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A 2D CLIMATE ENERGY BALANCE MODEL COUPLED WITH A 3D DEEP OCEAN MODEL

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Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor

Abstract. We study a three dimensional climate model which represents the coupling of the mean surface temperature with the ocean temperature. We prove the existence of a bounded weak solution by a fixed point argument.

1. Introduction

In the last decades, several works about global climate energy balance models (EBM) have appeared which study the evolution of the mean surface temperature of the Earth (see for example, Díaz [7], Díaz and Tello [9], Ghil and Childress [12], Hetzer [13], North [15], etc.). From the mathematical point of view, two dimensional EBM (latitude – longitude) has an spatial domain given by a Riemannian manifold without boundary $\mathcal{M}$ simulating the Earth surface, as follows

$$\begin{gathered} c(x)u_t - \operatorname{div}(k(x)|\nabla u|^{p-2}\nabla u) + R_e(x,u) \in R_a(x,u) \quad (0,T)\times\mathcal{M}, \\ u(x,0) = u_0(x) \quad \mathcal{M}, \end{gathered} \tag{1.1}$$

where u represents the mean surface temperature, R_e and R_a the emitted and absorbed energy, respectively. R_a depends on the planetary coalbedo β (which is eventually discontinuous on u). One dimensional models (early proposed) assume uniform temperature over each parallel. By calling x the sine of the latitude and doing a change to spherical coordinates, we obtain

$$\begin{gathered} c(x)u_t - (k(x)(1-x^2)^{p/2}|u_x|^{p-2}u_x)_x + R_e(x,u) \in R_a(x,u) \quad (0,T)\times(-1,1), \\ (1-x^2)^{p/2}|u_x|^{p-2}u_x = 0 \quad x \in \{-1,1\}, \\ u(x,0) = u_0(x) \quad (-1,1), \end{gathered} \tag{1.2}$$

In these models, the effect of the oceans is only considered in a implicit and empirical way in the spatial dependence of the coefficients. However, some works about the

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rapid climatic change in Glacial-Holocene transition (see p.e. Berger et al [4]) show that the cause could be the past changes in deep water formation. In this work we study a model including the effect of the deep Ocean based in the model proposed by Watts - Morantine [19]. In such 3D model, the atmosphere temperature comes to an energy balance model closed to (1.1).

2. The Mathematical Model

The model represents the evolution of the temperature in a global ocean Ω with constant depth H. The upper boundary of Ω is a manifold $\mathcal{M}$ simulating the Earth surface. The bottom of the ocean is a manifold $\mathcal{N}$. For the mathematical treatment, we assume that $\mathcal{M}$ and $\mathcal{N}$ are C^∞ two dimensional compact connected oriented Riemannian Manifold of $\mathbb{R}^3$ without boundary and $\operatorname{dist}(\mathcal{M},\mathcal{N}) = H$. For example $\mathcal{M}$ and $\mathcal{N}$ can be two spheres with the same center and radius R and $R - H$, respectively.

The shape of the spatial domain Ω suggest us to use a suitable coordinate system (θ, φ, z) where $z \in (-H, 0)$ is measured from $\mathcal{M}$ to $\mathcal{N}$ in the orthogonal direction to the tangent space $T_p\mathcal{M}$.

The governing equation for the ocean interior is a heat equation with transport,

$$\frac{\partial U}{\partial t} - \operatorname{div}(\nabla U) + w\frac{\partial U}{\partial z} = 0 \quad (t, x) \in (0, T) \times \Omega, \tag{2.1}$$

where U is the temperature, t the time, the variable $z \in (-H, 0)$ represents the depth and w is the vertical velocity.

At the bottom of the ocean ($z = -H$), the advective and diffusive transports must be equal

$$\hat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 \quad \text{in } (0, T) \times \mathcal{N}, \tag{2.2}$$

where $\hat{F}$ is linear on the gradient $\nabla_{\mathcal{N}} U$ and the gradient is understood in the sense of the Riemannian metric of $\mathcal{N}$. Analogously for the F and $\mathcal{M}$ below.

