7.1, and then in its full generality in §7.3, after the proof of Theorem 3.2, a part of which will be needed (stated below as Corollary 3.3).

The central result of this paper is the following compactness theorem. Indeed, it provides answers to some questions about the solution u of problem (1.7), such as $u \in X$ and its φ-positivity or φ-negativity. Moreover, it also guarantees that the spectrum of the operator $\mathcal{A} = -\Delta + q(x)\bullet$ is the same in each of the spaces $L^2(\mathbb{R}^N)$, X, and $X^{\odot}$.

Theorem 3.2. *Let* (H2) *be satisfied. Then we have the following three statements for the resolvent* $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ *of* $\mathcal{A}$ *on* $L^2(\mathbb{R}^N)$*:*

- (a) *If* $-\infty < \lambda < \Lambda$ *then both operators* $\mathcal{K}|_X : X \to X$ *and* $\mathcal{K}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$ *are compact (and positive, see* (3.1)*).*
- (b) *If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $\mathcal{A}$*, that is,* $\mathcal{A}v = \lambda v$ *for some* $v \in L^2(\mathbb{R}^N)$*,* $v \neq 0$*, then* $v \in X$ *(* $\subset L^2(\mathbb{R}^N) \subset X^{\odot}$*) and* $\lambda \in \mathbb{R}$*,* $\lambda \geq \Lambda$*.*
- (c) *If* $\lambda \in \mathbb{C}$ *is* not *an eigenvalue of* $\mathcal{A}$*, then the restriction* $\mathcal{K}|_X$ *of* $\mathcal{K}$ *to* X *is a bounded linear operator from* X *into itself and, moreover,* $\mathcal{K}$ *possesses a unique extension* $\mathcal{K}|_{X^{\odot}}$ *to a bounded linear operator from* $X^{\odot}$ *into itself. Again, both operators* $\mathcal{K}|_X : X \to X$ *and* $\mathcal{K}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$ *are compact.*

Part (a) is the most difficult one to prove. Since $\mathcal{K}|_X : X \to X$ is compact *if and only if* $\mathcal{K}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$ is compact, by Schauder's theorem (Edwards [8, Corollary 9.2.3, p. 621] or Yosida [23, Chapt. X, Sect. 4, p. 282]), it suffices to prove that either of them is compact. Thus, our proof of Part (a) begins with the compactness of the restriction of $\mathcal{K}|_X$ to (the corresponding subspace of) radially symmetric functions with $q(x) = Q(|x|)$ of class (Q) and only for $\lambda < \Lambda$, see Lemma 6.2. So we may apply Schauder's theorem to get the compactness of the restriction of $\mathcal{K}|_{X^{\odot}}$ to radially symmetric functions with $q = Q$. Then we extend this result to $\mathcal{K}|_{X^{\odot}}$ on $X^{\odot}$ with $q = Q$ again, see Proposition 6.1. Finally, from there we derive that $\mathcal{K}|_{X^{\odot}}$ is compact for any $q(x)$ satifying (H2), first only for $\lambda < \Lambda$ and then for any $\lambda \in \mathbb{C}$ that is not an eigenvalue of $\mathcal{A}$, see Section 7, §7.2. Parts (b) and (c) are proved immediately thereafter; they will be derived from Part (a) by standard arguments based on the Riesz-Schauder theory of compact linear operators (Edwards [8, §9.10, pp. 677–682] or Yosida [23, Chapt. X, Sect. 5, pp. 283–286]).

As a direct consequence of Theorem 3.2, Part (a), the following corollary establishes the equivalence of the ground states.

Corollary 3.3. *Let* (H2) *be satisfied. Then the ground states* φ_q*,* φ_{Q_1}*, and* φ_{Q_2} *corresponding to the potentials* q*,* Q_1*, and* Q_2*, respectively, are comparable, that is, there exist some constants* $0 < \gamma_1 \leq \gamma_2 < \infty$ *such that* $\gamma_1 \varphi_q \leq \varphi_{Q_j} \leq \gamma_2 \varphi_q$ *in* $\mathbb{R}^N$*;* $j = 1, 2$*. Equivalently, we have* $X_q = X_{Q_1} = X_{Q_2}$*.*

A classical use of the compactness result, Theorem 3.2, Part (c), is the *anti-maximum principle* for the Schrödinger operator $\mathcal{A} = -\Delta + q(x)\bullet$ which complements the ground-state positivity of Theorem 3.1.

Theorem 3.4. *Let* (H2) *be satisfied and let* $f \in X$ *satisfy* $\int_{\mathbb{R}^N} f\varphi \, dx > 0$*. Then there exists a number* $\delta \equiv \delta(f) > 0$ *such that, for every* $\lambda \in (\Lambda, \Lambda + \delta)$*, the inequality* $u \leq -c\varphi$ *is valid a.e. in* $\mathbb{R}^N$ *with some constant* $c \equiv c(f) > 0$*.*

This important consequence of Theorem 3.2 was presented also in Alziary, Fleckinger, and Takáč [3, Theorem 3.4] for a class of potentials $q(x)$ satisfying different conditions (without radial symmetry). For a radially symmetric potential satisfying (H1), the anti-maximun principle has been obtained previously in Alziary, Fleckinger, and Takáč [1, Theorem 2.1, p. 128] for $N = 2$ and [2, Theorem 2.1, p. 365] for $N \geq 2$. Furthermore, in [1, 2] the function f is assumed to be a "sufficiently smooth" perturbation of a radially symmetric function from X.

Theorem 3.4 is an immediate consequence of the spectral decomposition of the resolvent of $\mathcal{A}$ as

$$(\lambda I - \mathcal{A})^{-1} = (\lambda - \Lambda)^{-1}\mathcal{P} + \mathcal{H}(\lambda) \quad \text{for} \quad 0 < |\lambda - \Lambda| < \eta, \tag{3.3}$$

see, e.g., Sweers [20, Theorem 3.2(ii), p. 259] or Takáč [21, Eq. (6), p. 67]. Here, $\lambda \in \mathbb{C}$, $\eta > 0$ is small enough, $\mathcal{P}$ denotes the spectral projection onto the eigenspace spanned by φ, and $\mathcal{H}(\lambda) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a holomorphic family of compact linear operators parametrized by λ with $|\lambda - \Lambda| < \eta$. Moreover, $\mathcal{P}$ is selfadjoint and $\mathcal{P}\mathcal{H}(\lambda) = \mathcal{H}(\lambda)\mathcal{P} = 0$ on $L^2(\mathbb{R}^N)$. Formula (3.3) is used to prove the anti--maximum principle also in [1, Eq. (6), p. 124] and [2, Eq. (6), p. 361]. The main idea of the proof of Theorem 3.4 is to show that each of the linear operators $\{\mathcal{H}(\lambda) : |\lambda - \lambda_1| < \eta\}$ is bounded not only on $L^2(\mathbb{R}^N)$ but also on X. Clearly, given the Neumann series expansion of $\mathcal{H}(\lambda)$ for $|\lambda - \Lambda| < \eta$, it suffices to show that the restriction $\mathcal{H}(\Lambda)|_X$ of $\mathcal{H}(\Lambda)$ to X is a bounded linear operator on X. But this clearly follows from Theorem 3.2, Part (c), with a help from formula (6.32) in Kato [16, Chapt. III, §6.5, p. 180] or formula (1) in Yosida [23, Chapt. VIII, Sect. 8, p. 228].

In various common versions of the anti-maximum principle in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, besides the assumption $0 \leq f \not\equiv 0$ in Ω, it is only assumed that $f \in L^p(\Omega)$ for some $p > N$ (cf. Clément and Peletier [5, Theorem 1, p. 222], Sweers [20] or Takáč [21]). For $\Omega = \mathbb{R}^N$ the authors [2, Example 4.1, pp. 377–379] have constructed an example of a simple potential $q(r)$ and a function $f(r)$, both radially symmetric, $f \in L^2(\mathbb{R}^N) \setminus X$, and $0 \leq f \not\equiv 0$ in $\mathbb{R}^N$, in which even the inequality $u \leq 0$ a.e. in $\mathbb{R}^N$ (weaker than the anti-maximum principle of Theorem 3.4) is violated. More precisely, if $|\lambda - \Lambda| > 0$ is small enough, then even $u(r) > 0$ for every $r > 0$ large enough.

4. Preliminary results

In Alziary, Fleckinger, and Takáč [3], an important ingredient in comparing the ground states $\varphi_j(x) \equiv \varphi_{Q_j}(|x|)$ corresponding to the potentials $Q_j(|x|)$ $(j = 1, 2)$ is an asymptotic formula due to Hartman and Wintner [13, eq. (xxv), p. 49]. In contrast, here we take advantage of a generalized Titchmarsh' lemma (Lemma 4.1) and a comparison result (Lemma 4.3) to obtain the equivalence of the ground states, φ_1 and φ_2, under Hypothesis (H2).

