Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems (2012). *Electronic Journal of Differential Equations*, Conference 21 (2014), pp. 45–59. ISSN: 1072-6691. http://ejde.math.txstate.edu, http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ELLIPTIC PROBLEMS ON THE SPACE OF WEIGHTED WITH THE DISTANCE TO THE BOUNDARY INTEGRABLE FUNCTIONS REVISITED

JESÚS IDELFONSO DÍAZ, JEAN-MICHEL RAKOTOSON

Abstract. We revisit the regularity of very weak solution to second-order elliptic equations $Lu = f$ in Ω with $u = 0$ on $\partial\Omega$ for $f \in L^1(\Omega, \delta)$, $\delta(x)$ the distance to the boundary $\partial\Omega$. While doing this, we extend our previous results (and many others in the literature) by allowing the presence of distributions $f+g$ which are more general than Radon measures (more precisely with g in the dual of suitable Lorentz-Sobolev spaces) and by making weaker assumptions on the coefficients of L. One of the new tools is a Hardy type inequality developed recently by the second author. Applications to the study of the gradient of solutions of some singular semilinear equations are also given.

1. Introduction

In recent works [1, 11, 12, 13, 23, 24, 25] a complete study of the differentiability of very weak solutions (the so called Brezis' problem) was done. This problem reads as follows

$$u \in L^1(\Omega), \quad \int_\Omega u L^* \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega,$$

where f is an integrable function with the distance function to the boundary as weight.

In those works strong regularity on the data were assumed either for Ω (which was at least of class C^2 at least) or for the coefficients of the linear operator L (assumed to be in $C^{0,1}$) and always for $f \in L^1(\Omega, \delta)$. We want to show here that we can weaken all the data that we have considered namely we can replace f by $f - \sum_i \frac{\partial f_i}{\partial x_i}$, $f_i \in L^1(\Omega)$. This result can be seen as an extension of Stampacchia [27] and Brezis-Strauss results for $L^1(\Omega)$-data [4]. More precisely, we shall show that we can replace f by a more general datum $f + g$ with $g \in W^{-1}L^{N',\infty}(\Omega) = \big(W_0^1 L^{N,1}(\Omega)\big)^*$. Notice that since $W_0^1 L^{N,1}(\Omega) \subset C(\overline{\Omega})$ then $\mathcal{M}(\Omega) \subset W^{-1}L^{N',\infty}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the set of bounded Radon measures. Moreover, if $f \in L^1\big(\Omega, \delta(1 + |\log \delta|)\big)$ we can weaken the regularity of Ω, that is the boundary is of class $C^{1,\gamma}$ for some $\gamma \in (0, 1]$

2000 *Mathematics Subject Classification.* 35J25, 35J60, 35P30, 35J67.
Key words and phrases. Very weak solutions; semilinear elliptic equations;
distance to the boundary; weighted spaces measure; Hardy inequalities; Hardy spaces.
©2014 Texas State University - San Marcos.
Published February 10, 2014.

and to assume the coefficients of L less regular. These improvements have been obtained thanks to a better use of some old or new Hardy inequalities. As an application, we will show that the use of those Hardy inequalities to some singular semilinear problems allows to get results proved previously by different authors as Ghergu [16], Mancebo-Hernandez [15], Del Pino [10], Díaz-Hernandez-Rakotoson [13] , Gui-Lin [17].

Finally, we also give new applications of the new Hardy inequalities considered here, as the existence and regularity of the very weak solution to the nonlinear Dirichlet equation

$$-\Delta u = \frac{\widehat{a}(x)}{u^a(1+\log_+ u)^m + K(x,u)} = f(x) \quad \text{in } \Omega,$$

with $K(x,0)=0$.

2. Notation and preliminary results

For a Lebesgue measurable set E of Ω we denote by $|E|$ its measure and χ_E its characteristic function.

