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QUASIREVERSIBILITY FOR INHOMOGENEOUS ILL-POSED PROBLEMS IN HILBERT SPACES

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Abstract. In a Hilbert space $\mathcal{H}$, the inhomogeneous ill-posed abstract Cauchy problem is given by $\frac{du}{dt} = Au(t)+h(t)$, $u(0) = \chi$, $0 \le t < T$; where A is a positive self-adjoint linear operator acting on $\mathcal{H}$, $\chi \in \mathcal{H}$, and $h : [0,T) \to \mathcal{H}$. Using semigroup theory, we obtain Hölder continuous dependence for the control problem generated by the method of quasireversibility.

1. Introduction

In a Hilbert space $\mathcal{H}$, we consider the problem

$$\begin{gathered} \frac{du}{dt} = Au(t) + h(t), \quad 0 \le t < T, \\ u(0) = \chi, \end{gathered} \tag{1.1}$$

where A is a positive self-adjoint linear operator, $\chi \in \mathcal{H}$, and $h : [0,T) \to \mathcal{H}$. Since A is unbounded, the problem is ill-posed. Lattes and Lions introduced the method of quasireversibility in [16] in the 1960s as a way to generate approximate solutions to ill-posed problems. As part of their technique, they perturb the operator A to construct an approximate problem. We do the same, considering the approximate problem

$$\begin{gathered} \frac{dv}{dt} = f(A)v(t) + h(t), \\ v(0) = \chi, \end{gathered} \tag{1.2}$$

where $f(\lambda)$ is a real-valued Borel function bounded above. Since $f(A)$ is bounded above, the approximate problem (1.2) is well-posed with solution $v(t)$. Following work done by Ames and Hughes [6], we have proved that the solution to the ill-posed problem, if it exists, depends continuously on the solution to the approximate problem. These results have been obtained both in Hilbert space (including the nonlinear case) [7] and in Banach space [8].

As mentioned above, Lattes and Lions perturb the operator to define an approximate problem. However, they use the solution $v(t)$ to this approximate problem

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to generate data $w(T)$ used to solve the problem

$$\begin{aligned} \frac{dw}{dt} &= Aw(t), \quad 0 \le t < T, \\ w(T) &= v(T). \end{aligned} \tag{1.3}$$

As a final-value problem, this problem is well-posed. Lattes and Lions show that

$$\|u(0) - w(0)\| \le \epsilon.$$

Note that the method of Lattes and Lions does not give an approximation for $u(t)$ where $t > 0$. In [19], Miller points out an additional concern: that the norm of the operator $e^{f(A)}$ is large for small ϵ. Miller refines the method of quasireversibility, making additional assumptions on f in order to obtain a logarithmic convexity result for the difference of the solutions $u(t)$ and $w(t)$. He calls this approach a stabilized quasireversibility method. We show here that we are able to obtain the same results as Miller using our assumptions. In particular, we use the solution to (1.2) to create the final-value problem

$$\begin{aligned} \frac{dw}{dt} &= Aw(t) + h(t), \quad 0 \le t < T, \\ w(T) &= v(T). \end{aligned} \tag{1.4}$$

Under the appropriate stabilizing conditions, we show that there exist computable constants C and M, independent of $0 < \beta < 1$, such that

$$\|u(t) - w(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T},$$

where $u(t)$ and $w(t)$ are solutions to (1.1) and (1.4), respectively, assuming a solution to (1.1) exists.

2. Theory

In Hilbert space, the linear inhomogeneous ill-posed problem is given by

$$\begin{aligned} \frac{du}{dt} &= Au(t) + h(t), \\ u(0) &= \chi, \end{aligned} \tag{2.1}$$

for $0 \le t < T$, where A is a positive self-adjoint operator on a Hilbert space $\mathcal{H}$, $\chi \in \mathcal{H}$, and $h : [0, T) \to \mathcal{H}$. We assume that h is differentiable on $(0, T)$ and that $h' \in L^1((0, T); \mathcal{H})$. The following theorem states conditions under which a solution exists.

