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A BOUNDARY CONTROL PROBLEM WITH A NONLINEAR REACTION TERM

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Abstract. The authors study the problem $u_t = u_{xx} - au$, $0 < x < 1$, $t > 0$; $u(x,0) = 0$, and $-u_x(0,t) = u_x(1,t) = \phi(t)$, where $a = a(x,t,u)$, and $\phi(t) = 1$ for $t_{2k} < t < t_{2k+1}$ and $\phi(t) = 0$ for $t_{2k+1} < t < t_{2k+2}$, $k = 0,1,2,\dots$ with $t_0 = 0$ and the sequence t_k is determined by the equations $\int_0^1 u(x,t_k)dx = M$, for $k = 1,3,5,\dots$, and $\int_0^1 u(x,t_k)dx = m$, for $k = 2,4,6,\dots$, where $0 < m < M$. Note that the switching points t_k, are unknown. A maximum principal argument has been used to prove that the solution is positive under certain conditions. Existence and uniqueness are demonstrated. Theoretical estimates of the t_k and $t_{k+1} - t_k$ are obtained and numerical verifications of the estimates are presented.

1. Introduction

In this paper, we consider the problem

$$
\begin{gathered}
u_t = u_{xx} - a(x,t,u)u, \quad 0 < x < 1,\ t > 0, \\
-u_x(0,t) = u_x(1,t) = \phi(t), \quad t \geq 0, \\
u(x,0) = 0, \quad 0 \leq x \leq 1,
\end{gathered}
\tag{1.1}
$$

where $a(x,t,u)$ is a continuous function and

$$
0 \leq \alpha \leq a(x,t,u) \leq \beta \tag{1.2}
$$

for $(x,t) \in [0,1] \times [0,T]$ and $u \in \mathbb{R}$, and

$$
\phi(t) = \begin{cases} 1, & t_{2n} \leq t \leq t_{2n+1}, \\ 0, & t_{2n+1} \leq t \leq t_{2n+2}, \end{cases} \tag{1.3}
$$

where $\{t_n\}$ depends on

$$
\mu(t) = \int_0^1 u(x,t)dx, \tag{1.4}
$$

where

$$
\begin{gathered}
2\mu(t_{2n}) = m, \quad n = 1,2,\dots, \\
\mu(t_{2n+1}) = M, \quad n = 0,1,\dots,
\end{gathered}
$$

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with $0 < m < M$.

2. Existence of solutions

We study the existence of a solution $u(x,t)$ to (1.1) for a given stepwise boundary conditions $\phi(t)$. We assume that

$$|ua_u(x,t,u)| \le C, \tag{2.1}$$

uniformly for all x, t, u, which ensures that the source term $F(x,t,u) = -a(x,t,u)u$ is uniformly Lipschitz with respect to u; i.e.,

$$|F(x,t,u_1) - F(x,t,u_2)| \le C|u_1 - u_2|,$$

for all x, t, u_1, u_2. The constants C's in the above inequalities or that come in the sequel aren't necessarily the same.

Under certain smoothness conditions on $u(x,t)$, $a(x,t,u)$, problem (1.1) is equivalent to

$$\begin{aligned} u(x,t) = 2\int_0^t [\theta(x,t-\tau) + \theta(1-x,t-\tau)]\phi(\tau)d\tau \\ + \int_0^t \int_0^1 [\theta(x-\xi,t-\tau) + \theta(x+\xi,t-\tau)]F(\xi,\tau,u(\xi,\tau))d\xi d\tau. \end{aligned} \tag{2.2}$$