4.1. **Generalized Titchmarsh' lemma.** Throughout this paragraph we assume that $U : [R, \infty) \to (0, \infty)$ is monotone increasing and continuous, where $0 \leq R < \infty$. Hence, $U'(r) \geq 0$ for a.e. $r \geq R$. We generalize Titchmarsh' lemma (cf. Titchmarsh [22, Sect. 8.2, p. 165]) used in Alziary and Takáč [4], Lemma 3.2, p. 286.

Lemma 4.1. *Assume that $f, f' : [R, \infty) \to \mathbb{R}$ are locally absolutely continuous, $f(r) > 0$ for all $r \geq R$, and f satisfies*

$$-f'' + U(r)f \leq 0 \quad \text{for a.e. } r \geq R. \tag{4.1}$$

If $f' \le 0$ on $[R,\infty)$ then we must have

$$-f'/f \ge U(r)^{1/2} \quad \text{for all } r \ge R. \tag{4.2}$$

Proof. Upon the substitution $g = -(\log f)' = -f'/f$, where $g \ge 0$ on $[R,\infty)$, inequality (4.1) is equivalent to

$$g' \le g^2 - U(r) \quad \text{for all } r \ge R. \tag{4.3}$$

This follows from $g' = -f''/f + (f'/f)^2 = -f''/f + g^2$. Equivalently to (4.2), we claim that $g \ge U^{1/2}$ on $[R,\infty)$.

Indeed, if on the contrary

$$\delta \stackrel{\text{def}}{=} g(r_0)^2 - U(r_0) < 0 \quad \text{for some } r_0 \ge R,$$

then from (4.3) and the continuity of g and U at r_0 we deduce that there exists some $r_1 > r_0$ such that also

$$g'(r) \le g(r)^2 - U(r) < 0 \quad \text{for a.e. } r \in [r_0, r_1).$$

Consequently, with regard to $U(r) \ge U(r_0)$ for $r \ge r_0$, we get

$$g'(r) \le g(r)^2 - U(r) < g(r_0)^2 - U(r) \le g(r_0)^2 - U(r_0) = \delta < 0$$

for a.e. $r \in (r_0, r_1]$. Hence, we may take $r_1 > r_0$ arbitrarily large as long as $g(r_1) \ge 0$ holds; let us choose $r_1 > r_0$ so large that $g(r_1) = 0$. Then

$$g'(r_1) \le g(r_1)^2 - U(r_1) = -U(r_1) < 0$$

which implies $g(r_1 + s) < 0$ for every $s > 0$ small enough and thus contradicts our hypothesis $g \ge 0$ on $[R,\infty)$. Hence, we have proved $g(r)^2 - U(r) \ge 0$ for all $r \ge R$, which entails $-f'/f = g \ge U^{1/2}$ on $[R,\infty)$. □

Corollary 4.2. *In the situation of* Lemma 4.1 *above, the function f is decreasing, convex, and satisfies $f(s) \searrow 0$ and $f'(s) \nearrow 0$ as $s \nearrow \infty$. Moreover, we have*

$$\frac{f(s)}{f(r)} \le \exp\Big(- \int_r^s U(t)^{1/2}\,\mathrm{d}t \Big) \quad \text{whenever } R \le r \le s < \infty. \tag{4.4}$$

Proof. First, we integrate inequality (4.2) over the interval $[r,s]$ to get (4.4). Since $f(s) > 0$ for all $s \ge R$, and U is monotone increasing, inequality (4.4) forces $f(s) \searrow 0$ as $s \nearrow \infty$. Next, (4.1) guarantees that f is convex. It follows that $f'(s) \nearrow 0$ as $s \nearrow \infty$. □

4.2. **Asymptotic equivalence of solutions.** In this paragraph we compare positive solutions of homogeneous Schrödinger equations (or inequalities) in $[R,\infty)$ with different potentials (for $N = 1$). We start with a comparison result proved in M. Hoffmann-Ostenhof et al. [14, Lemma 3.2, p. 348].

Lemma 4.3. *Let $U_1, U_2 : [R,\infty) \to (0,\infty)$ be two continuous potentials satisfying $0 < \text{const} \le U_1 \le U_2$ for $r \ge R$, where $0 < R < \infty$. Let all $f_1, f_2, f_1', f_2' \in L^2(R,\infty)$ be locally absolutely continuous in $[R,\infty)$, and let also $f_1 > 0$ and $f_2 > 0$ for $r \ge R$. Finally, assume that*

$$-f_1'' + U_1(r) f_1 \ge 0 \quad \text{and} \quad -f_2'' + U_2(r) f_2 \le 0 \quad \text{for almost every } r > R.$$

Then we have

$$\frac{f_1}{f_2} \ge \frac{f_1(R)}{f_2(R)} \quad \text{and} \quad \frac{f_1'}{f_1} \ge \frac{f_2'}{f_2} \quad \text{for every } r \ge R.$$

In the next proposition we give a sufficient upper bound for the perturbation $Q_2(r) - Q_1(r)$ in (H2) that guarantees that all ground states φ_q, φ_{Q_1}, and φ_{Q_2} are comparable (see §7.2).

Proposition 4.4. *Let $U_1, U_2 : [R,\infty) \to (0,\infty)$ be monotone increasing and continuous, where $0 \le R < \infty$. Assume that $f_j, f_j' : [R,\infty) \to \mathbb{R}$ are locally absolutely continuous; $j = 1,2$, $f_j(r) > 0$ and $f_j'(r) \le 0$ for all $r \ge R$, and f_j satisfies*

$$-f_j'' + U_j(r) f_j = 0 \quad \textit{for a.e. } r \ge R. \tag{4.5}$$

Then for all $r \ge R$ we have

$$\begin{aligned} &\Big| \frac{\mathrm{d}}{\mathrm{d}r} \Big(\log \frac{f_1(r)}{f_2(r)} \Big) \Big| \\ &\le \int_r^\infty |U_1(s) - U_2(s)| \exp\Big(- \int_r^s [U_1(t)^{1/2} + U_2(t)^{1/2}] \,\mathrm{d}t \Big) \,\mathrm{d}s\,. \end{aligned} \tag{4.6}$$

Proof. Employing (4.5) we compute

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big[f_1 f_2 \Big(\frac{f_1'}{f_1} - \frac{f_2'}{f_2} \Big) \Big] = f_1'' f_2 - f_1 f_2'' = (U_1(r) - U_2(r))\, f_1 f_2.$$

Since $f_1'(r) f_2(r) \to 0$ and $f_1(r) f_2'(r) \to 0$ as $r \to \infty$, by Corollary 4.2, integration yields

$$-f_1(r) f_2(r) \Big(\frac{f_1'}{f_1} - \frac{f_2'}{f_2} \Big) = \int_r^\infty (U_1(s) - U_2(s))\, f_1(s) f_2(s) \,\mathrm{d}s,$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}r} \Big(\log \frac{f_1}{f_2} \Big) = -\frac{1}{f_1(r) f_2(r)} \int_r^\infty (U_1(s) - U_2(s))\, f_1(s) f_2(s) \,\mathrm{d}s$$

for all $r \ge R$. Consequently,

$$\Big| \frac{\mathrm{d}}{\mathrm{d}r} \Big(\log \frac{f_1(r)}{f_2(r)} \Big) \Big| \le \frac{1}{f_1(r) f_2(r)} \int_r^\infty |U_1(s) - U_2(s)|\, f_1(s) f_2(s) \,\mathrm{d}s$$

for all $r \ge R$. Finally, we apply inequality (4.4) with $U = U_j$ and $f = f_j$; $j = 1,2$, to the right-hand side of the inequality above to get (4.6). ☐

Remark 4.5. In the situation of Proposition 4.4 above, the functions $f_1(r)$ and $f_2(r)$ are *asymptotically equivalent near infinity* , meaning that the ratios $f_1(r)/f_2(r)$ and $f_2(r)/f_1(r)$ stay bounded as $r \to \infty$, provided the following condition is satisfied:

$$\begin{aligned} &\int_R^\infty \int_r^\infty |U_1(s) - U_2(s)| \exp\Big(- \int_r^s [U_1(t)^{1/2} + U_2(t)^{1/2}] \,\mathrm{d}t \Big) \,\mathrm{d}s \,\mathrm{d}r = \\ &\int_R^\infty |U_1(s) - U_2(s)| \int_R^s \exp\Big(- \int_r^s [U_1(t)^{1/2} + U_2(t)^{1/2}] \,\mathrm{d}t \Big) \,\mathrm{d}r \,\mathrm{d}s < \infty\,. \end{aligned} \tag{4.7}$$

This claim follows by integrating (4.6) with respect to $r \in [R,\infty)$. Note that condition (4.7) corresponds to (2.7) for Q_1 and Q_2.

Simple sufficient conditions for U_1 and U_2 are given in the Appendix, §8.2, which guarantee that (4.7) is satisfied.