The boundary condition at $z = 0$ comes from an energy balance

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2}\nabla_{\mathcal{M}} u) + \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + R_e(u) \in \frac{1}{\rho c} QS(x)\beta(U) \tag{2.3}$$

where $u : \mathcal{M} \to \mathbb{R}$ and

$$U\big|_{\mathcal{M}} = u. \tag{2.4}$$

Here, u represents the temperature on $\mathcal{M}$. R_e is the energy emitted by the surface $\mathcal{M}$ to the outer space by cooling. The diffusion operator for $p = 2$ is the Laplace - Beltrami operator on the manifold $\mathcal{M}$. The pioneering climate energy balance models considered the linear case $p = 2$, but, later, Stone [17] proposed the case $p = 3$ in order to consider the negative feedback in the eddy fluxes. In order to include both cases we consider $p \geq 2$. $\beta(u)$ the coalbedo function (a nondecreasing function of u of the type $\beta(u) = 0.7$ if $u > -10 + \epsilon$, $\beta(u) = 0.4$ if $u < -10 - \epsilon$ with $\epsilon \geq 0$). The coalbedo function will be treated in the general class of multivalued graphs. More precisely, we shall assume that β is a maximal monotone graph of $\mathbb{R}^2$ such that β is bounded (i.e. $m \leq b \leq M$ for any $b \in \beta(s)$ for any $s \in \mathbb{R}$). We recall that β was assumed to be *multivalued* at $u = -10$ by Budyko [6] and β *locally Lipschitz* by Sellers [16]. The function $S : \mathcal{M} \to \mathbb{R}$ is the insolation function and $Q > 0$ the Solar constant. This kind of models are very sensitive to small fluctuations of Solar and terrestrial parameters, see Hetzer [13], Arcoya, Diaz, Tello [1], Diaz, Tello [11] and the references therein. The equation describing the energy balance at $\mathcal{M}$ is similar to the equation studied in Diaz, Tello [9] except for the two terms describing the thermal communication with the deep ocean through the advective and diffusive transports of heat.

We also need to add the initial conditions at $t = 0$,

$$U(0,\cdot) = U_0(\cdot) \quad \text{and} \quad u(0,\cdot) = u_0(\cdot). \tag{2.5}$$

We notice that u is also an unknown in this model.

The model can be simplified by considering that $\mathcal{M}$ is the sphere of radius R and the temperature is constant over each parallel. In this case, the spatial domain Ω is reduced to a rectangle with boundary $\Gamma_0 \cup \Gamma_H \cup \Gamma_1$ and the model obtained (P_{2D}) is

$$\begin{gathered}
\frac{\partial U}{\partial t} - \frac{K_H}{R^2}\frac{\partial}{\partial x}((1-x^2)\frac{\partial U}{\partial x}) - K_v\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\Omega,\\
wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\Gamma_H,\\
D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\Big((1-x^2)^{p/2}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\Big) + K_V\frac{\partial U}{\partial n}\\
+ wx\frac{\partial U}{\partial x} + \mathcal{G}(U) \in \frac{1}{\rho c}QS(x)\beta(x,U) \quad \text{in } (0,T)\times\Gamma_0,\\
(1-x^2)^{p/2}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} = 0 \quad \text{in } (0,T)\times\Gamma_1,\\
U(0,x,z) = U_0(x,z) \quad \text{in } \Omega,\\
U(0,x,0) = u_0(x) \quad \text{in } \Gamma_0,
\end{gathered}$$

where x represents the sine of the latitude. The physical description and the numerical approximation if $p = 2$ and where β depends only on x is in Watts Morantine [19]. The proof of existence of bounded weak solution to this 2-D model is in Diaz and Tello [10]. The number of steady states of (P_{2D}) was studied in [11].

We notice that in the 3D model some physical constants were assumed equal to one.

The goal of the present work is to prove the existence of weak solutions to the 3D evolution model with dynamical and diffusive boundary condition described in (2.1)-(2.5).

3. Existence of weak solutions

The model represents the temperature evolution in a global ocean Ω. The unknown are $U : \Omega \to \mathbb{R}$ and $u : \mathcal{M} \to \mathbb{R}$. The independent spatial variables are (θ, φ, z) where $z \in (-H, 0)$. See [9] and [14] to the expression of the operators ∇

and div in the new coordinates. Let (P) be the problem

$$
\begin{gathered}
\frac{\partial U}{\partial t} - \operatorname{div}(\nabla U) + w\frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\Omega,\\
\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2}\nabla_{\mathcal{M}} u) + \frac{\partial U}{\partial n} + F(x,\nabla_{\mathcal{M}} u) + \mathcal{G}(u)\\
\in \frac{1}{\rho c} QS(x)\beta(U) \quad \text{in } (0,T)\times\mathcal{M}\\
U\big|_{\mathcal{M}} = u\\
\hat{F}(x,\nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 \quad \text{in } (0,T)\times\mathcal{N}\\
U(0,x,z) = U_0(x,z) \quad \text{in } \Omega,\\
u(0,\cdot) = u_0(\cdot) \quad \text{on } \mathcal{M},
\end{gathered}
\tag{3.1}
$$

where

- (H1) Ω is a bounded and open set of $\mathbb{R}^3$ with constant depth H and $\partial\Omega = \mathcal{M}\cup\mathcal{N}$. $\mathcal{M}$ and $\mathcal{N}$ are C^∞ two dimensional compact connected oriented Riemannian Manifold of $\mathbb{R}^3$ without boundary and $\operatorname{dist}(\mathcal{M},\mathcal{N}) = H$.

