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POSITIVE SOLUTIONS TO A NONLINEAR THIRD ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. We consider a third order three point boundary value problem. Some upper and lower estimates for positive solutions of the problem are proved. Sufficient conditions for the existence and nonexistence of positive solutions for the problem are obtained. An example is included to illustrate the results.

1. Introduction

Recently, second and higher order multi-point boundary value problems have attracted a lot of attention. In 2004, Henderson [5] considered the second order three point boundary value problem

$$\begin{gathered} u''(t) + f(u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u(p) - u(1) = 0. \end{gathered} \tag{1.1}$$

In 2006, Graef and Yang [3] studied the third order nonlocal boundary value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.2}$$

$$u(0) = u'(p) = u''(1) = 0. \tag{1.3}$$

For some other results on third order boundary value problems we refer the reader to the papers [1, 2, 4, 6, 8, 9]. Motivated by these works, in this paper we consider the third order three point nonlinear boundary value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.4}$$

$$u(0) = u(p) - u(1) = u''(1) = 0. \tag{1.5}$$

To our knowledge, the problem (1.4)–(1.5) has not been considered before. The boundary conditions (1.5) are closely related to some other boundary conditions. Firstly, (1.5) contains (1.1) as a part. We also note that $u(p)-u(1) = 0$ implies that

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there exists $\beta \in (p, 1)$ such that $u'(\beta) = 0$, and therefore the boundary conditions (1.5) imply

$$u(0) = u'(\beta) = u''(1) = 0.$$

Hence, boundary conditions (1.5) are closely related to the conditions (1.3). If we let $p \to 1^-$, then (1.5) "tends to"

$$u(0) = u'(1) = u''(1) = 0, \tag{1.6}$$

which are often referred to as the (1,2) focal boundary conditions.

In this paper, we are interested in the existence and nonexistence of positive solutions of the problem (1.4)–(1.5). By a *positive solution*, we mean a solution $u(t)$ to the boundary value problem such that $u(t) > 0$ for $0 < t < 1$. Throughout the paper, we assume that

(H1) The functions $f : [0, \infty) \to [0, \infty)$ and $g : [0, 1] \to [0, \infty)$ are continuous, and $g(t) \not\equiv 0$ on $[0, 1]$;
(H2) The constant p satisfies $1/2 < p < 1$.

We will use the following fixed point theorem, which is due to Krasnosel'skii, to prove the existence results.

Theorem 1.1 ([7]). *Let X be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let $\le$ be the partial order on X determined by P. Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let*

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

be a completely continuous operator such that, either

(K1) *$Lu \not\ge u$ if $u \in P \cap \partial\Omega_1$, and $Lu \not\le u$ if $u \in P \cap \partial\Omega_2$; or*
(K2) *$Lu \not\le u$ if $u \in P \cap \partial\Omega_1$, and $Lu \not\ge u$ if $u \in P \cap \partial\Omega_2$.*

Then L has a fixed point in $P \cap (\overline{\Omega}_2 - \Omega_1)$.

Before the Krasnosel'skii fixed point theorem can be used to obtain any existence result, we need to find some nice estimates to positive solutions to the problem (1.4)–(1.5) first. These *a priori* estimates are essential to a successful application of the Krasnosel'skii fixed point theorem. It is based on these estimates that we can define an appropriate cone on which Theorem 1.1 can be applied. Better estimates will result in sharper existence and nonexistence conditions.

We now fix some notation. Throughout we let $X = C[0, 1]$ with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)| \quad \text{for all } v \in X.$$

Clearly, X is a Banach space. Also, we define the constants

$$F_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x},$$
$$F_\infty = \limsup_{x\to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to +\infty} \frac{f(x)}{x}.$$

These constants will be used later in the statements of our existence theorems.

This paper is organized as follows. In Section 2, we obtain some *a priori* estimates to positive solutions to the problem (1.4)–(1.5). In Section 3, we define a positive cone of the Banach space X using the estimates obtained in Section 2, and apply Theorem 1.1 to establish some existence results for positive solutions of the problem

(1.4)–(1.5). In Section 4, we present some nonexistence results. An example is given at the end of the paper to illustrate our results.