Theorem 2.1 ([22, Corollary 4.2.10])**.** *Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If $h : [0, T) \to X$ is differentiable almost everywhere on $[0, T]$ and $h' \in L^1((0, T); X)$, then for every $\chi \in \mathrm{Dom}(A)$ the initial value problem* (2.1) *has a unique strong solution u on $[0, T]$ given by*

$$u(t) = T(t)\chi + \int_0^t T(t - s)h(s)ds. \tag{2.2}$$

Recall that self-adjoint operators bounded above generate C_0 semigroups. This yields the following corollary:

Corollary 2.2. *Let $\mathcal{H}$ be a Hilbert space, and let $A : \mathcal{H} \to \mathcal{H}$ be a closed, densely-defined linear operator. If A is self-adjoint and bounded above and $h : [0,T) \to \mathcal{H}$ is differentiable on $(0,T)$ with $h' \in L^1(0,T)$, then for every $\chi \in \operatorname{Dom}(A)$ the initial value problem* (2.1) *has a unique solution u on $[0,T]$ given by* (2.2)*.*

We approximate the inhomogeneous ill-posed problem with

$$\begin{gathered}\frac{dv}{dt} = f(A)v(t) + h(t),\\ v(0) = \chi,\end{gathered}$$

where f is a real-valued Borel function bounded above that approximates A in a suitable sense. Take $f(A) = A - \epsilon A^2$, following Lattes and Lions [16], Miller [18], and Ames [2], or $f(A) = A(I + \epsilon A)^{-1}$, following Showalter [24]. Since $f(A)$ is bounded above, by Corollary 2.2 the approximate problem is well-posed with solution

$$v(t) = \mathrm{e}^{tf(A)}\chi + \int_0^t \mathrm{e}^{(t-s)f(A)}h(s)\,ds.$$

Now consider the *final*-value problem given by

$$\begin{gathered}\frac{dw}{dt} = Aw(t) + h(t),\\ w(T) = v(T) = \mathrm{e}^{Tf(A)}\chi + \int_0^T \mathrm{e}^{(T-s)f(A)}h(s)\,ds.\end{gathered} \tag{2.3}$$

This problem is well-posed with solution

$$w(t) = \mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi + \int_0^T \mathrm{e}^{(T-s)f(A)}h(s)\,ds\Big) - \int_t^T \mathrm{e}^{(t-s)A}h(s)\,ds,$$

where $\chi \in \operatorname{Dom}(f(A))$. Under certain stabilizing conditions, we prove that

$$\|u(t) - w(t)\| \le C\beta^{1-\frac{t}{T}}M^{t/T},$$

where $0 < \beta < 1$ and C and M are computable constants independent of β.

Definition 2.3 ([6, Definition 1])**.** Let A be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $f : [0,\infty) \to \mathbb{R}$ be a Borel function, and assume that there exists $\omega \in \mathbb{R}$ such that $f(\lambda) \le \omega$ for all $\lambda \in [0,\infty)$. Then f is said to satisfy *Condition ($\mathcal{A}$)* if there exist positive constants β, δ, with $0 < \beta < 1$, for which $\operatorname{Dom}(A^{1+\delta}) \subseteq \operatorname{Dom}(f(A))$, and

$$\|(-A + f(A))\psi\| \le \beta\|A^{1+\delta}\psi\|, \tag{2.4}$$

for all $\psi \in \operatorname{Dom}(A^{1+\delta})$.