Here $\nabla_{\mathcal{M}}$ and div are understood in the sense of the Riemannian metric on $\mathcal{M}$. We study the existence of solutions of (3.1) under the following structure hypotheses:

- (H2) β is a bounded maximal monotone graph, i.e. $|v| \le M$ for all $v \in \beta(s)$, and all $s \in D(\beta) = \mathbb{R}$.
- (H3) $\mathcal{G}: \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\mathcal{G}(0) = 0$ and $|\mathcal{G}(\sigma)| \ge C|\sigma|^r$ for some $r > 0$.
- (H4) $S: \mathcal{M} \to \mathbb{R}$, $s_1 \ge S(x) \ge s_0 > 0$ a.e. $x \in \mathcal{M}$.
- (H5) $f \in L^\infty((0,T)\times\Omega)$,
- (H6) $F: \mathcal{M}\times T\mathcal{M} \to \mathbb{R}$ and $\hat{F}: \mathcal{N}\times T\mathcal{N} \to \mathbb{R}$ are linear on the tangent bundle spaces $T\mathcal{M}$ and $T\mathcal{N}$ with bounded coefficients.
- (H7) $w \in C^1(\overline{\Omega})$.

We define the functional space on the Riemannian manifold $\mathcal{M}$, as follows,

$$
V := \{u \in L^2(\mathcal{M}) : \nabla_{\mathcal{M}} u \in L^p(T\mathcal{M})\}
$$

where $T\mathcal{M} = \cup_{p\in\mathcal{M}} T_pM$ is the tangent bundle space (see Aubin [2]).

Definition 3.1. We say that the pair $(U,u) \in C([0,T]; L^2(\Omega)\times L^2(\mathcal{M}))$ is a bounded weak solution of (3.1) if

- (i) $(U,u) \in L^\infty((0,T)\times\Omega)\times L^\infty((0,T)\times\mathcal{M}) \cap L^2(0,T;H^1(\Omega))\times L^p(0,T;V)$,
- (ii) there exists $Z \in L^\infty((0,T)\times\mathcal{M})$ with $Z(t,x) \in \beta(u(t,x))$ a.e. $(t,x) \in (0,T)\times\mathcal{M}$ such that

$$\begin{aligned}
&\int_\Omega U(T,x)\phi(T,x)dA - \int_0^T \langle \phi_t(t,x), U(t,x)\rangle_{H^1(\Omega)'\times H^1(\Omega)} dt \\
&+ \int_0^T \int_\Omega \nabla U \nabla \phi \, dA\, dt + \int_0^T \int_\Omega w \frac{\partial U}{\partial z} \phi \, dA \, dt \\
&- \int_0^T \int_{\mathcal{M}} \frac{\partial U}{\partial n} \phi \, dS \, dt + \int_0^T \int_{\mathcal{N}} \hat{F}(x, \nabla_{\mathcal{N}}) \phi \, dS \, dt \\
&= \int_\Omega U_0(x) \phi(0,x) dA,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathcal{M}} u(T,x)\psi(T,x)dA - \int_0^T \langle \psi_t(t,x), u(t,x)\rangle_{V'\times V} dt \\
&+ \int_0^T \int_{\mathcal{M}} |\nabla u|^{p-2} \nabla u \nabla \psi \, dS\, dt + \int_0^T \int_{\mathcal{M}} \mathcal{G}(u) \psi \, dS \, dt \\
&+ \int_0^T \int_{\mathcal{M}} \frac{\partial U}{\partial n} \psi \, dS \, dt + \int_0^T \int_{\mathcal{M}} F(x, \nabla_{\mathcal{M}}) \psi \, dS \, dt \\
&= \int_0^T \int_{\mathcal{M}} QS(x) Z(t,x) \psi \, dA \, dt + \int_0^T \int_{\mathcal{M}} f \psi \, dA \, dt + \int_{\mathcal{M}} u_0(x) \psi(0,x) dS
\end{aligned}$$

for every test function $(\phi, \psi) \in L^2(0,T; H^1(\Omega)) \times L^p((0,T); W^{1,p}(\mathcal{M}))$ such that $(\phi_t, \psi_t) \in L^2(0,T; H^1(\Omega)') \times L^{p'}(0,T; V')$. Here $<,>_{V'\times V}$ denotes the duality product in $V' \times V$.

Theorem 3.2. *Let $U_0 \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\mathcal{M})$. Then there exists at least a bounded weak global solution of* (3.1).