The *decreasing rearrangement of a measurable function* $u:\Omega\to\mathbb{R}$ is given by

$$u_*:\Omega_*=\left]0,|\Omega|\right[\to\mathbb{R},\quad u_*(s)=\inf\{t\in\mathbb{R}:|u>t|\leqslant s\},$$
$$u_*(0)=\operatorname{ess\,sup}_\Omega u,\quad u_*(|\Omega|)=\operatorname{ess\,inf}_\Omega u,$$
$$|u|_{**}(t)=\frac{1}{t}\int_0^t |u|_*(s)ds \quad \text{for } t\in\Omega_*=\left]0,|\Omega|\right[.$$

We shall use the following Lorentz spaces (see [2, 26] for example), for $1<p\leqslant+\infty$, $1\leqslant q\leqslant+\infty$:

$$L^{p,q}(\Omega)=\Big\{v:\Omega\to\mathbb{R} \text{ measurable } |v|^q_{L^{p,q}}=\int_0^{|\Omega|}[t^{1/p}|v|_{**}(t)]^q\frac{dt}{t}<+\infty\Big\},$$

for $q<+\infty$.

$$L^{p,\infty}(\Omega)=\Big\{v:\Omega\to\mathbb{R} \text{ measurable } |v|_{L^{p,\infty}}=\sup_{t\leqslant|\Omega|}t^{1/p}|v|_{**}(t)<+\infty\Big\}.$$

We recall that

$$L^{p,p}(\Omega)=L^p(\Omega),\quad L^{p,s}(\Omega)\subset L^{r,q}(\Omega) \text{ once } r<p, \text{ for any } q,\ s\in[1,+\infty].$$

Finally we notice that

$$L^{p,1}(\Omega)\subset L^{p,s}(\Omega)\subset L^{p,q}(\Omega)\subset L^{p,\infty}(\Omega) \text{ if } 1\leqslant s<q\leqslant+\infty,$$

for any $p\in]1,+\infty]$. We denote $\partial_i=\frac{\partial}{\partial x_i}$, $\partial_{ij}=\frac{\partial^2}{\partial x_i\partial x_j}$, and by χ_E the characteristic function of a set $E\subset\Omega$.

For $\alpha>0$, we introduce now Zygmund spaces $L^\alpha_{\exp}(\Omega)$ and $L(\log L)$. They satisfy the following inclusions $L^\infty\subset L^\alpha_{\exp}\subset L^p\subset L(\log L)\subset L^1$ for any $p\in(1,+\infty)$. Although there are several equivalent formulations we prefer the following ones:

$$L^\alpha_{\exp}(\Omega)=\Big\{v:\Omega\to\mathbb{R} \text{ measurable: } \|v\|_\alpha=\sup_{t\leqslant|\Omega|}\frac{|v|_{**}(t)}{\big(1+\log\frac{|\Omega|}{t}\big)^\alpha}<+\infty\Big\},$$

It is a Banach space under the norm

$$\|v\|_{L^\alpha_{\exp}(\Omega)}=\sup_{0<t<|\Omega|}\frac{|v|_{**}(t)}{(1+\log\frac{|\Omega|}{t})^\alpha}.$$

$$W^1L^\alpha_{\exp}(\Omega)=\{v\in L^1(\Omega):|\nabla v|\in L^\alpha_{\exp}(\Omega)\},$$
$$W^1_0L^\alpha_{\exp}(\Omega)=W^1L^\alpha_{\exp}(\Omega)\cap W^{1,1}_0(\Omega).$$

$$L(\log L)=\Big\{v:\Omega\to\mathbb{R}\text{ measurable, }|v|_{L(\log L)}=\int_0^{|\Omega|}|v|_{**}(t)dt<+\infty\Big\}.$$

We note that $L^1_{\exp}(\Omega)=L_{\exp}(\Omega)$ and $L(\log L)$ are associate each other (see [2]).

In particular, one has a constant $c>0$ such that for all $f\in L_{\exp}(\Omega)$ and all $g\in L(\log L)$,

$$\int_\Omega |fg|dx\leqslant c|f|_{L_{\exp}(\Omega)}\cdot|g|_{L(\log L)}.$$

Finally, we define the Sobolev-Lorentz spaces

$$W^1(\Omega,|\cdot|_{p,q})=W^1L^{p,q}(\Omega)=\{v\in W^{1,1}(\Omega):|\nabla v|\in L^{p,q}(\Omega)\}$$

and

$$C^m_c(\Omega)=\{\varphi\in C^m(\Omega),\ \varphi\text{ has compact support in }\Omega\},$$
$$C^{0,1}(\overline{\Omega})=\{v:\overline{\Omega}\to\mathbb{R}\text{ is a Lipschitz function}\},$$
$$C^{m,1}(\overline{\Omega})=\{v\in C^m(\overline{\Omega}):D^\alpha v\in C^{0,1}(\overline{\Omega})\text{ for }|\alpha|=m\},$$
$$C^{0,\gamma}(\overline{\Omega})=\{v\in C^0(\overline{\Omega}):v\text{ is }\gamma\text{-Hölder continuous}\},$$
$$W^1_0L^{p,q}(\Omega)\text{ is the closure of }C^1_c(\Omega)\text{ in }W^1L^{p,q}(\Omega).$$