5. Two known compactness results

We state in this section two compactness results obtained in Alziary, Fleckinger, and Takáč [3, Section 6] which are essential for the proof of our main Theorem 3.2. We need local compactness from Proposition 5.1 together with a compactness result by comparison of two potentials from Proposition 5.2. These propositions are valid for any potential satisying (1.3) and, thus, we refer to [3, Sections 6 and 8] for the proofs.

5.1. **A local compactness result.**

The potential q is assumed to satisfy only conditions (1.3) in this section. If $B_R(0)$ is an open ball of radius R $(0 < R < \infty)$ in $\mathbb{R}^N$ centered at the origin, let $u|_{B_R(0)}$ denote the restriction of a function $u : \mathbb{R}^N \to \mathbb{R}$ to $B_R(0)$.

Proposition 5.1. *Assume that $q : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying* (1.3)*, and let $\lambda < \Lambda$. Then, given any $0 < R < \infty$, the restricted resolvent*

$$\mathcal{R}_R : X^{\odot} \to L^1(B_R(0)) : f \mapsto u|_{B_R(0)}$$

is compact, where $u = (\mathcal{A} - \lambda I)^{-1}|_{X^{\odot}} f$.

This proposition is proved in [3, Theorem 6.1].

5.2. **Compactness by comparison of two potentials.**

Let us consider two potentials, $q_j : \mathbb{R}^N \to \mathbb{R}$ for $j = 1, 2$, each assumed to be a continuous function satisfying only conditions (1.3) in place of q. We denote by $\Lambda_j = \Lambda_{q_j}$ the principal eigenvalue of the Schrödinger operator

$$\mathcal{A}_j = \mathcal{A}_{q_j} \stackrel{\text{def}}{=} -\Delta + q_j(x) \bullet \quad \text{on } L^2(\mathbb{R}^N). \tag{5.1}$$

The associated eigenfunction $\varphi_j = \varphi_{q_j}$ is normalized by $\varphi_j > 0$ throughout $\mathbb{R}^N$ and $\|\varphi_j\|_{L^2(\mathbb{R}^N)} = 1$. Finally, we write $X_j = X_{q_j}$ and $X_j^{\odot} = L^1(\mathbb{R}^N; \varphi_j \, \mathrm{d}x)$.

The following comparison result is natural (and holds without any growth conditions other than (1.3)).

Proposition 5.2. *Assume $q_1 \le q_2$ in $\mathbb{R}^N$. Then the following statements hold:*

(a) $0 < \Lambda_1 \le \Lambda_2 < \infty$.

(b) *For each $\lambda < \Lambda_1$,*

$$f \ge 0 \ \text{ in } L^2(\mathbb{R}^N) \quad \Longrightarrow \quad (\mathcal{A}_{q_2} - \lambda I)^{-1} f \le (\mathcal{A}_{q_1} - \lambda I)^{-1} f \ \text{ in } L^2(\mathbb{R}^N).$$

(c) *Given any $\lambda < \Lambda_1$, if the restriction $(\mathcal{A}_{q_1} - \lambda I)^{-1}|_{X_1} : X_1 \to X_1$ of the resolvent $(\mathcal{A}_{q_1} - \lambda I)^{-1}$ to X_1 is weakly compact, then $(\mathcal{A}_{q_j} - \lambda I)^{-1}|_{X_1}$ is also compact for $j = 1, 2$.*

(c') *Given any $\lambda < \Lambda_1$, if the extension $(\mathcal{A}_{q_1} - \lambda I)^{-1}|_{X_1^{\odot}} : X_1^{\odot} \to X_1^{\odot}$ of the resolvent $(\mathcal{A}_{q_1} - \lambda I)^{-1}$ to $X_1^{\odot}$ is weakly compact, then $(\mathcal{A}_{q_j} - \lambda I)^{-1}|_{X_1^{\odot}}$ is also compact for $j = 1, 2$.*

Corollary 5.3. *Assume $q_1 \le q_2$ in $\mathbb{R}^N$. If the weak compactness condition (the "if" part) in* (c) *or* (c')*,* Proposition 5.2*, is satisfied, for some $\lambda < \Lambda_1$, then we have $\sup_{\mathbb{R}^N}(\varphi_2/\varphi_1) < \infty$ or, equivalently, $X_2 \hookrightarrow X_1$ is a continuous embedding.*

Proposition 5.2 and Corollary 5.3 are proved in Alziary, Fleckinger, and Takáč [3], Proposition 8.1 and Corollary 8.2, respectively. The proof of Proposition 5.2 makes use of Proposition 5.1.

6. Compactness for potentials of class (Q)

Throughout this section we consider only a radially symmetric potential q of class (Q), $q(x) = Q(|x|)$ for all $x \in \mathbb{R}^N$. All symbols $\mathcal{A}$, Λ, φ, X, $X^{\odot}$, etc. are considered only for this special type of potential. Under the hypotheses in (Q), we are able to show the following special case of Theorem 3.2, Part (a):

Proposition 6.1. *Both operators* $(\mathcal{A}-\lambda I)^{-1}|_X : X \to X$ *and* $(\mathcal{A}-\lambda I)^{-1}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$ *are compact.*

Also the compactness of $(\mathcal{A}-\lambda I)^{-1}|_X : X \to X$ is equivalent to that of $(\mathcal{A}-\lambda I)^{-1}|_{X^{\odot}} : X^{\odot} \to X^{\odot}$, by Schauder's theorem (Edwards [8, Corollary 9.2.3, p. 621] or Yosida [23, Chapt. X, Sect. 4, p. 282]); we will prove the latter one.

We split the proof of Proposition 6.1 into two paragraphs, §6.1 and §6.2. We set $\mathcal{K} = (\mathcal{A}-\lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$. In the first paragraph, §6.1, we restrict the operators $\mathcal{K}|_X$ and $\mathcal{K}|_{X^{\odot}}$ to the corresponding subspaces of radially symmetric functions and show that Proposition 6.1 is valid in these subspaces; see Lemmas 6.2 and 6.3 below. In the second paragraph, §6.2, we take advantage of Lemma 6.3 to prove the compactness of $\mathcal{K}|_{X^{\odot}}$ in Proposition 6.1.

6.1. **Compactness on the space of radial functions.** Throughout this paragraph, we denote by $X_{\rm rad}$, $L^2_{\rm rad}(\mathbb{R}^N)$, and $X^{\odot}_{\rm rad}$, respectively, the subspaces of X, $L^2(\mathbb{R}^N)$, and $X^{\odot}$ that consist of all radially symmetric functions from these spaces. All these subspaces are closed. Moreover, since the potential Q is radially symmetric, all subspaces above are invariant under the operator $\mathcal{K}|_{X^{\odot}}$. We denote by $\mathcal{K}|_{X_{\rm rad}}$, $\mathcal{K}|_{L^2_{\rm rad}(\mathbb{R}^N)}$, and $\mathcal{K}|_{X^{\odot}_{\rm rad}}$, respectively, the restrictions of $\mathcal{K}|_{X^{\odot}}$ to the spaces $X_{\rm rad}$, $L^2_{\rm rad}(\mathbb{R}^N)$, and $X^{\odot}_{\rm rad}$. These restrictions have similar properties as $\mathcal{K}|_X$, $\mathcal{K}$, and $\mathcal{K}|_{X^{\odot}}$, respectively, above.

Lemma 6.2. *Under the hypotheses in* (Q) *the operator* $\mathcal{K}|_{X_{\rm rad}} : X_{\rm rad} \to X_{\rm rad}$ *is compact.*

By Schauder's theorem again (Edwards [8, Corollary 9.2.3, p. 621] or Yosida [23, Chapt. X, Sect. 4, p. 282]), this lemma is equivalent to

Lemma 6.3. *Under the hypotheses in* (Q) *the operator* $\mathcal{K}|_{X^{\odot}_{\rm rad}} : X^{\odot}_{\rm rad} \to X^{\odot}_{\rm rad}$ *is compact.*

The following simple consequence of (H1) on the growth of the ground state φ is needed in the proof of Lemma 6.2:

Lemma 6.4. *Under* (H1) *the function* $P : (r_0,\infty) \to (0,\infty)$*, defined by* $P(r) = 2\,Q(r)^{1/2}$ *for* $r > r_0$*, is monotone increasing,* $\int_{r_0}^{\infty} P(r)^{-1}\,\mathrm{d}r < \infty$*, and*

$$p(r) \overset{\rm def}{=} -\frac{N-1}{r} - 2(\log\varphi(r))' \geq P(r) > 0 \quad \textit{holds for all } r > r_0. \tag{6.1}$$

Proof. Using the radial Schrödinger equation for φ,

$$-\varphi''(r) - \frac{N-1}{r}\,\varphi'(r) + q(r)\varphi(r) = \Lambda\varphi(r) \quad \text{for } r \geq R,$$

we observe that the function $v(r) = r^{(N-1)/2}\varphi(r) > 0$ must satisfy

$$-v''(r) + \Big(q(r) - \Lambda + \frac{(N-1)(N-3)}{4r^2}\Big)\,v(r) = 0 \quad \text{for } r \geq R.$$