Note that $f(A)$ and $A^{1+\delta}$ are defined by the functional calculus for self-adjoint operators that follows from the Spectral Theorem. Set

$$g(\lambda) = -\lambda + f(\lambda). \tag{2.5}$$

Lemma 2.4 ([6, Lemma 1])**.** *For all $t \ge 0$,*

$$\mathrm{e}^{tg(A)} = \mathrm{e}^{-tA}\mathrm{e}^{tf(A)}.$$

We will use this repeatedly in our proofs, together with the fact that this relationship holds for all $\alpha \in \mathbb{C}$:

$$\mathrm{e}^{\alpha g(A)} = \mathrm{e}^{-\alpha A}\mathrm{e}^{\alpha f(A)}.$$

Recall that A is unbounded and thus not defined everywhere. We need to regularize our data so that it is in the domain of these operators. In Hilbert space, we use the resolution of the identity for this regularization. As mentioned above, we also rely on the functional calculus for unbounded operators that follows from the Spectral Theorem. Before stating our result and its proof, we review these ideas.

Theorem 2.5. Spectral Theorem for Unbounded Self-Adjoint Operators [11, Theorem XII.2.3] *Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Then its spectrum is real and there is a uniquely determined regular countably additive self-adjoint spectral measure E defined on the Borel sets of the plane, vanishing on the complement of the spectrum, and related to T by the equations*

(a) $\mathrm{Dom}(T) = \{x \in \mathcal{H} : \int_{\sigma(T)} \lambda^2 d(E(\lambda)x, x) < \infty\}$, *and*

(b) $Tx = \lim_{n\to\infty} \int_{-n}^{n} \lambda dE(\lambda)x$, *where* $x \in \mathrm{Dom}(T)$.

Definition 2.6 ([11, Definition XII.2.4])**.** The unique spectral measure associated with a self-adjoint operator T as in the above theorem is called the *resolution of the identity* for T.

3. Results

We assume that there exists a strong solution $u(t)$ to the ill-posed inhomogeneous problem given in (2.1).

Theorem 3.1. *Let A be a positive self-adjoint operator acting on a Hilbert space $\mathcal{H}$ and let f satisfy Condition ($\mathcal{A}$). Assume that $h(t) : [0, T) \to \mathcal{H}$ is continuously differentiable with $h'(t) \in L^1(0, T)$ and $h(t) \in \mathrm{Dom}(\mathrm{e}^{TA})$ for all $t \in [0, T)$. Also, assume that there exists a constant γ, independent of β and ω, such that $(g(A)\psi, \psi) \le \gamma(\psi, \psi)$, for all $\psi \in \mathrm{Dom}(g(A))$. Further, suppose that $\chi \in \mathrm{Dom}(\mathrm{e}^{TA})$ and $\|e^{TA}\chi\| \le L$, $\|e^{TA}h(t)\| \le N$ for all $t \in [0, T)$. Then there exist constants C and M, independent of β, such that for $0 \le t < T$,*

$$\|u(t) - w(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T}.$$

As discussed above, since A is unbounded we need to regularize our data so that it is in the domain of the operators with which we are working. We use the resolution of the identity for this regularization. Let $\{E(\cdot)\}$ represent the resolution of the identity for the linear operator A. Set $e_n = \{\lambda \in [0, \infty) : |g(\lambda)| \le n\}$. Using the definition of g given above in (2.5), where f satisfies Condition ($\mathcal{A}$), we see that e_n is a bounded set since

$$\begin{aligned} e_n &= \{\lambda \in [0, \infty) : |g(\lambda)| \le n\} \\ &\subseteq \{\lambda : 0 \le \lambda \le n + \omega\}. \end{aligned}$$

Let $E_n = E(e_n)$. The following lemma is used repeatedly throughout this work.