We shall use some other functional spaces but we prefer to postpone their introduction to the precise moment in which they will be used. We shall denote by c various constants depending only on the data. The notation $\approx$ stands for equivalence of nonnegative quantities; that is,

$$\Lambda_1\approx\Lambda_2\iff\exists\, c_1>0,\ c_2>0\text{ such that }c_1\Lambda_2\leqslant\Lambda_1\leqslant c_2\Lambda_2.$$

$B(x;r)$ will denote the ball of $\mathbb{R}^N$ centered at x of radius $r>0$.

3. Hardy type inequalities and their applications

3.1. **Revisiting and improving old results.** For simplicity, we shall consider the linear operator L defined by

$$Lv=-\sum_{i,j=1}^N\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial v}{\partial x_j}\Big)$$

with $a_{ij}\in C^{0,\gamma}(\overline{\Omega})$, for some $\gamma\in[0,1]$, $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j\geqslant b_0|\xi|^2$, for all $\xi\in\mathbb{R}^N$, $x\in\Omega$ and $b_0>0$. Its formal adjoint is given by

$$L^*v=-\sum_{i,j=1}^N\frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial v}{\partial x_i}\Big). \tag{3.1}$$

We recall the following definition.

Definition 3.1 (Very weak solution (v.w.s.)). Assume $\gamma=1$, let $f\in L^1(\Omega,\delta)$. A very weak solution of the Dirichlet problem $Lu=f$, $u=0$ on $\partial\Omega$ is a function $u\in L^1(\Omega)$ satisfying

$$\int_\Omega uL^*\varphi dx=\int_\Omega f\varphi dx,\quad\forall\varphi\in C^2(\overline{\Omega}),\quad\varphi=0\text{ on }\partial\Omega.$$

Definition 3.2 (weak solution (w.s.))**.** Assume $\gamma \in [0,1]$, let $f \in L^1(\Omega,\delta)$. A weak solution of the Dirichlet problem $Lu = f$, $u = 0$ on $\partial\Omega$ is a function $u \in W_0^{1,1}(\Omega)$ satisfying

$$\int_\Omega a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

The first Hardy inequality that we have used in [11] was not the usual one which we can get in the text books (see Theorem 3.3 below) but a simpler one which can be easily proved by the mean value theorem.

Theorem 3.3 (L^∞-Hardy-Sobolev inequality)**.** *Assume that Ω is a bounded Lipschitz open set. Then*

$$\forall \varphi \in W_0^{1,\infty}(\Omega), \quad \frac{|\varphi(x)|}{\delta(x)} \leqslant |\nabla \varphi|_\infty \quad \text{for any } x \in \Omega. \tag{3.2}$$

Such relation was also given in [3] and gives a justification to the right hand side term in definition 3.1 (which can be extended to right hand side term in $L^1(\Omega,\delta) + W^{-1}L^{N',\infty}(\Omega)$ in an obvious way). An application of such inequality is the following existence result extending the result of [11] and the one by Brezis -Strauss [4], to the case of more general sourcing terms.

Theorem 3.4. *Let $f \in L^1(\Omega,\delta)$, $f_i \in L^1(\Omega)$, $i = 1,\dots,N$, let L with $\gamma = 1$, and let Ω be a $C^{1,1}$ open bounded set. Then there exists an unique function*

$$u \in \begin{cases} L^{N',\infty}(\Omega),\ N' = \frac{N}{N-1} & \text{if } N \geqslant 2, \\ L^\infty(\Omega) & \text{if } N = 1, \end{cases}$$

such that

$$\int_\Omega u L^* \varphi \, dx = \int_\Omega f \varphi \, dx + \sum_{i=1}^N \int_\Omega f_i \frac{\partial \varphi}{\partial x_i} dx, \tag{3.3}$$

for all $f \in C^2(\overline{\Omega})$, $\varphi = 0$ on $\partial\Omega$.