It follows from (2.5) that

$$-v''(r) + Q(r)v(r) \le 0 \quad \text{for } r \ge R.$$

Next, we claim that $v'(r) \le 0$ for all $r \ge R$. Indeed, since $v''(r) \ge Q(r)v(r) \ge 0$, the derivative v' is nondecreasing on $[R, \infty)$. Therefore, if $v'(r_0) > 0$ for some $r_0 \ge R$, then $v'(r) \ge v'(r_0) > 0$ for every $r \ge r_0$, which contradicts $\int_{r_0}^{\infty} v(r)\, r^{(N-1)/2}\, \mathrm{d}r < \infty$. Finally, we may apply the generalized Titchmarsh' lemma to conclude that the function $g = -(\log v)' = -v'/v$ satisfies $g(r) \ge Q(r)^{1/2}$ for all $r \ge R$. We compute

$$g(r) = -\frac{\mathrm{d}}{\mathrm{d}r} \log\Big(r^{(N-1)/2}\varphi(r)\Big) = -\frac{N-1}{2r} - \frac{\mathrm{d}}{\mathrm{d}r} \log\varphi(r) = \tfrac{1}{2}\, p(r)$$

with $g(r) \ge Q(r)^{1/2}$ for all $r \ge R$. This proves (6.1). The remaining claims follow from the properties of class (Q). □

We prove Lemma 6.2 directly using Arzelà-Ascoli's compactness criterion for continuous functions on the one point compactification $\mathbb{R}_+^* = \mathbb{R}_+ \cup \{\infty\}$ of $\mathbb{R}_+$. The metric on $\mathbb{R}_+^*$ is defined by

$$d(x,y) \overset{\text{def}}{=} \begin{cases} \frac{|x-y|}{1+|x-y|} & \text{for } x, y \in \mathbb{R}_+; \\ 1 & \text{for } 0 \le x < y = \infty \ \text{ or } \ 0 \le y < x = \infty; \\ 0 & \text{for } \ x = y = \infty. \end{cases}$$

We denote by $C(\mathbb{R}_+^*)$ the Banach space of all continuous functions on the compact metric space $\mathbb{R}_+^*$ endowed with the supremum norm from $L^\infty(\mathbb{R}_+)$.

Proof of Lemma 6.2. Given $f, u \in X_{\mathrm{rad}}$, $u = \mathcal{K}f$ is equivalent with the ordinary differential equation

$$-u''(r) - \frac{N-1}{r}\, u'(r) + q(r)u(r) = \lambda u(r) + f(r) \quad \text{for } 0 < r < \infty$$

supplemented by the conditions

$$\lim_{r\to 0+} u'(r) = 0 \quad \text{and} \quad \sup_{0<r<\infty} \left|\frac{u(r)}{\varphi(r)}\right| \le (\Lambda - \lambda)^{-1} \cdot \sup_{0<r<\infty} \left|\frac{f(r)}{\varphi(r)}\right|.$$

Clearly, the former one is a boundary condition at zero that follows from the radial symmetry, whereas the latter one follows from the weak maximum principle.

Substituting $g = f/\varphi$ and $v = u/\varphi$, combined with

$$-\varphi''(r) - \frac{N-1}{r}\, \varphi'(r) + q(r)\varphi(r) = \Lambda\varphi(r) \quad \text{for } 0 < r < \infty,$$

we have equivalently

$$-v''(r) - \frac{N-1}{r}\, v'(r) - 2(\log\varphi(r))'\, v'(r) + (\Lambda - \lambda)v(r) = g(r) \quad \text{for } 0 < r < \infty \tag{6.2}$$

subject to the conditions

$$\lim_{r\to 0+} v'(r) = 0 \quad \text{and} \quad \sup_{0<r<\infty} |v(r)| \le (\Lambda - \lambda)^{-1} \cdot \sup_{0<r<\infty} |g(r)|. \tag{6.3}$$

Then $\mathcal{K}|_{X_{\mathrm{rad}}}$ is compact on X_{rad} if and only if the linear operator $\mathcal{K}_\varphi : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$, defined by

$$\mathcal{K}_\varphi g \overset{\text{def}}{=} v = \varphi^{-1} \cdot \mathcal{K}(g\varphi) \quad \text{for } g \in L^\infty(\mathbb{R}_+),$$

is compact.

We will apply Arzelà-Ascoli's compactness criterion in the Banach space $C(\mathbb{R}_+^*)$ in order to show that the image $\mathcal{K}_\varphi\left(\overline{\mathcal{B}}_{L^\infty(\mathbb{R}_+)}\right)$ of the unit ball

$$\overline{\mathcal{B}}_{L^\infty(\mathbb{R}_+)} = \left\{g \in L^\infty(\mathbb{R}_+) : \|g\|_{L^\infty(\mathbb{R}_+)} \le 1\right\}$$

has compact closure in $C(\mathbb{R}_+^*)$. Since $L^\infty(\mathbb{R}_+)$ is a Banach lattice, it suffices to show that $\mathcal{K}_\varphi\Big(\overline{\mathcal{B}}^+_{L^\infty(\mathbb{R}_+)}\Big)$ has compact closure in $C(\mathbb{R}_+^*)$, where

$$\overline{\mathcal{B}}^+_{L^\infty(\mathbb{R}_+)} = \left\{g \in \overline{\mathcal{B}}_{L^\infty(\mathbb{R}_+)} : g \ge 0 \text{ in } \mathbb{R}_+\right\}.$$

Clearly, the function v from (6.2) and (6.3) above satisfies $v \in C^1(\mathbb{R}_+)$; we will show also $v \in C(\mathbb{R}_+^*)$. Therefore, we need to show that the linear operator

$$\mathcal{K}_\varphi : L^\infty(\mathbb{R}_+) \to C(\mathbb{R}_+^*) \subset L^\infty(\mathbb{R}_+)$$

is compact.

So let $g \in L^\infty(\mathbb{R}_+)$ be arbitrary with $0 \le g(r) \le 1$ for $r \in \mathbb{R}_+$. Hence, $v = \mathcal{K}_\varphi g$ satisfies $v \in C^1(\mathbb{R}_+)$ and also $0 \le v(r) \le (\Lambda - \lambda)^{-1}$, by (3.2). It follows that the function

$$g^\sharp \stackrel{\text{def}}{=} g - (\Lambda - \lambda)v$$

satisfies $-1 \le g^\sharp \le 1$, and the derivative $w \stackrel{\text{def}}{=} v'$ verifies the ordinary differential equation

$$-w'(r) - \frac{N-1}{r}\, w(r) - 2(\log\varphi(r))'\, w(r) = g^\sharp(r) \quad \text{for } 0 < r < \infty \tag{6.4}$$

subject to the conditions

$$\lim_{r\to 0+} w(r) = 0 \quad \text{and} \quad \sup_{0<r<\infty}\left|\int_0^r w(s)\,\mathrm{d}s\right| \le (\Lambda - \lambda)^{-1}.$$

The latter condition has been obtained from

$$\int_0^r w(s)\,\mathrm{d}s = v(r) - v(0) \quad \text{with} \quad 0 \le v(r) \le (\Lambda - \lambda)^{-1}$$

for all $r \ge 0$. Since w is continuous, this condition implies that there exists a sequence $\{r_n\}_{n=1}^\infty \subset \mathbb{R}_+$ such that $r_n \to \infty$ and $w(r_n) \to 0$ as $n \to \infty$.

The differential equation (6.4) is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1}\varphi(r)^2\, w(r)\right) = r^{N-1}\varphi(r)^2\, g^\sharp(r) \quad \text{for } 0 < r < \infty.$$

After integration, we thus arrive at

$$r^{N-1}\varphi(r)^2\, w(r) - s^{N-1}\varphi(s)^2\, w(s) = \int_r^s t^{N-1}\varphi(t)^2\, g^\sharp(t)\,\mathrm{d}t$$

whenever $0 \le r, s < \infty$. Applying $\lim_{s\to 0+} w(s) = 0$ we obtain

$$r^{N-1}\varphi(r)^2\, w(r) = -\int_0^r t^{N-1}\varphi(t)^2\, g^\sharp(t)\,\mathrm{d}t \quad \text{for all } r \ge 0. \tag{6.5}$$

Taking $s = r_n$ and letting $n \to \infty$ we obtain also

$$r^{N-1}\varphi(r)^2\, w(r) = \int_r^\infty t^{N-1}\varphi(t)^2\, g^\sharp(t)\,\mathrm{d}t \quad \text{for all } r \ge 0. \tag{6.6}$$

Here we have used the facts that $s^{N-1}\varphi(s)^2 \to 0$ as $s \to \infty$ together with $r_n \to \infty$ and $w(r_n) \to 0$ as $n \to \infty$. Recall the normalization $\int_0^\infty \varphi(t)^2\, t^{N-1}\,\mathrm{d}t = \sigma_{N-1}^{-1}$, where σ_{N-1} stands for the surface area of the unit sphere in $\mathbb{R}^N$. Below we will take advantage of formulas (6.5) and (6.6) to estimate $|w(r)|$ as $r \to 0+$ and $r \to \infty$, respectively.