2. Green Function and Estimates of Positive Solutions

In this section, we give the Green function for the problem (1.4)–(1.5) and prove some estimates for positive solutions of the problem.

We need the indicator function χ to write the expression for the Green's function for the problem (1.4)–(1.5). Recall that if $[a,b] \subset R := (-\infty,+\infty)$ is a closed interval, then the indicator function χ of $[a,b]$ is given by

$$\chi_{[a,b]}(t) = \begin{cases} 1, & \text{if } t \in [a,b], \\ 0, & \text{if } t \notin [a,b]. \end{cases}$$

Now we define the function $G : [0,1] \times [0,1] \to [0,\infty)$ by

$$\begin{aligned} G(t,s) = {} & \frac{t(1+p)-t^2}{2} - \frac{t(1-s)^2}{2(1-p)} + \frac{t(p-s)^2}{2(1-p)}\chi_{[0,p]}(s) \\ & + \frac{(t-s)^2}{2}\chi_{[0,t]}(s). \end{aligned} \tag{2.1}$$

Then $G(t,s)$ is the Green function associated with the problem (1.4)–(1.5). Moreover, the problem (1.4)–(1.5) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1. \tag{2.2}$$

It can be shown that $G(t,s) \ge 0$ if $(t,s) \in [0,1]^2$.

The following result is based on Lemmas 2.1 and 2.2 of Graef and Yang [3].

Lemma 2.1. *If $u'''(t) \ge 0$ for $0 \le t \le 1$, and $u(0) = u'(\beta) = u''(1) = 0$, where $\beta \in (1/2,1)$ is a constant, then $u(t) \ge 0$ for $0 \le t \le 1$, and*

$$\frac{2\beta t - t^2}{\beta^2}u(\beta) \le u(t) \le u(\beta) \quad \text{for } 0 \le t \le 1.$$

Throughout this paper we let

$$a(t) = \begin{cases} 2t - t^2, & \text{if } t \le \frac{2p}{1+p}, \\ \frac{2pt-t^2}{p^2}, & \text{if } t \ge \frac{2p}{1+p}. \end{cases}$$

It can be shown that

$$a(t) \ge \min\{t, 1-t\}, \quad 0 \le t \le 1.$$

The proof of the last inequality is straightforward and therefore is omitted.

Lemma 2.2. *If $u \in C^3[0,1]$ is such that*

$$u'''(t) \ge 0, \quad 0 \le t \le 1, \tag{2.3}$$

and

$$u(0) = u(p) - u(1) = u''(1) = 0,$$

then $u(t) \ge a(t)\|u\|$ for $0 \le t \le 1$.

Proof. Since $u(p) = u(1)$, there exists $\beta \in (p, 1) \subset (1/2, 1)$ such that $u'(\beta) = 0$. Since $u'''(t) \geq 0$ for $0 \leq t \leq 1$ and $u(0) = u'(\beta) = u''(1) = 0$, by Lemma 2.1 we have

$$\frac{2\beta t - t^2}{\beta^2} u(\beta) \leq u(t) \leq u(\beta) = \|u\|, \quad 0 \leq t \leq 1.$$

If $0 \leq t \leq 2p/(1+p)$, then

$$\begin{aligned}
u(t) - a(t)\|u\| &\geq u(\beta)\left[\frac{2\beta t - t^2}{\beta^2} - (2t - t^2)\right] \\
&= \frac{u(\beta)}{\beta^2} t(1-\beta)[2\beta - t(1+\beta)] \\
&\geq \frac{u(\beta)}{\beta^2} t(1-\beta)\left[2\beta - \frac{2p}{1+p}(1+\beta)\right] \\
&= \frac{2tu(\beta)(1-\beta)}{\beta^2(1+p)}(\beta - p) \geq 0.
\end{aligned}$$