Proof. We follow the same scheme as in [11]. Consider $g_{ki} = T_k(f_i)$, $T_k(f) = g_k$ be the truncation at level k. Then there exists $\varphi_k \in W^2L^{N,1}(\Omega) \cap H_0^1(\Omega)$ a solution of

$$L^* \varphi_k = \chi_E \operatorname{sign}(u_k), \tag{3.4}$$

where $E \subset \Omega$ is a measurable set and $u_k \in H_0^1(\Omega)$ satisfies,

$$L u_k = g_k - \sum_{i=1}^N \frac{\partial}{\partial x_i} g_{ik} \quad \text{in } H^{-1}(\Omega). \tag{3.5}$$

Then

$$\langle \varphi_k, L u_k \rangle = \int_\Omega g_k \varphi_k + \sum_{i=1}^N \int_\Omega g_{ik} \frac{\partial \varphi_k}{\partial x_{ik}} dx, \tag{3.6}$$

$$\langle \varphi_k, L u_k \rangle = \int_\Omega u_k L^* \varphi_k = \int_E |u_k| dx, \tag{3.7}$$

$$\int_E |u_k| dx \leqslant \Big(\int_\Omega |g_k| \delta \, dx \Big) |\varphi_k(x) \delta(x)^{-1}|_\infty + |\nabla \varphi_k|_\infty \int_\Omega \Big(\sum g_{ik}^2 \Big)^{1/2} dx. \tag{3.8}$$

Using Theorem 3.3,

$$\int_E |u_k|dx \leqslant \Big[\Big(\int_\Omega |g_k|\delta dx\Big) + \sum_{i=1}^n \int_\Omega |g_{ik}|dx\Big]|\nabla\varphi_k|_\infty. \tag{3.9}$$

By $W^{2,p}$ regularity of φ_k, we have

$$|\nabla\varphi_k|_\infty \leqslant c|\chi_E|_{L^{N,1}} \leqslant c|E|^{1/N}. \tag{3.10}$$

Thus

$$\begin{gathered} |u_k|_{L^{N',\infty}(\Omega)} \leqslant c\Big[\int_\Omega |g|\delta\, dx + \sum_{i=1}^N \int_\Omega |g_i|dx\Big] \quad \text{if } N \geqslant 2, \\ |u_k|_\infty \leqslant c\Big[\int_\Omega |g|\delta\, dx + \sum_{i=1}^N \int_\Omega |g_i|dx\Big] \quad \text{if } N = 1. \end{gathered} \tag{3.11}$$

We conclude as in [11] by applying the Hardy-Littlewood inequality. □

In [21], we give a more general framework for the right hand side. The second Hardy type inequality that we used in [11] is as follows.

Theorem 3.5 (Hardy-Sobolev-Lorentz inequality). *Let $0 < \alpha < 1$ and assume that Ω is a bounded Lipschitz open set. Then there exists $c > 0$ such that*

$$\frac{|\psi(x)|}{\delta^\alpha(x)} \leqslant c|\nabla\psi|_{L^{\frac{N}{1-\alpha}}(\Omega)}, \tag{3.12}$$

for all $\psi \in W_0^1 L^{\frac{N}{1-\alpha}}(\Omega)$ and all $x \in \Omega$.

Remark 3.6. If $\alpha = 0$, we have

$$|\psi(x)| \leqslant c|\nabla\psi|_{L^{N,1}(\Omega)}, \quad \forall x \in \Omega,\ \forall\psi \in W_0^1 L^{N,1}(\Omega). \tag{3.13}$$

Theorem 3.5 implies the following major lemma.

Lemma 3.7. *Let $0 < \alpha < 1$. Under the same assumption as in Theorem 3.5, we have*

$$L^1(\Omega, \delta^\alpha) \hookrightarrow W^{-1,\frac{N}{N-1+\alpha}}(\Omega).$$

If $\alpha = 0$, then

$$L^1(\Omega) \hookrightarrow W^{-1}L^{N',\infty}(\Omega) = \big(W_0^1 L^{N,1}(\Omega)\big)^*.$$