Because of $|g^\sharp| \le 1$, eq. (6.5) yields $|w| \le w_0^\sharp$ where $w_0^\sharp : \mathbb{R}_+ \to \mathbb{R}_+$ is the function defined by $w_0^\sharp(0) = 0$ and

$$r^{N-1}\varphi(r)^2\, w_0^\sharp(r) = \int_0^r t^{N-1}\varphi(t)^2\,\mathrm{d}t \quad \text{for all } r > 0.$$

Using $\lim_{r\to 0+}\varphi(r) = \varphi(0) > 0$ we conclude that

$$\begin{aligned}\lim_{r\to 0+}\frac{w_0^\sharp(r)}{r} &= \lim_{r\to 0+}\frac{1}{r}\int_0^r \Big(\frac{t}{r}\Big)^{N-1}\Big(\frac{\varphi(t)}{\varphi(r)}\Big)^2\,\mathrm{d}t\\ &= \lim_{r\to 0+}\frac{1}{r}\int_0^r \Big(\frac{t}{r}\Big)^{N-1}\,\mathrm{d}t = \frac{1}{N}\,.\end{aligned}$$

Because of $|g^\sharp| \le 1$, eq. (6.6) yields $|w| \le w_\infty^\sharp$ where $w_\infty^\sharp : (0,\infty) \to \mathbb{R}_+$ is the function defined by

$$w_\infty^\sharp(r) = r^{-(N-1)}\varphi(r)^{-2}\int_r^\infty t^{N-1}\varphi(t)^2\,\mathrm{d}t \quad \text{for all } r > 0. \tag{6.7}$$

Next, we wish to show that $w_\infty^\sharp(r) \le P(r)^{-1}$ holds for all $r > r_0$, where $P(r) = 2\,Q(r)^{1/2}$. To this end, notice first that eq. (6.7) is equivalent with

$$w_\infty^\sharp(r) = \int_r^\infty \exp\Big(-\int_r^s p(t)\,\mathrm{d}t\Big)\,\mathrm{d}s, \quad r > r_0,$$

where $p(t)$ is given by formula (6.1). In particular, the function $w_\infty^\sharp(r)$ satisfies the differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}r}\,w_\infty^\sharp(r) + p(r)\,w_\infty^\sharp(r) = 1 \quad \text{for } r_0 < r < \infty.$$

By Lemma 6.4, we have $p(t) \ge P(t) \ge P(r)$ whenever $r_0 < r \le t < \infty$. Hence, we can estimate

$$\begin{aligned}w_\infty^\sharp(r) &\le \int_r^\infty \exp\Big(-\int_r^s P(r)\,\mathrm{d}t\Big)\,\mathrm{d}s\\ &= \int_r^\infty e^{-P(r)\,(s-r)}\,\mathrm{d}s = \int_0^\infty e^{-P(r)\,s}\,\mathrm{d}s = P(r)^{-1}.\end{aligned}$$

To summarize our estimates for the functions $w_0^\sharp : \mathbb{R}_+ \to \mathbb{R}_+$ and $w_\infty^\sharp : (0,\infty) \to \mathbb{R}_+$ in the inequalities $|w| \le w_0^\sharp$ for $r \ge 0$ and $|w| \le w_\infty^\sharp$ for $r > r_0$, we observe that both functions $w_0^\sharp$ and $w_\infty^\sharp$ are continuously differentiable and satisfy the estimates

$$|w(r)| \le w_0^\sharp(r) \le C\,r \quad \text{for } 0 \le r \le r_0, \tag{6.8}$$

$$|w(r)| \le w_\infty^\sharp(r) \le P(r)^{-1} \quad \text{for } r_0 < r < \infty, \tag{6.9}$$

where $C > 0$ is a constant. Recall that $\int_{r_0}^\infty P(r)^{-1}\,\mathrm{d}r < \infty$, by condition (2.4).

Consequently, for g ranging over $L^\infty(\mathbb{R}_+)$ with $0 \le g \le 1$ in $\mathbb{R}_+$, the set of functions $v = \mathcal{K}_\varphi g \in C^1(\mathbb{R}_+)$ defined above is *uniformly equicontinuous* on the compact metric space $\mathbb{R}_+^*$, thanks to

$$v(r) = v(0) + \int_0^r w(s)\,\mathrm{d}s = v(\infty) - \int_r^\infty w(s)\,\mathrm{d}s \quad \text{for } 0 \le r < \infty.$$

The limit $v(\infty) = \lim_{r\to\infty} v(r) \in \mathbb{R}$ exists by (6.9). Furthermore, owing to $0 \le v(r) \le (\Lambda - \lambda)^{-1}$ for $r \in \mathbb{R}_+$, this set is also *uniformly bounded* on $\mathbb{R}_+^*$. Thus, by Arzelà-Ascoli's compactness criterion, the set $\mathcal{K}_\varphi\Big(\overline{\mathcal{B}}^+_{L^\infty(\mathbb{R}_+)}\Big)$ has compact closure in $C(\mathbb{R}_+^*)$.

We have proved that the linear operator $\mathcal{K}|_{X_{\mathrm{rad}}}$ is compact on X_{rad} and, moreover, its image satisfies $\mathcal{K}(X_{\mathrm{rad}}) \subset C(\mathbb{R}_+^*)$. $\square$

6.2. **Compactness on the entire space** X**.** We keep the assumption $q(x) = Q(|x|)$ for all $x \in \mathbb{R}^N$. Recall $\lambda < \Lambda$ and $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$. This time we will show first that the operator $\mathcal{K}|_{X^\odot}$ is compact on $X^\odot$. We derive this result from the compactness of its restriction $\mathcal{K}|_{X^\odot_{\mathrm{rad}}}$ to $X^\odot_{\mathrm{rad}}$ which we have already established in the previous paragraph.

To prove the compactness of $\mathcal{K}|_{X^\odot}$, we will apply the well-known compactness criterion of Fréchet and Kolmogorov in the Lebesgue space $X^\odot = L^1(\mathbb{R}^N; \varphi\,\mathrm{d}x)$; see Edwards [8, Theorem 4.20.1, p. 269] or Yosida [23, Chapt. X, Sect. 1, p. 275]. We denote by

$$\overline{\mathcal{B}}_{X^\odot} = \big\{ f \in X^\odot : \|f\|_{X^\odot} \le 1 \big\}$$

the closed unit ball centered at the origin in the Banach lattice $X^\odot = L^1(\mathbb{R}^N; \varphi\,\mathrm{d}x)$.

Lemma 6.5. *Given any $\varepsilon > 0$, there exists a number $R \equiv R(\varepsilon) \in (0,\infty)$ such that for every $f \in \overline{\mathcal{B}}_{X^\odot}$ and $u = \mathcal{K}|_{X^\odot} f$ we have*

$$\int_{|x|\ge R} |u(x)|\,\varphi(|x|)\,\mathrm{d}x \le \varepsilon. \tag{6.10}$$

The proof of this lemma is given in Alziary, Fleckinger, and Takáč [3], proof of Lemma 7.4, where [3, Lemma 7.3] has to be replaced by our Lemma 6.3.

Proof of Proposition 6.1. It suffices to prove that $\mathcal{K}|_{X^\odot}$ is compact. Let $0 < R < \infty$. Since the restricted resolvent $\mathcal{R}_R : X^\odot \to L^1(B_R(0)) : f \mapsto u|_{B_R(0)}$ is compact, by Proposition 5.1, so is $\mathcal{R}_R^\natural : X^\odot \to X^\odot : f \mapsto \chi_{B_R(0)}\, u$, where $u = \mathcal{K}|_{X^\odot} f$ and $\chi_{B_R(0)}$ denotes the characteristic function of the open ball $B_R(0) \subset \mathbb{R}^N$. Moreover, applying Lemma 6.5, we get $\mathcal{R}_R^\natural \to \mathcal{K}|_{X^\odot}$ uniformly on $\overline{\mathcal{B}}_{X^\odot}$ as $R \to \infty$. We invoke a well-known approximation theorem (Edwards [8, Theorem 9.2.6, p. 622] or Yosida [23, Chapt. X, Sect. 2, p. 278]) to conclude that also the limit operator $\mathcal{K}|_{X^\odot} : X^\odot \to X^\odot$ must be compact. $\square$

7. Positivity and compactness for $q(x)$ nonradial

We begin with the proof of Theorem 3.1 for potentials q of class (Q) only.