Lemma 3.2. *Let A be a self-adjoint operator with E the resolution of the identity for A and $e_n = \{\lambda \in [0, \infty) : |g(\lambda)| \le n\}$. Let $\tau \in \mathcal{H}$. Then $E_n\tau \in \mathrm{Dom}(f(A))$, where f is a complex Borel function defined E-almost everywhere on the real axis and bounded on bounded sets.*

Our proof begins with approximations $u_n(t)$ and $w_n(t)$. Set $\chi_n = E_n\chi$ and $h_n(s) = E_n h(s)$. Note that $f(\lambda) = \mathrm{e}^{t\lambda}$ is a Borel function bounded on bounded sets, so by Lemma 3.2 χ_n, $h_n \in \operatorname{Dom}(A) \cap \operatorname{Dom}(\mathrm{e}^{tA})$. Define

$$u_n(t) = E_n u(t),$$
$$w_n(t) = E_n w(t).$$

Lemma 3.3 ([7, Lemma 9])**.**

$$u_n(t) = \mathrm{e}^{tA}\chi_n + \int_0^t \mathrm{e}^{(t-s)A} h_n(s) ds.$$

We have an analogous result for w_n:

Lemma 3.4.

$$w_n(t) = \mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi_n + \int_0^T \mathrm{e}^{(T-s)f(A)} h_n(s)\, ds\Big) - \int_t^T \mathrm{e}^{(t-s)A} h_n(s)\, ds.$$

Proof. Note that we may write the final value problem given in (2.3) as an initial value problem by replacing t with $T-t$. Then the differential equation becomes

$$\frac{dw}{dt} = -Aw(t) + h(t).$$

Since $-A$ is bounded above, $-A$ generates a C_0 semigroup. Then by Corollary 2.2, the problem given in (2.3) has a unique solution given by

$$w(t) = \mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi + \int_0^T \mathrm{e}^{(T-s)f(A)} h(s)\, ds\Big) - \int_t^T \mathrm{e}^{(t-s)A} h(s).$$

Thus

$$\begin{aligned} E_n w(t) &= E_n\Big[\mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi + \int_0^T \mathrm{e}^{(T-s)f(A)} h(s)\, ds\Big) - \int_t^T \mathrm{e}^{(t-s)A} h(s)\Big] \\ &= \mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi_n + \int_0^T \mathrm{e}^{(T-s)f(A)} h_n(s)\, ds\Big) - \int_t^T \mathrm{e}^{(t-s)A} h_n(s), \end{aligned}$$

so

$$w_n(t) = \mathrm{e}^{(t-T)A}\Big(\mathrm{e}^{Tf(A)}\chi_n + \int_0^T \mathrm{e}^{(T-s)f(A)} h_n(s)\, ds\Big) - \int_t^T \mathrm{e}^{(t-s)A} h_n(s).$$

□

Proof of Theorem 3.1. To obtain our result we use the Three Lines Theorem (cf. [23, p. 33]), which requires us to extend u_n and w_n into the complex strip $\{\alpha = t + \mathrm{i}\eta : 0 \le t \le T, \eta \in \mathbb{R}\}$. To do so, set

$$u_n(\alpha) = \mathrm{e}^{\mathrm{i}\eta A} u_n(t)$$
$$w_n(\alpha) = \mathrm{e}^{\mathrm{i}\eta A} w_n(t),$$

Note that the complex-valued function $u_n - w_n$ is analytic and continuous on the strip. Define

$$\phi_n(\alpha) = (u_n(\alpha) - w_n(\alpha), \tau),$$

where $(\cdot,\cdot)$ is the inner product in $\mathcal{H}$ and τ is an arbitrary element in $\mathcal{H}$. To use the Three Lines Theorem, we must show that $\phi_n(\alpha)$ is bounded in the strip. For $\alpha = t + \mathrm{i}\eta$ we have