Let us show that the integral equation (2.2) has a solution by considering the operator

$$\begin{aligned} Hu = 2\int_0^t [\theta(x,t-\tau) + \theta(1-x,t-\tau)]\phi(\tau)d\tau \\ + \int_0^t \int_0^1 [\theta(x-\xi,t-\tau) + \theta(x+\xi,t-\tau)]F(\xi,\tau,u(\xi,\tau))d\xi d\tau, \end{aligned}$$

on the set of functions

$$B_\eta = \{u(x,t) \in C([0,1]\times[0,\eta]), \quad \|u\|_\eta < \infty\},$$

where

$$\|u\|_\eta = \sup_{0\le x\le 1,\, 0\le t\le \eta} |u(x,t)|\,.$$

The set B_η is a Banach space. The mapping H maps B_η into into itself [1]. Furthermore,

$$|Hu_1 - Hu_2| \le Ct\,\|u_1 - u_2\|_t\,,$$

which implies

$$\|Hu_1 - Hu_2\|_\eta \le C\eta\,\|u_1 - u_2\|_\eta\,.$$

If we select $\eta < 1/C$, then H is a contraction map on B_η, Thus H has a unique fixed point $u \in B_\eta$, which solves (2.2). Since F is uniformly Lipschitz, the solution u can be extended on any time interval $[0,T]$ (see [1]).

3. Maximum Principle

In this section, we use the maximum principle to prove that problem (1.1) has a nonnegative solution. To achieve this, we establish the following lemmas.

Lemma 3.1. *Let $D = \{(x,t) : 0 < x < 1; 0 < t \le T\}$ and $a(x,t,u)$ satisfy condition* (1.2). *The solution u of*

$$\begin{gathered} u_t = u_{xx} - a(x,t,u)u, \quad \text{in } D, \\ u(x,0) \ge 0, \quad u(x,0) \ge 0,\ 0 \le x \le 1, \\ -u_x(0,t) = u_x(1,t) = 1, \quad 0 \le t \le T, \end{gathered} \tag{3.1}$$

is nonnegative on $\overline{D}$.

Proof. To prove that $u(x,t) \ge 0$ in $\overline{D}$, let us assume the converse, i.e., $u(x,t) < 0$ at some point in $\overline{D}$. The continuity of $u(x,t)$ on $\overline{D}$ implies the existence of a negative minimum in $\overline{D}$. If $\min_{\overline{D}} u = u(0,\bar{t})$, for some $0 < \bar{t} \le T$, then the boundary condition $u_x(0,t) = -1$ implies $u_x < 0$ in neighborhood of $(0,\bar{t})$, so $u(x,\bar{t}) < u(0,\bar{t})$ for some small x, which contradicts the fact that $u(0,\bar{t})$ is the minimum.

A similar argument can be used to prove that the minimum can never happen at $x = 1$. So, u has its negative minimum at $(\overline{x},\bar{t})$ in the interior of D. This implies $u_t(\overline{x},\bar{t}) \le 0$ and $u_{xx}(\overline{x},\bar{t}) \ge 0$. Therefore, $u_t - u_{xx} + au$ is negative at $(\overline{x},\bar{t})$, which is a contradiction. Thus, we proved the lemma. □

Next, we consider the problem

$$\begin{gathered} u_t = u_{xx} - a(x,t,u)u, \quad (x,t) = D, \\ u_x(0,t) = u_x(1,t) = 0, \quad 0 \le t \le T, \\ u(x,0) \ge 0, \quad 0 \le x \le 1, \end{gathered} \tag{3.2}$$

and $a(x,t,u)$ satisfies condition (1.2). We establish the following lemma for a closely related problem.

Lemma 3.2. *For a positive constant γ, the solution $v(x,t;\gamma)$ of*

$$\begin{gathered} v_t = v_{xx} - \gamma v, \quad \text{in } D, \\ v_x(0,t) = v_x(0,t) = 0, \quad t \ge 0, \\ v(x,0) > 0, \quad 0 \le x \le 1 \end{gathered}$$

is positive for all $(x,t) \in \overline{D}$ where γ is a positive constant.