Proof. (implicitly given in [11]) Let $\varphi \in W_0^{1,\frac{N}{1-\alpha}}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ $(\frac{N}{1-\alpha} > N)$. Then, we write

$$\begin{aligned} \int_\Omega |f|\,|\varphi|dx &= \int_\Omega |f|\,\delta^\alpha|\varphi|\,\delta^{-\alpha} \\ &\leqslant \Big|\,|\varphi|\delta^{-\alpha}\Big|_\infty \int_\Omega |f|\delta^\alpha \\ &\leqslant c|\nabla\varphi|_{L^{\frac{N}{1-\alpha}}(\Omega)} \int_\Omega |f|\delta^\alpha \\ &= c\|\varphi\| \int_\Omega |f|\delta^\alpha, \end{aligned}$$

$$\sup_{\|\varphi\|\leqslant 1} \int_\Omega f\varphi \leqslant c\int_\Omega |f|\delta^\alpha.$$

The second inequality corresponding to $\alpha = 0$ follows from the same argument using relation (3.13). □

A similar inequality related to $W_0^1 L^\alpha_{\exp}(\Omega)$ can be provided (see [24]), for any $\alpha > 0$,

$$\frac{|\varphi(x)|}{\delta(x)\big(1+|\log\delta(x)|\big)^\alpha} \leqslant c\|\nabla\varphi\|_{L^\alpha_{\exp}(\Omega)}, \quad \forall x \in \Omega.$$

Lemma 3.7 has been also observed by Amrouche after reading [11] (personal communication). The following theorem can be found in [6] and extended in [23, 25] for Lorentz spaces.

Theorem 3.8. *Assume that $\Omega \in C^1$, and the coefficients a_{ij} are bounded with a vanishing mean oscillation(for instance a_{ij} continuous in $\overline{\Omega}$). Let $u \in W_0^{1,1}(\Omega)$ be the weak solution of*

$$Lu = \operatorname{div}(F) \quad \text{in the sense of distributions} \tag{3.14}$$

whenever $F \in L^{p,q}(\Omega)^N$, $1 < p < +\infty$, $1 \leqslant q \leqslant +\infty$. Then, there exists a constant $c(\Omega) > 0$ such that

$$|\nabla u|_{L^{p,q}(\Omega)} \leqslant c|F|_{L^{p,q}(\Omega)^N}. \tag{3.15}$$

We can apply Theorem 3.8 to Lemma 3.7, to deduce that for $f \in L^1(\Omega,\delta^\alpha)$ and $0 < \alpha < 1$,

$$\text{there exist } F \in L^p(\Omega)^N \text{ with } p = \frac{N}{N-1+\alpha}, \text{ such that } f = \operatorname{div}(F). \tag{3.16}$$

If $f \in L^1(\Omega)$, then $f \in W^{-1}L^{N',\infty}(\Omega)$, according to the inequality (3.13),

$$\text{there exists } F \in L^{N',\infty}(\Omega) \text{ such that } f = \operatorname{div}(F). \tag{3.17}$$

With relations (3.16) and (3.17), we have the following result.

Theorem 3.9. *The very weak solution (v.w.s.) of $Lu = f$, $u = 0$ on $\partial\Omega$ satisfies*

- *$u \in W_0^1 L^{N(\alpha)}(\Omega)$ with $N(\alpha) = \frac{N}{N-1+\alpha}$, for $0 < \alpha < 1$,*
$$|\nabla u|_{L^{N(\alpha)}} \leqslant c|f|_{L^1(\Omega,\delta^\alpha)};$$
- *$u \in W_0^1 L^{N',\infty}(\Omega)$, for $\alpha = 0$,*
$$|\nabla u|_{L^{N',+\infty}} \leqslant c|f|_{L^1(\Omega)}.$$

Therefore, the v.w.s is then a weak solution so the assumption on the coefficients can be relaxed, say a_{ij} being bounded but in $VMO(\Omega)$ is enough (see e.g. [28] for a treatment of $VMO(\Omega)$). For treating the limit case $\alpha \to 1$, the following Hardy inequalities were introduced in [24].

Theorem 3.10 ([24])**.** *Assume that Ω is a bounded Lipschitz open set and let $\beta > 0$. Then there exists $c(\Omega) > 0$, such that*

$$\frac{|\varphi(x)|}{\delta(x)(1+|\log\delta(x)|)^\beta} \leqslant c(\Omega)\|\nabla\varphi\|_{L^\beta_{\exp}(\Omega)}, \quad \forall \varphi \in W_0^1 L^\beta_{\exp}(\Omega).$$

The proof is based on the following Lemma (see [24]).