$$
\begin{aligned}
&\|\phi_n(\alpha)\| \\
&\le \|u_n(\alpha) - w_n(\alpha)\|\|\tau\| \\
&= \|e^{i\eta A}\|\|u_n(t) - w_n(t)\|\|\tau\| \\
&= \Big\| \Big(e^{tA}\chi_n + \int_0^t e^{(t-s)A} h_n(s) ds \Big) - \Big(e^{tA} e^{T(-A+f(A))}\chi_n \\
&\quad + e^{(t-T)A} \int_0^T e^{(T-s)f(A)} h_n(s) ds - \int_t^T e^{(t-s)A} h_n(s) ds \Big) \Big\| \, \|\tau\| \\
&\le \Big\| \big(I - e^{T(-A+f(A))}\big) e^{tA}\chi_n + \int_0^T \Big(e^{(t-s)A} - e^{(t-T)A} e^{(T-s)f(A)} \Big) h_n(s)\, ds \Big\| \, \|\tau\| \\
&\le \Big(\|\big(I - e^{Tg(A)}\big) e^{tA}\chi_n\| + \Big\| \int_0^T \Big(e^{(t-s)A} - e^{(t-s+s-T)A} e^{(T-s)f(A)} \Big) h_n(s)\, ds \Big\| \Big) \|\tau\| \\
&\le \Big(\|\big(I - e^{Tg(A)}\big) e^{tA}\chi_n\| + \Big\| \int_0^T e^{(t-s)A} \Big(I - e^{(T-s)(-A+f(A))} \Big) h_n(s)\, ds \Big\| \Big) \|\tau\| \\
&\le \Big(\|\big(I - e^{Tg(A)}\big) e^{tA}\chi_n\| + \Big\| \int_0^T \Big(I - e^{(T-s)g(A)} \Big) e^{(t-s)A} h_n(s)\, ds \Big\| \Big) \|\tau\|.
\end{aligned}
$$

Recall that $(g(A)\psi, \psi) \le \gamma(\psi, \psi)$ for all $\psi \in \operatorname{Dom}(g(A))$, and so $g(A)$ is the generator of a strongly continuous semigroup $\{e^{tg(A)}\}_{t\ge 0}$ of bounded operators with $\|e^{tg(A)}\| \le e^{\gamma t}$. Thus $\| \big(I - e^{tg(A)}\big) \psi\| \le K\|\psi\|$ for all $t \in [0, T]$, where K is a constant. Using our assumptions that $\|e^{TA}\chi\| \le L$ and $\|e^{TA} h(t)\| \le N$, we have

$$
\begin{aligned}
\|\phi_n(\alpha)\| &\le \Big(\|\big(I - e^{Tg(A)}\big) e^{tA}\chi_n\| + \Big\| \int_0^T \Big(I - e^{(T-s)g(A)} \Big) e^{(t-s)A} h_n(s)\, ds \Big\| \Big) \|\tau\| \\
&\le \Big(K\|e^{tA}\chi_n\| + K \int_0^T \|e^{(t-s)A} h_n(s)\|\, ds \Big) \|\tau\| \\
&\le K \Big(\|e^{TA}\chi_n\| + \int_0^T \|e^{TA} h_n(s)\|\, ds \Big) \|\tau\| \\
&\le K(L + TN)\|\tau\|,
\end{aligned}
\tag{3.1}
$$

and thus ϕ_n is bounded. Hence we may apply the Three Lines Theorem to this inner product. By the Three Lines Theorem,

$$
|(\phi_n(t), \tau)| \le M(0)^{1-\frac{t}{T}} M(T)^{t/T}
$$

for $0 \le t \le T$, where

$$
M(t) = \max_{\alpha = t + i\eta, \eta \in \mathbb{R}} |(\phi_n(\alpha), \tau)|.
$$