7.1. **Positivity for potentials of class** (Q)**.** Throughout this paragraph we consider only a radially symmetric potential q of class (Q), $q(x) = Q(|x|)$ for all $x \in \mathbb{R}^N$. Therefore, all symbols $\mathcal{A}$, Λ, φ, X, $X^\odot$, etc. are considered only for this special type of potential. We now prove Theorem 3.1 for the Schrödinger equation

$$-\Delta u + Q(|x|)u = \lambda u + f(x) \quad \text{in } X^\odot. \tag{7.1}$$

Proposition 7.1. *Let $Q(r)$ be of class* (Q) *and $-\infty < \lambda < \Lambda$. Assume that $f \in X^\odot$ satisfies $f \ge 0$ almost everywhere and $f \not\equiv 0$ in $\mathbb{R}^N$. Then the (unique) solution $u \in X^\odot$ to the Schrödinger equation* (7.1) *(in the sense of distributions on*

$\mathbb{R}^N$) is given by $u = (\mathcal{A} - \lambda I)^{-1}|_{X^\odot} f$ and satisfies $u \geq c\varphi$ almost everywhere in $\mathbb{R}^N$, with some constant $c \equiv c(f) > 0$.

This proposition is proved in Alziary and Takáč [4, Theorem 2.1, p. 284] under a slightly different growth hypothesis on the monotone increasing potential $q(x) \equiv q(r)$.

Proof. The proof of our proposition follows the same pattern as does the proof of Theorem 2.1 in [4, pp. 289–290]. We leave the details to the reader. □

The results of the previous section (§5.2) allow us to finally remove the restriction that q be radially symmetric, i.e., we consider a potential $q : \mathbb{R}^N \to \mathbb{R}$ that satisfies (H2).

7.2. **Compactness of $\mathcal{K}|_X$ for q nonradial.**

Proof of Theorem 3.2. Part (a): According to (H2), potentials q, Q_1, and Q_2 satisfy (2.6), that is,

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_{12}\, Q_1(|x|) \quad \text{for all } x \in \mathbb{R}^N.$$

Consequently, these potentials satisfy also conditions (1.3) in place of q. We denote by Λ_q, Λ_{Q_1}, and Λ_{Q_2} the principal eigenvalues of the Schrödinger operators $\mathcal{A}_q$, $\mathcal{A}_{Q_1}$, and $\mathcal{A}_{Q_2}$ with potentials q, Q_1, and Q_2, respectively. The associated eigenfunctions φ_q, φ_{Q_1}, and φ_{Q_2} are normalized by being positive throughout $\mathbb{R}^N$ and having the $L^2(\mathbb{R}^N)$ norm $= 1$.

First, we have $0 < \Lambda_{Q_1} \leq \Lambda_q \leq \Lambda_{Q_2} < \infty$, by Proposition 5.2, Part (a). From Proposition 6.1 we infer that, given any $\lambda < \Lambda_{Q_1}$, the restriction $(\mathcal{A}_{Q_1} - \lambda I)^{-1}|_{X_{Q_1}} : X_{Q_1} \to X_{Q_1}$ of the resolvent $(\mathcal{A}_{Q_1} - \lambda I)^{-1}$ to X_{Q_1} is compact (hence, also weakly compact). By Proposition 5.2, Part (c), the same is true of the restrictions $(\mathcal{A}_q - \lambda I)^{-1}|_{X_{Q_1}}$ and $(\mathcal{A}_{Q_2} - \lambda I)^{-1}|_{X_{Q_1}}$ to X_{Q_1}. Hence, we can apply Corollary 5.3 to conclude that $\sup_{\mathbb{R}^N}(\varphi_q/\varphi_{Q_1}) < \infty$ and $\sup_{\mathbb{R}^N}(\varphi_{Q_2}/\varphi_{Q_1}) < \infty$. Equivalently, both $X_q \hookrightarrow X_{Q_1}$ and $X_{Q_2} \hookrightarrow X_{Q_1}$ are continuous embeddings.

Second, let us denote

$$V_j(r) \stackrel{\text{def}}{=} Q_j(r) - \Lambda + \frac{(N-1)(N-3)}{4r^2} \quad \text{for } r > r_0;\ j = 1, 2, \tag{7.2}$$

where $0 < r_0 < \infty$ is large enough, such that $|(N-1)(N-3)|/4r_0^2 \leq 1$ and $V_j(r) > 0$ for all $r > r_0$. Consequently,

$$Q_j(r) - \Lambda - 1 \leq V_j(r) \leq Q_j(r) - \Lambda + 1 \quad \text{for all } r > r_0;\ j = 1, 2.$$

Therefore, we may apply first Remarks 2.2, Part (b), and 4.5, and then Lemma 4.3 to conclude that $\varphi_{Q_j}(r)$ is comparable with any positive solution $f_j(r)$ of

$$-f_j'' + Q_j(r) f_j = 0 \quad \text{for a.e. } r > r_0,$$

such that $f_j(r) \to 0$ as $r \to \infty$. The functions f_1 and f_2 are comparable by results from paragraph §4.2, Proposition 4.1 and Remark 4.5 for $U_j = Q_j$; $j = 1, 2$. Thus, we have obtained

$$0 < c' \leq \frac{\varphi_{Q_1}(r)}{\varphi_{Q_2}(r)} \leq c'' < \infty \quad \text{for all } r > r_0, \tag{7.3}$$

where c' and c'' are some constants. This formula yields $\sup_{\mathbb{R}^N}(\varphi_{Q_1}/\varphi_{Q_2}) < \infty$ or, equivalently, $X_{Q_1} \hookrightarrow X_{Q_2}$ is a continuous embedding.

Finally, let us rewrite the equation $\mathcal{A}_q\varphi_q = \Lambda_q\varphi_q$ for $\varphi_q \in X_{Q_1} = X_{Q_2}$ as

$$-\Delta\varphi_q + Q_2(|x|)\varphi_q = f(x) \quad \text{in } X^{\odot}_{Q_2},$$

where

$$f(x) = [Q_2(|x|) - q(x) + \Lambda_q]\,\varphi_q(x) \ge \Lambda_q\varphi_q(x) > 0, \quad x \in \mathbb{R}^N,$$

by condition (2.6). Notice that

$$\sup_{\mathbb{R}^N}(\varphi_q/\varphi_{Q_1}) < \infty, \quad \sup_{\mathbb{R}^N}(\varphi_{Q_2}/\varphi_{Q_1}) < \infty, \quad \text{and} \quad \sup_{\mathbb{R}^N}(\varphi_{Q_1}/\varphi_{Q_2}) < \infty,$$

combined with $\int_{\mathbb{R}^N} Q_2\,\varphi^2_{Q_2}\,\mathrm{d}x < \infty$, yield $\int_{\mathbb{R}^N} Q_2\,\varphi_q\,\varphi_{Q_2}\,\mathrm{d}x < \infty$, that is, $Q_2\,\varphi_q \in X^{\odot}_{Q_2} = L^1(\mathbb{R}^N;\varphi_{Q_2}\,\mathrm{d}x)$. Consequently, also $(Q_2 - q)\varphi_q \in X^{\odot}_{Q_2}$ which guarantees $f \in X^{\odot}_{Q_2}$. We apply Proposition 7.1 with $Q = Q_2$ and $\lambda = 0 < \Lambda_Q$ to conclude that $\inf_{\mathbb{R}^N}(\varphi_q/\varphi_{Q_2}) > 0$ or, equivalently, $X_{Q_2} \hookrightarrow X_q$ is a continuous embedding.

Summarizing the results proved in this section for φ_q, φ_{Q_1}, and φ_{Q_2}, we arrive at $X_q = X_{Q_1} = X_{Q_2}$, i.e., $\gamma_1\varphi_q \le \varphi_{Q_1}, \varphi_{Q_2} \le \gamma_2\varphi_q$ everywhere in $\mathbb{R}^N$, where $0 < \gamma_1 \le \gamma_2 < \infty$ are some constants. As we already know that the restriction $(\mathcal{A}_q - \lambda I)^{-1}|_{X_{Q_1}}$ to X_{Q_1} is compact, Part (a) follows immediately.

Part (b): In the remaining part of the proof we abbreviate $\mathcal{A} = \mathcal{A}_q$, $\Lambda = \Lambda_q$, and $X = X_q$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}$, that is, $\mathcal{A}v = \lambda v$ for some $v \in L^2(\mathbb{R}^N)$, $v \ne 0$. Since $\mathcal{A}$ is positive definite and selfadjoint on $L^2(\Omega)$, its inverse $\mathcal{A}^{-1}$ is bounded on $L^2(\mathbb{R}^N)$. Property (1.3) implies that $\mathcal{A}^{-1}$ is also compact. Consequently, $\lambda \in \mathbb{R}$ and $\lambda \ge \Lambda > 0$. Given $v \in L^2(\mathbb{R}^N)$, $v \ne 0$, it follows that equation $\mathcal{A}v = \lambda v$ is equivalent with $\mathcal{A}^{-1}v = \lambda^{-1}v$. By Part (a), also the restriction $\mathcal{A}^{-1}|_X$ to X is compact. Now we can apply Lemma 8.4 with $\mathcal{T} = \mathcal{A}^{-1}|_X$ compact on X to obtain the conclusion of Part (b).