Lemma 3.11 ([24])**.** *Let Ω be an open set in $\mathbb{R}^N$, $r > 0$, $B(x,r) \subset \Omega$, $u \in W^1 L^\alpha_{\exp}(\Omega)$. Then*

$$\operatorname{Osc}_{B(x,r)} u \leqslant \frac{\alpha_N^{1-\frac{1}{N}}}{\alpha_{N-1}} e^{1/N} N^{\alpha+1} |\Omega|^{1/N} \Gamma\big(\alpha+1;\omega_N(r)\big)\|\nabla u\|_{L^\alpha_{\exp}(\Omega)},$$

where

$$\Gamma(x;a) = \int_a^{+\infty} e^{-t} \cdot t^{x-1} dt,$$

We recall that there exists a constant $c_N(\alpha) > 0$ such that

$$\Gamma\big(\alpha+1;\omega_N(r)\big) \leqslant c_N(\alpha) r\big(1+|\log r|\big)^{\alpha}.$$

In particular for $\alpha = 1$, if we consider $u \in W^1 L_{\exp}(\Omega)$, then

$$\operatorname{Osc}_{B(x,r)} u \leqslant c_N r\big(1+|\log r|\big)\|\nabla u\|_{L_{\exp}}.$$

Theorem 3.12. *Assume that $\Omega \in C^{1,\alpha}$ for some $0 < \alpha \leqslant 1$, $a_{ij} \in C^{0,\alpha}(\overline{\Omega})$. Then, for $f \in L^1(\Omega, \delta(1+\log\delta|))$, there is an unique very weak solution of $Lu = f$ satisfying $u \in W_0^{1,1}(\Omega)$, and then*

$$\sum_{i,j} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega f\varphi\, dx,$$

i.e. u is also a weak solution of $Lu = f$.

Remark 3.13. When $f \in L^1(\Omega, \delta^\alpha)$ with $0 \leqslant \alpha < 1$, the weak solution exists under the assumption that $a_{ij} \in \mathrm{VMO}(\Omega) \cap L^\infty(\Omega)$; see [25].

Proof of Theorem 3.12. Let $u_k \in H_0^1(\Omega)$ be the solution of $Lu_k = T_k(f) = f_k$. According to Campanato's regularity results [8] and John-Nirenberg inequality [18, 28], we have $\varphi_k \in W_0^1 L_{\exp}(\Omega)$ satisfying

$$\sum_{i,j} \int_\Omega a_{ij}(x) \frac{\partial \varphi_k}{\partial x_j} \frac{\partial \psi}{\partial x_j}\, dx = \int_\Omega H(\nabla u_k)\nabla\psi\, dx, \quad \forall \psi \in H_0^1(\Omega),$$

with $H(\nabla u_k) = \frac{\nabla u_k}{|\nabla u_k|}$ if $\nabla u_k \neq 0$ and zero otherwise. Therefore, we can argue as in [11, 24] to obtain

$$\begin{aligned}
\int_\Omega |\nabla u_k| dx &= \sum_{i,j} \int_\Omega a_{ij}(x) \frac{\partial \varphi_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} dx \\
&= \int_\Omega f\varphi_k \leqslant \Big|\frac{\varphi_k}{\delta(1+\log\delta|)}\Big|_\infty \int_\Omega |f|\delta(1+\log\delta|)\, dx \\
&\leqslant c|\nabla\varphi_k|_{L_{\exp}} \int_\Omega |f|\delta(1+\log\delta|) dx.
\end{aligned}$$

Since $|\nabla\varphi_k|_{L_{\exp}} \leqslant c(\Omega)$ we get the desired result. $\square$

Remark 3.14. The Campanato results are given under the assumptions that $a_{ij} = a_{ji}$ but a closer look at his proof shows that these assumptions can be removed to obtain the BMO regularity.

The following result is given in [24].

Theorem 3.15 (Hardy inequality in weighted space)**.** *Let Ω be a bounded Lipschitz open set. Then there exists a constant $c(\Omega) > 0$ such that*

$$\int_\Omega \frac{|\psi(x)|}{\delta(x)}\, dx \leqslant c(\Omega) \int_\Omega |\nabla\psi|(x)(1+|\log\delta(x)|) dx.$$

for all $\psi \in W_0^1 L\big(\Omega, (1+|\log\delta|)\big)$, where

$$W_0^1 L\big(\Omega, (1+|\log\delta|)\big) = \big\{\varphi \in W_0^{1,1}(\Omega) : \int_\Omega |\nabla\varphi|\,|\log\delta(x)| dx < +\infty\big\}.$$

We point out that other different versions of the Hardy inequality in weighted spaces can be found in Brezis-Marcus [5]. From Theorem 3.15 we derive the following result (see [14, 24]).