If $2p/(1+p) \leq t \leq 1$, then

$$\begin{aligned}
u(t) - a(t)\|u\| &\geq u(\beta)\left[\frac{2\beta t - t^2}{\beta^2} - \frac{2pt - t^2}{p^2}\right] \\
&= \frac{tu(\beta)(\beta - p)}{\beta^2 p^2}[(p+\beta)t - 2p\beta] \\
&= \frac{tu(\beta)(\beta - p)}{\beta^2 p^2}(pt + \beta(t - 2p)) \\
&\geq \frac{tu(\beta)(\beta - p)}{\beta^2 p^2}\left[\frac{2p^2}{1+p} + \beta\left(\frac{2p}{1+p} - 2p\right)\right] \\
&= \frac{tu(\beta)(\beta - p)}{\beta^2 p^2} \frac{2p^2}{1+p}(1-\beta) \geq 0.
\end{aligned}$$

Thus, we have proved that $u(t) \geq a(t)\|u\|$ on $[0, 1]$. □

The following lemma is immediate.

Lemma 2.3. *If $u \in C^3[0,1]$ is such that $u'''(t) \geq 0$, $0 \leq t \leq 1$, and*

$$u(0) = u(p) - u(1) = u''(1) = 0,$$

then

$$u(2p/(1+p)) \geq \frac{4p}{(1+p)^2}\|u\|,$$

or equivalently,

$$\|u\| \leq \frac{(1+p)^2}{4p} u(2p/(1+p)).$$

We can summarize our findings in the following theorem.

Theorem 2.4. *Suppose that* (H1) *and* (H2) *hold. If $u \in C^3[0,1]$ satisfies* (2.3) *and the boundary conditions* (1.5)*, then $u(t) \geq a(t)\|u\|$ on $[0,1]$. In particular, if $u \in C^3[0,1]$ is a nonnegative solution to the boundary value problem* (1.4)–(1.5)*, then $u(t) \geq a(t)\|u\|$ on $[0,1]$.*

Now we define

$$P=\{v\in X : a(t)\|v\|\le v(t) \quad \text{on} \quad [0,1]\}.$$

Clearly, P is a positive cone of the Banach space X. Define an operator $T : P\to X$ by

$$Tu(t)=\int_0^1 G(t,s)g(s)f(u(s))ds,\quad 0\le t\le 1,\ \text{for all } u\in X.$$

It is well known that $T : P\to X$ is a completely continuous operator. And by the same arguments as those used to prove Theorem 2.4, we can show that $T(P)\subset P$. We also note that if $v\in P$, then

$$\|v\|\le \frac{(1+p)^2}{4p}v(2p/(1+p)).$$

Now the integral equation (2.2) is equivalent to the equality

$$Tu=u,\quad u\in P,$$

so in order to solve the problem (1.4)–(1.5), we only need to find a fixed point of T in P.

3. Existence of Positive Solutions

We begin by defining the constants

$$A=\int_0^1 G(2p/(1+p),s)g(s)a(s)\,ds \quad \text{and} \quad B=\int_0^1 G(2p/(1+p),s)g(s)\,ds.$$

Our first existence result is the following.

Theorem 3.1. *If*

$$BF_0\frac{(1+p)^2}{4p}<1<Af_\infty,$$

then the problem (1.4)–(1.5) *has at least one positive solution.*

Proof. Choose $\epsilon>0$ such that $(F_0+\epsilon)B(1+p)^2/4p<1$. Then there exists $H_1>0$ such that

$$f(x)\le (F_0+\epsilon)x \quad \text{for} \quad 0<x\le H_1.$$

For each $u\in P$ with $\|u\|=H_1$, we have

$$\begin{aligned}
(Tu)(2p/(1+p)) &= \int_0^1 G(2p/(1+p),s)g(s)f(u(s))\,ds\\
&\le \int_0^1 G(2p/(1+p),s)g(s)(F_0+\epsilon)u(s)\,ds\\
&\le (F_0+\epsilon)\|u\|\int_0^1 G(2p/(1+p),s)g(s)ds\\
&= (F_0+\epsilon)\|u\|B\\
&\le B(F_0+\epsilon)\frac{(1+p)^2}{4p}u(2p/(1+p))\\
&< u(2p/(1+p)),
\end{aligned}$$