The main idea for the proof is to construct an operator $\mathcal{T}$ and to find a fixed point which is the solution of (3.1). The proof consist of several steps.

Step 1. For every $h \in L^\infty((0,T) \times \mathcal{M})$ we consider the problem (P_h) by replacing the coalbedo term in (3.1) by h. The proof of the existence of solution of (P_h) is inspired in Diaz and Jimenez [8] and Bejaranu, Diaz and Vrabie [3]. We define the vectorial operator $\mathcal{A}$

$$\mathcal{A}(U,u) \longmapsto (AU, Bu)$$

on the domain

$$D(\mathcal{A}) = \{(U,u) \in L^2(\Omega) \times L^2(\mathcal{M}) : AU \in L^2(\Omega), Bu \in L^2(\mathcal{M}), U_{|\mathcal{M}} = u\},$$

where

$$AU = -\operatorname{div}(\nabla U) + w \frac{\partial U}{\partial z},$$

$$Bu = -\operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + \frac{\partial U}{\partial n} + F(x, \nabla_{\mathcal{M}} u) + \mathcal{G}(u).$$

The existence of solution of (P_h) is a consequence of the following properties of $\mathcal{A}$.

Lemma 3.3. *There exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$, we have:*

(i) *$\mathcal{A} + \lambda I$ is T-accretive in $L^2(\Omega) \times L^2(\mathcal{M})$.*
(ii) *$R(\mathcal{A} + \lambda I) = L^2(\Omega) \times L^2(\mathcal{M})$.*

Note that (i) allows us to prove a comparison principle for the system

$$\begin{gathered}
\lambda U + AU = f \quad \text{in } L^2(\Omega) \\
\lambda u + Bu = g \quad \text{in } L^2(\mathcal{M}) \\
U_{|\mathcal{M}} = u \\
\widehat{F}(x, \nabla_{\mathcal{N}} U) + \frac{\partial U}{\partial z} = 0 \quad \mathcal{N}.
\end{gathered} \tag{3.2}$$

In fact, if $f_1 \le f_2$ and $g_1 \le g_2$ then the solutions of (3.2) with $f = f_1$, $g = g_1$ and of (3.2) with $f = f_2$, $g = g_2$ satisfy

$$U_1 \le U_2, \quad \text{and} \quad u_1 \le u_2.$$

Remark 3.4. Lemma 3.3 and similar arguments to those in [9, Lemma 3] allow us to prove the existence of a maximal and minimal solutions.

To prove (ii) in Lemma 3.3 we notice that the operator B can be expressed as $B_1 + B_2 + B_3$, where B_1 and B_2 are maximal monotone operators in $L^2(\mathcal{M})$,

$$B_1 u = -\operatorname{div}(|\nabla_{\mathcal{M}} u|^{p-2} \nabla_{\mathcal{M}} u) + \mathcal{G}(u)$$

and the pseudo-differential operator

$$B_2 u = \frac{\partial U}{\partial n},$$

where U is the solution of the problem

$$\begin{gathered}
\lambda U + AU = f \quad \text{in } L^2(\Omega) \\
U_{|\mathcal{M}} = u.
\end{gathered}$$

The operator B_3 is defined by

$$B_3 u = F(\nabla_{\mathcal{M}} u).$$

This operator is not necessarily monotone but it is dominated (in some sense) by the operators B_1 and B_2. Consequently, it is possible to apply the abstracts results of perturbation of maximal monotone operators (see e.g. [5, Proposition 2.10]), and we arrive to the conclusion.

Step 2. We follow the proof in [9, Theorem 3]. We define the operator $\mathcal{T} : h \to g$ where $g \in \beta(u_h)$ and u_h is the solution of (P_h). It is easy to see that every fixed point of $\mathcal{T}$ is a solution of (3.1).

We prove that $\mathcal{T}$ satisfies the hypotheses of Kakutani fixed point Theorem (see p.e. Vrabie [18]). We denote $X = L^p((0,T), L^2(\mathcal{M}))$ then

- (i) $K = \{h \in L^p((0,T), L^\infty(\Omega)) : ||h(t)|| \le C_0 \text{ a.e. } t \in (0,T)\}$ is a nonempty, convex and weakly compact set of X;
- (ii) $\mathcal{T} : K \mapsto 2^X$ with nonempty, convex and closed values such that $\mathcal{T}(g) \subset K$, $\forall g \in K$;
- (iii) graph($\mathcal{T}$) is weakly$\times$weakly sequentially closed.

Consequently, $\mathcal{T}$ has at least one fixed point in K which is a solution of (3.1). This completes the proof of Theorem 3.2.

Remark 3.5. The boundedness of the solutions is a consequence of the existence of super-solutions and sub-solutions which are constants.

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