Corollary 3.16. *Under the hypothesis of Theorem 3.15, one has a constant* $c(\Omega) > 0$ *such that for all* $\psi \in W_0^1 L(\log L)$,

$$\int_\Omega \frac{|\psi(x)|}{\delta(x)} dx \leqslant c(\Omega) \int_{\Omega_*} |\nabla \psi|_{**}(t) dt.$$

The link with the very weak solution and those theorems is contained the following theorem.

Theorem 3.17 ([24])**.** *Let* $f \in L^1(\Omega, \delta)$, $f \geqslant 0$ *and* u *the very weak solution of* $Lu = f$, $u = 0$ *on* $\partial\Omega$. *Then*

$$\frac{u}{\delta} \in L^1(\Omega) \text{ if and only if } f \in L^1(\Omega, \delta(1 + |\log \delta|)).$$

As a consequence of Theorem 3.15 and Theorem 3.17, we have the following result.

Theorem 3.18 ([1, 24])**.** *If* $f \in L^1(\Omega, \delta) \backslash L^1(\Omega, \delta(1 + \log \delta|))$, $f \geqslant 0$, *then the very weak solution of* $Lu = f$, $u = 0$ *on* $\partial\Omega$ *verifies*

$$\int_\Omega [\nabla u| \log_+ |\nabla u| \, dx = +\infty, \quad = \int_\Omega |\nabla u| \, |\log \delta| \, dx = +\infty,$$

where $\log_+ \sigma = \begin{cases} \log \sigma & \text{if } \sigma \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$

For the application of the above result, we want to select few previous results and derive additional properties. For instance let us consider the following equation treated by Ghergu see [16] (see [13] for a similar problem).

Theorem 3.19. [16] *Let* $p \geqslant 0$, $A > \operatorname{diam}(\Omega)$, $a \in \mathbb{R}$. *Then the problem*

$$\begin{gathered} -\Delta u = \delta(x)^{-2} [A - \log \delta(x)]^{-a} u^{-p}, \\ u > 0, \quad u \in C(\overline{\Omega}) \cap C^2(\Omega), \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

has a solution if and only if $a > 1$. *Moreover, if* $a > 1$ *then*

$$c_1 \big[A - \log \delta(x)\big]^{\frac{1-a}{1+p}} \leqslant u(x) \leqslant c_2 \big[A - \log \delta(x)\big]^{\frac{1-a}{1+p}} \quad \text{for any } x \in \Omega\,.$$

The gradient behaviour is not included in this theorem, nevertheless we have the following result.

Theorem 3.20. • *If* $a > 2 + p$ *then* $u \in W_0^{1,1}(\Omega)$.
• *If* $1 < a \leqslant 2 + p$ *then*

$$\int_\Omega |\nabla u| \log(1 + |\nabla u|) dx = +\infty, \quad \int_\Omega |\nabla u| \, |\log \delta(x)| dx = +\infty.$$

Proof. Indeed, $f(x) = \delta(x)^{-2}[A - \log \delta(x)]^{-a} u^{-p}$ is equivalent to $f_0(x) = \delta(x)^{-2}[A - \log \delta(x)]^{-\frac{a+p}{1+p}}$ according to the growth of u. Then a direct computation shows that: $\int_\Omega f_0(x) \Big[A - \log \delta(x)\Big] dx$ is finite if and only if $a > 2 + p$. Hence for $a \leqslant 2 + p$ we can apply the blow-up phenomena given in Theorem 3.18 or in [24]. □

Next we have corollary to Theorem 3.20.

Corollary 3.21. *Assume that Ω is B the unit ball of $\mathbb{R}^N$. Then the solution u given in Theorem 3.19 is radial and then, $u \in W^{1,1}(\Omega)$. If $1 < a \leqslant 2 + p$, then $\sum_{i=0}^{N} |\frac{\partial u}{\partial x_i}|_{\mathcal{H}^1(B)} = +\infty$. Here $\mathcal{H}^1(B)$ denotes the Hardy space defined in* [24].