which means $Tu\not\geq u$. If we let $\Omega_1=\{u\in X \mid \|u\|<H_1\}$, then

$$Tu\not\geq u,\quad \text{for any } u\in P\cap\partial\Omega_1.$$

To construct Ω_2, we first choose $c \in (0, 1/4)$ and $\delta > 0$ such that

$$(f_\infty - \delta) \int_c^{1-c} G(2p/(1+p), s) g(s) a(s)\, ds > 1.$$

Now, there exists $H_3 > 0$ such that $f(x) \geq (f_\infty - \delta)x$ for $x \geq H_3$. Let $H_2 = H_1 + H_3/c$. If $u \in P$ with $\|u\| = H_2$, then for $c \leq t \leq 1 - c$, we have

$$u(t) \geq \min\{t, 1-t\}\|u\| \geq cH_2 \geq H_3.$$

So, if $u \in P$ with $\|u\| = H_2$, then

$$\begin{aligned}
(Tu)(2p/(1+p)) &\geq \int_c^{1-c} G(2p/(1+p), s) g(s) f(u(s))\, ds \\
&\geq \int_c^{1-c} G(2p/(1+p), s) g(s) (f_\infty - \delta) u(s) ds \\
&\geq (f_\infty - \delta)\|u\| \int_c^{1-c} G(2p/(1+p), s) g(s) a(s)\, ds \\
&> \|u\| \\
&\geq u(2p/(1+p)),
\end{aligned}$$

which means $Tu \not\leq u$. So, if we let $\Omega_2 = \{u \in X : \|u\| < H_2\}$, then $\Omega_1 \subset \Omega_2$, and

$$Tu \not\leq u, \quad \text{for any } u \in P \cap \partial\Omega_2.$$

Therefore, condition (K1) of Theorem 1.1 is satisfied, and so there exists a fixed point of T in P. This completes the proof of the theorem. $\square$

Our next theorem is a companion result to the one above.

Theorem 3.2. *If*

$$BF_\infty \frac{(1+p)^2}{4p} < 1 < Af_0,$$

then the problem (1.4)–(1.5) *has at least one positive solution.*

Proof. We first choose $\epsilon > 0$ such that

$$A(f_0 - \epsilon) > 1.$$

There exists $H_1 > 0$ such that $f(x) \geq (f_0 - \epsilon)x$ for $x \geq H_1$. If $u \in P$ with $\|u\| = H_1$, then

$$\begin{aligned}
(Tu)(2p/(1+p)) &\geq \int_0^1 G(2p/(1+p), s) g(s) f(u(s))\, ds \\
&\geq \int_0^1 G(2p/(1+p), s) g(s) (f_0 - \epsilon) u(s) ds \\
&\geq (f_0 - \epsilon)\|u\| \int_0^1 G(2p/(1+p), s) g(s) a(s)\, ds \\
&> \|u\| \\
&\geq u(2p/(1+p)),
\end{aligned}$$

which means $Tu \not\leq u$. So, if we let $\Omega_1 = \{u \in X \mid \|u\| < H_1\}$, then

$$Tu \not\leq u, \quad \text{for any } u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta \in (0,1)$ such that

$$((F_\infty+\delta)B+\delta)\frac{(1+p)^2}{4p}<1.$$

There exists $H_3>0$ such that $f(x)\le (F_\infty+\delta)x$ for $x\ge H_3$. If we let $M=\max_{0\le x\le H_3} f(x)$, then $f(x)\le M+(F_\infty+\delta)x$ for $x\ge 0$. Let

$$K=M\int_0^1 G(2p/(1+p),s)g(s)ds+1,$$

and let $H_2=H_1+K\big(\frac{4p}{(1+p)^2}-(F_\infty+\delta)B\big)^{-1}$. Now for each $u\in P$ with $\|u\|=H_2$, we have