Proof. Let $w = e^{\gamma t} v$. Then

$$\begin{gathered} w_t = w_{xx}, \quad \text{in } D, \\ w_x(0,t) = w_x(1,t) = 0, \quad t \ge 0, \\ w(x,0) = v(x,0) > 0, \quad 0 \le x \le 1. \end{gathered}$$

If $w \le 0$, then w has a minimum that's not positive either at $x = 0$ or $x = 1$ for some $t = t_0 \in (0,T]$, which implies by the strong maximum principle [15], $w_x(0,t_0) > 0$ or $w_x(1,t_0) < 0$, which is a contradiction. Hence, $w > 0$, and therefore, $v > 0$. □

Lemma 3.3. *The solution $z(x,t,\epsilon)$ of*
$$
\begin{gathered}
z_t = z_{xx} \quad \text{in } D, \\
z_x(0,t) = \epsilon, \quad 0 \le t \le T, \\
z_x(1,t) = -\epsilon, \quad 0 \le t \le T, \\
z(x,0) = 0, \quad 0 \le x \le 1,
\end{gathered}
$$
satisfies the inequality
$$
-C\epsilon < z(x,t) \le 0 \quad \text{in } \overline{D},
$$
where the positive constant C is a linear function of G.

Proof. This is a straightforward application of the strong minimum principle and a simple comparison with
$$
u(x,t) = -2\epsilon t + \epsilon x(1-x).
$$
□

Lemma 3.4. *The solution u of* (3.2) *satisfies the inequality*
$$
0 < v(x,t;\beta) \le u(x,t) \le v(x,t;\alpha) \quad \text{in } \overline{D},
$$
where α and β are the lower and upper bound of $a(x,t,u)$, respectively.

Proof. First consider $v(x,t;\beta) + z(x,t;\epsilon)$. For a fixed T, we can chose ϵ sufficiently small so that $v + z > 0$ in $\overline{D}$. Consider $w = u - v - z$. Clearly, w satisfies
$$
\begin{gathered}
w_t = w_{xx} - aw - (a-\beta)v \quad \text{in } D, \\
w_x(0,t) = -\epsilon, \quad 0 \le t \le T, \\
w_x(1,t) = \epsilon, \quad 0 \le t \le T, \\
w(x,0) = 0, \quad 0 \le x \le 1.
\end{gathered}
$$
Suppose $w < 0$ somewhere in $\overline{D}$. Then the boundary conditions force a negative minimum in D, where
$$
w_t - w_{xx} + aw + (a-\beta)v < 0,
$$
which contradicts the equation
$$
w_t - w_{xx} + aw + (a-\beta)v = 0 \quad \text{in } D.
$$
Thus, $w \ge 0$ which implies that
$$
u(x,t) \ge v(x,t,\beta) + z(x,t,\epsilon)
$$
for all $\epsilon > 0$ sufficiently small. Hence,
$$
u(x,t) \ge v(x,t;\beta).
$$
Likewise, by considering $w = v - z - u$, the inequality
$$
v(x,t;\alpha) \ge u(x,t),
$$
follows by a similar argument. □

Theorem 3.5. *The solution u of* (1.1) *is nonnegative.*

Proof. By applying, successively, Lemma 3.1 and 3.4 on each time stage where we keep the flux u_x either zero or one, and the conclusion follows. □

4. Existence of the Time Switches

If we formally differentiate (1.4), we obtain

$$\mu'(t) = 2\phi(t) - \int_0^1 a(x,t,u)u dx. \tag{4.1}$$

To prove the existence of t_1, let $\phi(t) = 1$ for $t > 0$. In view of hypothesis (1.2), equation (4.1) implies the estimate

$$\mu'(t) \geq 2 - \beta \int_0^1 u\, dx;$$

that is,

$$\mu'(t) \geq 2 - \beta\mu(t), \quad t \geq 0.$$

By applying Gronwal's inequality, we get

$$\mu(t) \geq \frac{2}{\beta}[1 - e^{-\beta t}].$$

Since $\mu(t)$ is continuous, then there exists a $t_1 > 0$ such that

$$\mu(t_1) = M,$$

for any $0 < M < \frac{2}{\beta}$.