Part (c): Assume that $\lambda \in \mathbb{C}$ is not an eigenvalue of $\mathcal{A}$. With regard to Part (a) we may restrict ourselves to the case $\lambda \notin (-\infty, \Lambda)$. Hence, by the Riesz-Schauder theory applied to $\mathcal{A}^{-1}$, which is compact on $L^2(\mathbb{R}^N)$, λ is in the resolvent set of $\mathcal{A}$ and the resolvent $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ is compact on $L^2(\mathbb{R}^N)$. We refer to Edwards [8, Theorem 9.10.2, p. 679] or Yosida [23, Chapt. X, Theorem 5.1, p. 283] for the Riesz-Schauder theory. Consequently, the following identities hold on $L^2(\mathbb{R}^N)$:

$$\mathcal{K}\left(\lambda^{-1}I - \mathcal{A}^{-1}\right) = \left(\lambda^{-1}I - \mathcal{A}^{-1}\right)\mathcal{K} = \lambda^{-1}\mathcal{A}^{-1}\,. \tag{7.4}$$

In particular, λ^{-1} cannot be an eigenvalue of $\mathcal{A}^{-1}$. So λ^{-1} is not an eigenvalue of $\mathcal{A}^{-1}|_X$ either. The restriction $\mathcal{A}^{-1}|_X$ being compact on X, by Part (a), we may apply Lemma 8.4 with $\mathcal{T} = \mathcal{A}^{-1}|_X$ again to conclude that the restriction $\lambda^{-1}I - \mathcal{A}^{-1}|_X$ of $\lambda^{-1}I - \mathcal{A}^{-1}$ to X has a bounded inverse, say, $\mathcal{L} = \left(\lambda^{-1}I - \mathcal{A}^{-1}|_X\right)^{-1}$. Hence, from (7.4) we deduce $\mathcal{K}|_X = \lambda^{-1}\mathcal{L}\left(\mathcal{A}^{-1}|_X\right)$ which shows that also $\mathcal{K}|_X$ is compact on X as claimed.

The proof of Theorem 3.2 is now complete. □

7.3. **Positivity for a nonradial potential $q(x)$.**

Proof of Theorem 3.1. Let $-\infty < \lambda < \Lambda_q$ and $u = (\mathcal{A}_q - \lambda I)^{-1}|_{X^{\odot}_q} f$. Since $0 \le f \in X^{\odot}_q$, we may apply the weak maximum principle to get $0 \le u \in X^{\odot}_q$. Hence, it suffices to prove our theorem for $g = \min\{f, \varphi_q\}$ in place of f, that is, for $0 \le f \le \varphi_q$ a.e. and $f \not\equiv 0$ in $\mathbb{R}^N$. This forces also $0 \le u \le (\Lambda_q - \lambda)^{-1}\varphi_q$ a.e. and $u \not\equiv 0$ in $\mathbb{R}^N$, by the weak maximum principle again.

Similarly as in the proof of Theorem 3.2, Part (a) above, let us rewrite the equation $\mathcal{A}_q u = \lambda u + f$ for $u \in X_q$, with $f \in X_q$, $X_q = X_{Q_1} = X_{Q_2}$, as

$$-\Delta u + Q_2(|x|)u = \lambda u + g(x) \quad \text{in } X_{Q_2}^{\odot},$$

where

$$g(x) = [Q_2(|x|) - q(x)]\, u(x) + f(x) \geq f(x), \quad x \in \mathbb{R}^N,$$

by condition (2.6) and $u \geq 0$ a.e. in $\mathbb{R}^N$. Again, we combine Corollary 3.3 with

$$\int_{\mathbb{R}^N} Q_2\, \varphi_{Q_2}^2 \,\mathrm{d}x < \infty$$

to get $\int_{\mathbb{R}^N} Q_2\, u\, \varphi_{Q_2} \,\mathrm{d}x < \infty$, that is, $Q_2\, u \in X_{Q_2}^{\odot} = L^1(\mathbb{R}^N; \varphi_{Q_2}\,\mathrm{d}x)$. Consequently, also $(Q_2 - q)u \in X_{Q_2}^{\odot}$ which guarantees $g \in X_{Q_2}^{\odot}$. We apply Proposition 7.1 with $Q = Q_2$ and $\lambda < \Lambda_q \leq \Lambda_Q = \Lambda_{Q_2}$ to conclude that $\inf_{\mathbb{R}^N}(u/\varphi_{Q_2}) > 0$ or, equivalently, $u \geq c\varphi_q$ a.e. in $\mathbb{R}^N$, with some constant $c \equiv c(f) > 0$. □

8. Appendix

8.1. **An example of a monotone radial potential.** Here we give an example of a radially symmetric potential $q(x) = q(r)$ which belongs to class (Q), but does *not* belong to the analogue of this class defined in Alziary, Fleckinger, and Takáč [3, eqs. (13) and (14)]. More precisely, it fails to satisfy condition (2) for *any* $\gamma > 0$ ([3, eq. (14)]). This example illustrates how "large" class (Q) actually is.

For $r \in \mathbb{R}_+$ we set either $q'(r) = 0$ or else $q'(r) = 2\, q(r)^{3/2}$, which yields a "very fast" growth of $q(r)$ on a sequence of pairwise disjoint, nonempty intervals $(n - \varrho_n, n + \varrho_n)$; $n = 1, 2, 3, \dots$, of total length $2\sum_{n=1}^{\infty} \varrho_n = 1$, where $\varrho_n \to 0$ sufficiently fast as $n \to \infty$, say, $\varrho_n = \mathcal{O}(1/n^3)$.

Example 8.1 ([3, Example 3.6])**.** We define $q : \mathbb{R}_+ \to (0, \infty)$ by $q(r) = \theta(r)^{-2}$ for $r \in \mathbb{R}_+$, where $\theta : \mathbb{R}_+ \to (0, 1]$ is a monotone decreasing, piecewise linear, continuous function defined as follows: Let $\{\varrho_n\}_{n=1}^{\infty} \subset (0, 1/2)$ be a sequence of numbers satisfying

$$\sum_{n=1}^{\infty} \varrho_n = 1/2\,. \tag{8.1}$$

Given $r \geq 0$, we set $\theta(0) = 1$ and

$$\frac{\mathrm{d}\theta}{\mathrm{d}r}(r) = \begin{cases} -1 & \text{if } |r - n| < \varrho_n \text{ for some } n \in \mathbb{N}; \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathbb{N} = \{1, 2, 3, \dots\}$. Setting $R_0 = 1$ and abbreviating

$$R_n = 1 - 2\sum_{k=1}^{n} \varrho_k > 0 \quad \text{for } n = 1, 2, \dots,$$

we compute for $r \geq 0$:

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1 - \varrho_1; \\ R_{n-1} - ((r - n) + \varrho_n) & \text{if } |r - n| < \varrho_n \text{ for some } n \in \mathbb{N}; \\ R_n & \text{if } \varrho_n \leq r - n \leq 1 - \varrho_{n+1} \text{ for some } n \in \mathbb{N}. \end{cases}$$

Clearly, $\theta : \mathbb{R}_+ \to (0,1]$ is monotone decreasing, piecewise linear, and continuous. It satisfies $\theta(r) \to 0$ as $r \to 0+$, by (8.1). Next, we compute

$$\begin{aligned}\int_{1-\varrho_1}^{\infty} \theta(r)\,\mathrm{d}r &= \sum_{n=1}^{\infty}(R_{n-1} - \varrho_n)\cdot 2\varrho_n + \sum_{n=1}^{\infty} R_n(1-\varrho_{n+1}-\varrho_n) \\ &< 1 + 2\sum_{n=1}^{\infty} R_n\,.\end{aligned}$$

Furthermore, for any $\gamma > 0$ we get

$$\begin{aligned}&\int_0^{\infty} \Big|\frac{\mathrm{d}}{\mathrm{d}r}\Big(q(r)^{-1/2}\Big)\Big|^{\gamma} q(r)^{1/2}\,\mathrm{d}r \\ &= \sum_{n=1}^{\infty} \int_{|r-n|<\varrho_n} \left[R_{n-1} - ((r-n)+\varrho_n)\right]^{-1}\,\mathrm{d}r > 2\sum_{n=1}^{\infty} \varrho_n\, R_{n-1}^{-1}\,.\end{aligned}$$

We will have an example of a potential $q(r)$ with the desired properties as soon as we find a sequence $\{\varrho_n\}_{n=1}^{\infty} \subset (0,1/2)$ that satisfies the following conditions:

$$\sum_{n=1}^{\infty} \varrho_n = 1/2\,, \quad \sum_{n=1}^{\infty} R_n < \infty\,, \quad \text{and} \quad \sum_{n=1}^{\infty} \varrho_n\, R_{n-1}^{-1} = \infty\,. \tag{8.2}$$

A simple choice of such ϱ_n's is, for instance,

$$\varrho_n = \frac{1}{2}\Big(\frac{1}{n^2} - \frac{1}{(n+1)^2}\Big) = \frac{1+\frac{1}{2n}}{n(n+1)^2} \quad \text{for } n = 1,2,\dots,$$

which renders

$$R_n = (n+1)^{-2} \quad \text{and} \quad \varrho_n\, R_{n-1}^{-1} = \frac{1+\frac{1}{2n}}{n(1+\frac{1}{n})^2} \quad \text{for } n = 1,2,\dots.$$

It is easy to see that these ϱ_n's satisfy all conditions in (8.2).