$$\begin{aligned}
(Tu)(2p/(1+p)) &= \int_0^1 G(2p/(1+p),s)g(s)f(u(s))\,ds\\
&\le \int_0^1 G(2p/(1+p),s)g(s)(M+(F_\infty+\delta)u(s))\,ds\\
&< K+(F_\infty+\delta)\int_0^1 G(2p/(1+p),s)g(s)u(s)\,ds\\
&\le K+(F_\infty+\delta)\|u\|\int_0^1 G(2p/(1+p),s)g(s)\,ds\\
&\le K+(F_\infty+\delta)B\|u\|\\
&\le \Big(\frac{4p}{(1+p)^2}-(F_\infty+\delta)B\Big)H_2+(F_\infty+\delta)BH_2\\
&= \frac{4p}{(1+p)^2}\|u\|\\
&\le u(2p/(1+p)),
\end{aligned}$$

which means $Tu\ngeq u$. So, if we let $\Omega_2=\{u\in X \mid \|u\|<H_2\}$, then

$$Tu\ngeq u,\quad \text{for any } u\in P\cap\partial\Omega_2.$$

By Theorem 1.1, T has a fixed point in $P\cap(\overline{\Omega}_2-\Omega_1)$. Therefore, problem (1.4)–(1.5) has at least one positive solution, and this completes the proof of the theorem. $\square$

4. Nonexistence Results and Example

In this section, we give some sufficient conditions for the nonexistence of positive solutions.

Theorem 4.1. *Suppose that* (H1) *and* (H2) *hold. If* $\frac{(1+p)^2}{4p}Bf(x)<x$ *for all* $x\in(0,+\infty)$, *then the problem* (1.4)–(1.5) *has no positive solutions.*

Proof. Assume to the contrary that $u(t)$ is a positive solution of problem (1.4)–(1.5). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(2p/(1+p)) &= \int_0^1 G(2p/(1+p), s)g(s)f(u(s))\,ds \\ &< \frac{4p}{(1+p)^2}B^{-1}\int_0^1 G(2p/(1+p), s)g(s)u(s)\,ds \\ &\le \frac{4p}{(1+p)^2}B^{-1}\|u\|\int_0^1 G(2p/(1+p), s)g(s)ds \\ &\le \frac{4p}{(1+p)^2}\|u\| \\ &\le u(2p/(1+p), \end{aligned}$$

which is a contradiction. □

In a similar fashion, we can prove the next theorem.

Theorem 4.2. *Suppose that* (H1) *and* (H2) *hold. If* $Af(x) > x$ *for all* $x \in (0, +\infty)$, *then the problem* (1.4)–(1.5) *has no positive solutions.*

We conclude the paper with an example.

Example 4.3. Consider the third-order boundary-value problem

$$u'''(t) = g(t)f(u(t)), \quad 0 < t < 1, \tag{4.1}$$

$$u(0) = u(3/4) - u(1) = u''(1) = 0, \tag{4.2}$$

where

$$\begin{gathered} g(t) = (1+t)/10, \quad 0 \le t \le 1, \\ f(u) = \lambda u\frac{1+3u}{1+u}, \quad u \ge 0. \end{gathered}$$

Here $\lambda > 0$ is a parameter. We easily see that $F_0 = f_0 = \lambda$ and $F_\infty = f_\infty = 3\lambda$. Calculations indicate that

$$A = \frac{5268393409}{216850636800}, \quad B = \frac{33611}{1229312}.$$

From Theorem 3.1 we see that if

$$13.7203 \approx \frac{1}{3A} < \lambda < \frac{48}{49B} \approx 35.8282,$$

then problem (4.1)–(4.2) has at least one positive solution. From Theorems 4.1 and 4.2, we see that if

$$\lambda < \frac{16}{49B} \approx 11.9427 \quad \text{or} \quad \lambda > \frac{1}{A} \approx 41.1607,$$

then problem (4.1)–(4.2) has no positive solutions.

This example shows that our existence and nonexistence conditions work very well.

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