Next, we prove the existence of t_2 by taking $\phi(t) = 0$ for $t > t_1$. This implies

$$\mu'(t) = -\int_0^1 a(x,t,u)u(x,t)dx, \quad t > t_1.$$

Using the estimate on $a(x,t,u)$, we obtain

$$\mu'(t) \leq -\alpha\mu(t), \quad t \geq t_1.$$

Gronwal's inequality implies

$$\mu(t) \leq \mu(t_1)e^{-\alpha(t-t_1)} = Me^{-\alpha(t-t_1)}, \quad t \geq t_1.$$

Since $\mu(t)$ is continuous, then there exists a $t_2 > t_1$ such that

$$\mu(t_2) = m,$$

where $0 < m < M$.

For $t > t_2$, we take $\phi(t) = 1$. This gives the estimate

$$\mu'(t) \geq 2 - \beta\mu(t), \quad t \geq t_2.$$

Using the condition $\mu(t_2) = m$ and Gronwal's inequality, we get

$$\mu(t) \geq \frac{2}{\beta} - \big(\frac{2}{\beta} - m\big)e^{-\beta(t-t_2)}, \quad t \geq t_2.$$

Note that the coefficient $\frac{2}{\beta} - m$ is positive, which implies the existence of $t_3 > t_2$ such that

$$\mu(t_3) = M.$$

We inductively get for $t > t_{2n}$ and $\phi(t) = 1$,

$$\mu(t) \geq \frac{2}{\beta} - \big(\frac{2}{\beta} - m\big)e^{-\beta(t-t_{2n})}, \quad t \geq t_{2n}, \tag{4.2}$$

which implies the existence of t_{2n+1} such that $\mu(t_{2n+1}) = M$.

Also, for $t > t_{2n+1}$ and $\phi(t) = 0$, we have,

$$\mu(t) \le Me^{-\alpha(t-t_{2n+1})}, \quad t \ge t_{2n+1}, \tag{4.3}$$

which ensures the existence of t_{2n+2} such that $\mu(t_{2n+2}) = m$.

Estimate (4.2) implies

$$M = \mu(t_{2n+1}) \ge \frac{2}{\beta} - (\frac{2}{\beta} - m)e^{-\beta(t_{2n+1}-t_{2n})},$$

which gives rise to

$$t_{2n+1} - t_{2n} \le \frac{1}{\beta} \ln \frac{2 - m\beta}{2 - M\beta}. \tag{4.4}$$

Similarly, if we employ (4.3), we can get

$$t_{2n+2} - t_{2n+1} \le \frac{1}{\alpha} \ln \frac{M}{m}.$$

5. Numerical Example

In this section, we consider a finite difference method to discretize the problem

$$\begin{gathered} u_t = cu_{xx} - \sin u, \quad 0 < x < 1,\ 0 < t \le T, \\ -u_x(0,t) = u_x(1,t) = \phi(t), \quad 0 < t \le T, \\ u(x,0) = 0, \quad 0 < x < 1, \end{gathered}$$

where the boundary control function is

$$\phi(t) = \begin{cases} 10, & t_{2n} \le t \le t_{2n+1}, \\ 0, & \text{elsewhere}, \end{cases} \tag{5.1}$$

and $\{t_n\}$ depends on

$$\mu(t) = \int_0^1 u(x,t)dx, \tag{5.2}$$

where

$$\begin{gathered} 2\mu(t_{2n}) = 1, \quad n = 1, 2, \dots, \\ \mu(t_{2n+1}) = 2, \quad n = 0, 1, \dots. \end{gathered}$$

The time limit and the diffusivity constant are taken as $T = 40$ and $c = 0.05$.