Using properties of semigroups and Condition $(\mathcal{A})$, we have

$$
M(0) \le \Big(\|\big(I - e^{Tg(A)}\big)\chi_n\| + \Big\| \int_0^T \Big(I - e^{(T-s)g(A)} \Big) e^{(-s)A} h_n(s)\, ds \Big\| \Big) \|\tau\|
$$

$$
\begin{aligned}
&\le \Big(\Big\| \int_0^T e^{\sigma g(A)} g(A)\chi_n \, d\sigma \Big\| + \int_0^T \| \big(I - e^{(T-s)g(A)} \big) e^{(-s)A} h_n(s) \| \, ds \Big) \|\tau\| \\
&\le \Big(e^{\gamma T} \int_0^T \|g(A)\chi_n\| \, d\sigma + \int_0^T \Big\| \int_0^{T-s} e^{\sigma g(A)} g(A) e^{(-s)A} h_n(s) \, d\sigma \Big\| \, ds \Big) \|\tau\| \\
&\le \Big(\beta T e^{\gamma T} \|A^{1+\delta}\chi_n\| + \int_0^T \int_0^{T-s} e^{\gamma\sigma} \|g(A) e^{(-s)A} h_n(s)\| \, d\sigma ds \Big) \|\tau\| \\
&\le \Big(\beta T e^{\gamma T} \|A^{1+\delta}\chi_n\| + \int_0^T e^{\gamma(T-s)} \int_0^{T-s} \beta \|A^{1+\delta} e^{(-s)A} h_n(s)\| \, d\sigma ds \Big) \|\tau\| \\
&\le \Big(\beta T e^{\gamma T} \|A^{1+\delta}\chi_n\| + \int_0^T e^{\gamma(T-s)} \beta (T-s) \|A^{1+\delta} e^{(-s)A} h_n(s)\| \, ds \Big) \|\tau\| \\
&\le \beta T e^{\gamma T} \Big(\|A^{1+\delta}\chi_n\| + \int_0^T \|A^{1+\delta} e^{(-s)A} h_n(s)\| \, ds \Big) \|\tau\| \\
&\le \beta T e^{\gamma T} \Big(\|A^{1+\delta}\chi_n\| + \int_0^T \|A^{1+\delta} h_n(s)\| \, ds \Big) \|\tau\| \\
&\le k\beta T e^{\gamma T} \Big(\|e^{TA}\chi_n\| + \int_0^T \|e^{TA} h_n(s)\| \, ds \Big) \|\tau\|,
\end{aligned}
$$

where k is a positive constant. Also, from (3.1) we have

$$
\begin{aligned}
M(T) &\le \Big(\|\big(I - e^{Tg(A)}\big) e^{TA}\chi_n\| + \Big\| \int_0^T \big(I - e^{(T-s)g(A)} \big) e^{(T-s)A} h_n(s) \, ds \Big\| \Big) \|\tau\| \\
&\le K \Big(\|e^{TA}\chi_n\| + \int_0^T \|e^{TA} h_n(s)\| \, ds \Big) \|\tau\|.
\end{aligned}
$$

Thus by the Three Lines Theorem,

$$
\begin{aligned}
|\phi_n(t)| \le &\Big\{ k\beta T e^{\gamma T} \Big(\|e^{TA}\chi_n\| + \int_0^T \|e^{TA} h_n(s)\| \, ds \Big) \Big\}^{1-\frac{t}{T}} \\
&\times \Big\{ K \Big(\|e^{TA}\chi_n\| + \int_0^T \|e^{TA} h_n(s)\| \, ds \Big) \Big\}^{t/T} \|\tau\|.
\end{aligned}
$$

Letting $n \to \infty$ and using our stability assumptions, we have

$$
|\phi(t)| \le \{k\beta T e^{\gamma T}(L + TN)\}^{1-\frac{t}{T}} \{K(L+TN)\}^{t/T} \|\tau\| \le C\beta^{1-\frac{t}{T}} M^{t/T},
$$

where C and M are computable constants independent of β and

$$
\phi(t) = (u(t) - w(t), \tau).
$$

Taking the supremum over all $\tau \in \mathcal{H}$ with $\|\tau\| \le 1$, we have

$$
\|u(t) - w(t)\| \le C\beta^{1-\frac{t}{T}} M^{t/T},
$$

our desired result. □

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