8.2. **Potentials with at most (exponential) power growth.** Here we verify the claims from cases (i) and (ii) in Remark 2.2, Part (a). Let us set $S(r) = Q_1(r)^{1/2} + Q_2(r)^{1/2}$ for $r \geq r_0$. Motivated by the computations in (2.11), we simply compare the end point value $S(s)$ with the average value $(s-r)^{-1}\int_r^s S(t)\,\mathrm{d}t$ for all $r_0 \leq r < s < \infty$.

Potentials with at most power growth.

Lemma 8.2. *Assume that $S(t) = Q_1(t)^{1/2} + Q_2(t)^{1/2}$ $(t \geq r_0)$ satisfies*

$$S(s) \leq \frac{C}{s-r}\int_r^s S(t)\,\mathrm{d}t \quad \textit{for all } \ r_0 \leq r < s < \infty, \tag{8.3}$$

where $C \geq 1$ is a constant. If condition (2.8) is valid then also condition (2.7) is satisfied. In particular, given any constant $\alpha \geq 0$, the function $S(t) = \mathrm{const}\cdot t^{\alpha} \geq 0$ obeys inequality (8.3) with $C = 1+\alpha$.

Proof. From (8.3) we deduce

$$\begin{aligned}\int_{r_0}^{s} \exp\Big(-\int_r^s S(t)\,\mathrm{d}t\Big)\,\mathrm{d}r &\le \int_{r_0}^{s} \exp\Big(-C^{-1}(s-r)S(s)\Big)\,\mathrm{d}r \\ &= \frac{C}{S(s)}\big[1-\exp\big(-C^{-1}(s-r_0)S(s)\big)\big] \\ &\le \frac{C}{S(s)}\end{aligned}$$

for all $s \ge r_0$. Consequently, (2.7) follows from

$$\begin{aligned}&\int_{r_0}^{\infty} (Q_2(s)-Q_1(s)) \int_{r_0}^{s} \exp\Big(-\int_r^s [Q_1(t)^{1/2}+Q_2(t)^{1/2}]\,\mathrm{d}t\Big)\,\mathrm{d}r\,\mathrm{d}s \\ &\le \int_{r_0}^{\infty} (Q_2(s)-Q_1(s))\,\frac{C}{S(s)}\,\mathrm{d}s \\ &= C\int_{r_0}^{\infty} \Big(Q_2(s)^{1/2}-Q_1(s)^{1/2}\Big)\,\mathrm{d}s < \infty\,,\end{aligned}$$

by condition (2.8).

If $S(t) = c\cdot t^{\alpha}$, with some constants $c, \alpha \ge 0$, then we have

$$\begin{aligned}(s-r)S(s) = c(s-r)s^{\alpha} &= c\,s^{1+\alpha}(1-(r/s)) \\ &\le c\,s^{1+\alpha}\big(1-(r/s)^{1+\alpha}\big) \\ &= c(s^{1+\alpha}-r^{1+\alpha}) \\ &= c(1+\alpha)\int_r^s t^{\alpha}\,\mathrm{d}t \\ &= (1+\alpha)\int_r^s S(t)\,\mathrm{d}t\end{aligned}$$

for all $r_0 \le r \le s < \infty$. ∎

Potentials with at most exponential power growth. Now we weaken condition (8.3) as follows.

Lemma 8.3. *Assume that $S(t) = Q_1(t)^{1/2} + Q_2(t)^{1/2}$ $(t \ge r_0)$ satisfies*

$$\frac{S(s)}{1+\log^+ S(s)} \le \frac{C}{s-r}\int_r^s S(t)\,\mathrm{d}t \quad \text{for all } r_0 \le r < s < \infty, \tag{8.4}$$

where $\log^+ = \max\{\log, 0\}$ and $C \ge 1$ is a constant. Then condition (2.9) implies that also condition (2.7) holds. In particular, given any constants $\alpha, \beta \ge 0$ and $\gamma > 0$, the function $S(t) = \gamma\cdot\exp(\beta t^{\alpha}) > 0$ obeys inequality (8.4) with $C = \alpha_1(1+\log^+\gamma^{-1})$ where $\alpha_1 = \max\{\alpha, 1\}$.

Proof. From (8.4) we deduce

$$\begin{aligned}\int_{r_0}^{s} \exp\Big(-\int_r^s S(t)\,\mathrm{d}t\Big)\,\mathrm{d}r &\le \int_{r_0}^{s} \exp\Big(-\frac{(s-r)S(s)}{C\,(1+\log^+ S(s))}\Big)\,\mathrm{d}r \\ &= \frac{C\,(1+\log^+ S(s))}{S(s)}\Big[1-\exp\Big(-\frac{(s-r)S(s)}{C\,(1+\log^+ S(s))}\Big)\Big] \\ &\le \frac{C\,(1+\log^+ S(s))}{S(s)}\end{aligned}$$

for all $s \geq r_0$. Consequently, (2.7) follows from

$$\begin{aligned}
&\int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{r_0}^{s} \exp\Big(- \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}] \, \mathrm{d}t \Big) \, \mathrm{d}r \, \mathrm{d}s \\
&\leq \int_{r_0}^{\infty} (Q_2(s) - Q_1(s)) \, \frac{C\,(1 + \log^+ S(s))}{S(s)} \, \mathrm{d}s \\
&= C \int_{r_0}^{\infty} \Big(Q_2(s)^{1/2} - Q_1(s)^{1/2} \Big) \Big[1 + \log^+ (Q_1(r)^{1/2} + Q_2(r)^{1/2}) \Big] \, \mathrm{d}r < \infty \,,
\end{aligned}$$

by condition (2.9).

We leave the example $S(t) = \gamma \cdot \exp(\beta t^\alpha)$ to the reader as an easy exercise. □

8.3. **Extensions of certain symmetric operators.** We present a few obvious, but necessary facts about extensions of bounded symmetric operators defined on X to $L^2(\mathbb{R}^N)$ and $X^{\odot}$. The following lemma is an easy consequence of the Riesz--Thorin interpolation theorem (M. Reed and B. Simon [18, Sect. IX.4, Theorem IX.17, p. 27]). It is applied to the resolvent $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$, for $\lambda < \Lambda$, which is bounded on X by inequality (3.2), and to similar operators as well.

Lemma 8.4. *Let q, φ, X, and $X^{\odot}$ be as in* Section 3. *Assume that $\mathcal{T} : X \to X$ is a bounded linear operator that satisfies the symmetry condition*

$$\int_{\mathbb{R}^N} (\mathcal{T} f) \, \bar{g} \, \mathrm{d}x = \int_{\mathbb{R}^N} f \, \overline{(\mathcal{T} g)} \, \mathrm{d}x \quad \text{for all } f, g \in X. \tag{8.5}$$

Then $\mathcal{T}$ possesses a unique extension $\mathcal{T}|_{X^{\odot}}$ to a bounded linear operator on $X^{\odot}$, $\mathcal{T}$ is the adjoint of $\mathcal{T}|_{X^{\odot}}$, and $\mathcal{T}|_{X^{\odot}}$ restricts to a bounded selfadjoint operator $\mathcal{T}|_{L^2(\mathbb{R}^N)}$ on $L^2(\mathbb{R}^N)$. Moreover, the operator norms of $\mathcal{T}|_{L^2(\mathbb{R}^N)}$, $\mathcal{T}|_{X^{\odot}}$, and $\mathcal{T}$, respectively, satisfy

$$\| \mathcal{T}|_{L^2(\mathbb{R}^N)} \|_{L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)} \leq \| \mathcal{T}|_X \|_{X^{\odot} \to X^{\odot}} = \| \mathcal{T} \|_{X \to X} < \infty \,. \tag{8.6}$$

The spectrum of $\mathcal{T}|_{L^2(\mathbb{R}^N)}$ is contained in the spectrum of $\mathcal{T}$. Finally, if $\mathcal{T}$ is compact, then so is $\mathcal{T}|_{L^2(\mathbb{R}^N)}$ and their spectra coincide; in particular, if $\mathcal{T}|_{L^2(\mathbb{R}^N)} v = \lambda v$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $v \in L^2(\mathbb{R}^N) \setminus \{0\}$, then $\lambda \in \mathbb{R}$ and $v \in X$.

This lemma is proved in Alziary, Fleckinger, and Takáč [3, Lemma 4.3].

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