Let's consider the space and time discretization

(i) $\Delta x = \frac{1}{J}$, $x_j = j\Delta x$, $j = 0, 1, \dots, J$
(ii) $\Delta t = \frac{T}{N}$, $\tau_n = n\Delta t$, $n = 0, \dots, N$

where $J = 50$ and $N = 400$. The integer N is chosen large enough so that the time step Δt is much smaller than an estimated differences between two consecutive values of the time switches.

We consider the backward implicit finite difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = c\frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{(\Delta x)^2} - \sin U_j^n,$$

which can be written as

$$-\nu U_{j-1}^{n+1} + (1 + 2\nu)U_j^{n+1} - \nu U_{j+1}^{n+1} = U_j^n - \Delta t \sin U_j^n, \tag{5.3}$$

n	T_n	$T_n - T_{n-1}$	n	T_n	$T_n - T_{n-1}$
1	5.2000	5.1000	8	21.6000	1.2000
2	6.4000	1.2000	9	25.5000	3.9000
3	10.2000	3.8000	10	26.7000	1.2000
4	11.4000	1.2000	11	30.6000	3.9000
5	15.3000	3.9000	12	31.8000	1.2000
6	16.5000	1.2000	13	35.6000	3.8000
7	20.4000	3.9000	14	36.8000	1.2000

TABLE 1. The Time switches T_n and the differences $T_n - T_{n-1}$. Note the differences between any two consecutive times tend to alternate between 1.2 and 3.8 or 3.9.

where $\nu = c\Delta t/(\Delta x)^2$, $j = 1, \dots, J-1$ and $n = 0, 1, \dots, N-1$. The initial condition is $U_j^0 = 0$ for $j = 1, \dots, J-1$, and the boundary conditions are

$$-\frac{U_1^n - U_0^n}{\Delta x} = \frac{U_J^n - U_{J-1}^n}{\Delta x} = \phi(\tau_n), \tag{5.4}$$

for $n = 0, 1, \dots, N$. The total mass integral is calculated by the following trapezoidal rule

$$\mu_n = \frac{h}{2} \sum_{j=0}^{N-1} \left(U_j^{n+1} + U_{j+1}^n \right). \tag{5.5}$$

The numerical experiment is carried out in the following way. We start by setting the flux at $\phi = 10$ then we solve a tridiagonal system coming out of the difference method. We evaluate the total mass μ_n and compare it with the upper threshold $M = 2$. We move to the next time step while keeping the flux at $\phi = 10$, if $\mu_n < M$, or switch it to $\phi = 0$, if $\mu_n \geq M$. At the moment, say τ_{n_1}, for some integer n_1, when the total mass exceeds M for the first time, we take $T_1 = \tau_{n_1}$ as an approximation for the first time switch. With $\phi = 0$, we move on our solution through the time, as long as μ_n does not fall below the threshold $m = 1$. By the moment, when $\mu_{n_2} \leq m$, for some integer n_2, we set $T_2 = \tau_{n_2}$, and we switch the flux back to $\phi = 10$ at the next step. We keep switching the flux on ($\phi = 10$) and off ($\phi = 0$) and calculating the time switches T_k until the end of the run when $\tau_n = 40$.

Table (1) shows the times switches T_n. As we can see there, the difference between any two consecutive time switches has a tendency to alternate between 3.8, 3.9 and 1.2. For the same set of data, graphs (1) through (5) show the concentration versus the space at consecutive time steps. The graphs are obtained for different stages, where at each stage the flux is kept constant at the end points. A profile of the concentrations at $x = 0.5$ for various times is shown in graph (6) with the same specified data. Graph (7) shows the total mass computed through (5.5) versus the time. Note the slow increase and the sharp fall in the graph due to the sink term $\sin U_j^n$.

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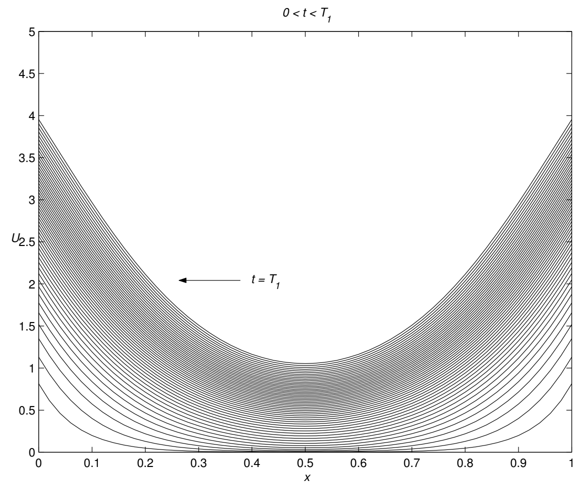

FIGURE 1. The first stage where the flux ϕ is held at 10 at the end points. Each curve shows the concentration profile at various discrete time steps $\tau_n = n\Delta t$. As the time goes on, the level of concentrations gets higher

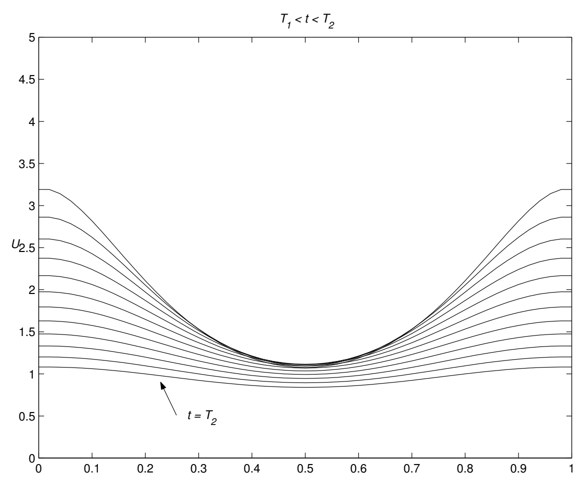

FIGURE 2. The second stage where the flux ϕ is held at 0 at the end points. As the time goes on, the level of concentrations decreases. Notice the fluctuations when the concentration is dropped suddenly to 0 at the beginning of the stage.

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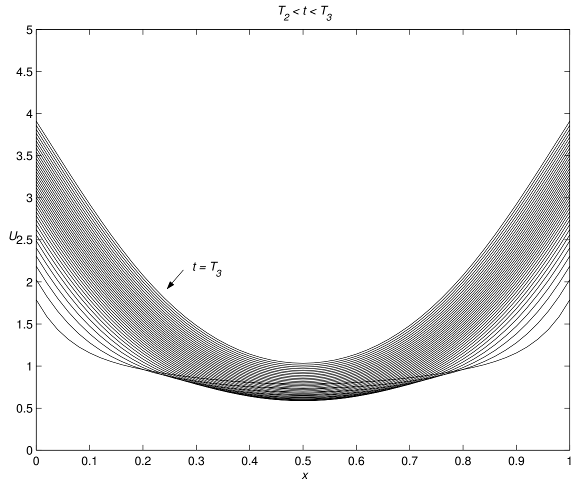

FIGURE 3. The third stage where the flux ϕ is switched to 10 at the end points. Each curve shows the concentration profile at various discrete time steps. Notice the fluctuations due to the sudden change on the concentrations. After a little while, the concentrations levels increase monotonically.

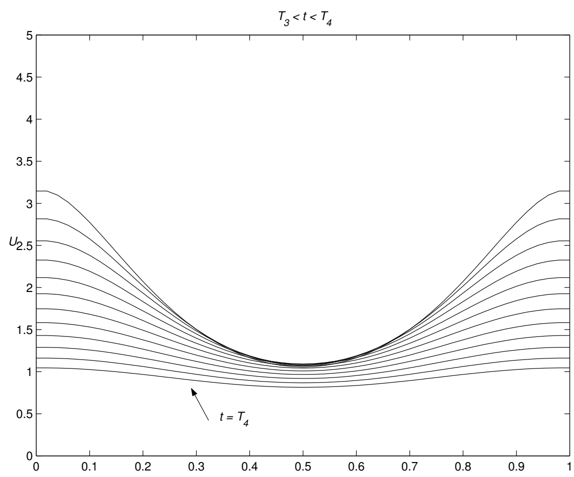

FIGURE 4. The fourth stage where the flux ϕ is switched to 0 at the end points. Notice the similarity with the second stage.

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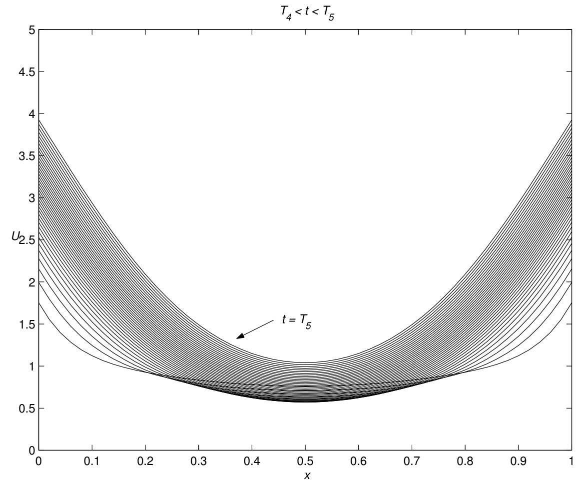

FIGURE 5. The fifth stage where the flux ϕ is switched to 10 at the end points. Notice the similarity with the third stage.

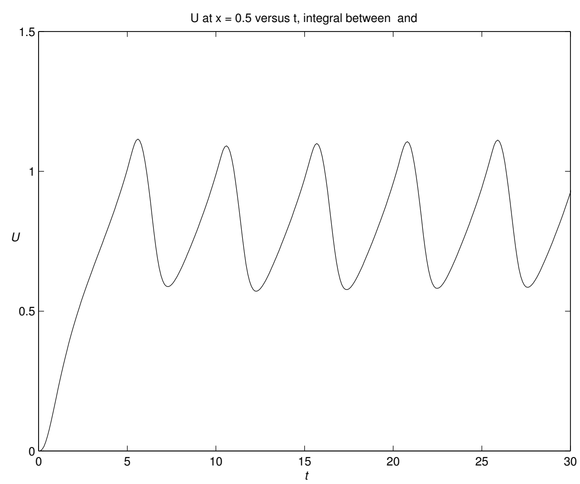

FIGURE 6. The concentration profile U at $x = 0.5$ versus the time shows periodic behavior due to the periodic change of the boundary conditions

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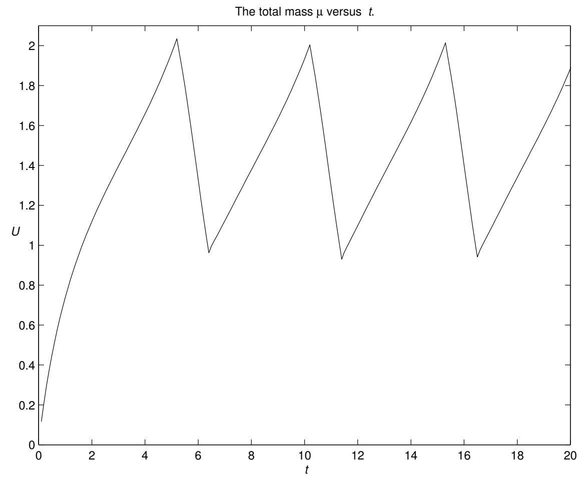

FIGURE 7. The total mass computed via equation (5.5) versus the time. Note the slow increase and the sharp fall in the graph due to the sink term $\sin U_j^n$.

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