Proof. To prove that the solution is radial, is a slight modification of the Ghergu's argument adding to the fixed point set the constraint "radial function". Since u is radial therefore $u \in W_0^{1,1}(B)$ and when $a \leqslant 2 + p$ the right hand side is not in $L^1(\Omega, \delta(1 + |\log \delta|))$. We then conclude as in Theorem 3.20. □

Remark 3.22. The space $W_0^1\mathcal{H}^1(\Omega)$ is included in $W_0^1L^1(\Omega)$, and it contains $W_0^1L(\log L)(\Omega)$.

There are many Hardy inequalities that we can develop using the same argument as in [19, 24]. Here are two of them.

Theorem 3.23 (Hardy inequality with weights)**.** *Let $\Omega \in C^{0,1}$ and $a > 0$. Then there exists $c_a(\Omega) > 0$ such that*

$$\int_\Omega \frac{|\psi(x)|}{\delta^a(x)}\, dx \leqslant c_a(\Omega) \int_\Omega |\nabla\psi|(x)\delta^{1-a}(x)dx \quad \text{if } a > 1,\ \forall \psi \in C_c^1(\Omega)$$

Proof. The idea of proof is the same as it is done in [19, 24]. Using the same notation as in those references, since $\Omega \in C^{0,1}$. Then

$$\int_{\Omega_i} \frac{|\psi(x)|}{\delta^a(x)}\, dx \leqslant \frac{c_i}{1-a} \int_{\mathcal{O}_i} dx_i' \int_{a_i(x_i')}^{a_i(x_i')+\beta} |\psi| \frac{\partial}{\partial x_{iN}} (x_{iN} - a_i(x_i'))^{1-a} dx_{iN}.$$

By integration by part and dropping non positive term ($a > 1$), we have

$$\begin{aligned}\int_{\Omega_i} \frac{\psi(x)|}{\delta^a(x)} &\leqslant c_i(\Omega) \int_{\mathcal{O}_i} \int_{a_i(x_i')}^{a_i(x_i')+\beta} \frac{\partial}{\partial x_{iN}} |\psi| \big(x_{iN} - a_i(x_i')\big)^{1-a} dx_{iN} \\ &\leqslant c_i \int_{\Omega_i} |\nabla\psi|(x)\delta(x)^{1-a} dx.\end{aligned}$$

The same argument holds for the second inequality. □

Here are some applications of those Hardy inequalities.

Theorem 3.24. *Let $u \in C(\overline{\Omega}) \cap H^2_{\rm loc}(\Omega)$ solution of*

$$0 \leqslant Lu \leqslant c_a\delta(x)^{-a} \quad \text{with } 1 < a < 2,\ u = 0 \text{ on } \delta\Omega,$$

the coefficients of L are Lipschitz in Ω, that is $\gamma = 1$. Then,

$$u \in W_0^1 L^{\frac{1}{a-1},\infty}(\Omega).$$

Moreover, there exist a constant $C_a > 0$ independent of u such that

$$|u|_{W_0^1 L^{\frac{1}{a-1},\infty}(\Omega)} \leqslant C_a$$

Remark 3.25. The above inequality was considered in [16] the novelty here is the regularity of the gradient and its estimate.

The main tool for deriving such result is the following lemma.

Lemma 3.26. *Assume that Ω is an open bounded Lipschitz set. One has $\delta^{1-a} \in L^{\frac{1}{a-1},\infty}(\Omega)$ whenever $a > 1$.*

Proof. We set $v = \delta^{1-a}$. Then it is sufficient to show that there exists $c_0 < +\infty$ such that
$$t \leqslant c_0|v > t|^{1-a}, \quad \forall t > 0.$$
But this is equivalent to prove the existence of c_0 such that
$$\text{meas}\left\{x : \delta(x) < t^{-\frac{1}{a-1}}\right\} \leqslant c_0^{\frac{1}{a-1}} t^{-\frac{1}{a-1}} \quad \text{if } a > 1.$$
Setting $\lambda = t^{-1/(a-1)}$, we have to prove that
$$\text{meas}\{x : \delta(x) < \lambda\} \leqslant c_0^{\frac{1}{a-1}} \lambda \quad \forall 0 < \lambda < |\delta|_\infty, \text{ for some } c_0.$$
Since Ω is smooth (say $C^{0,1}$) we then have
$$\text{meas}\{x : \delta(x) < \lambda\} = O(\lambda) \quad \text{as } \lambda \to 0.$$
Which implies the result. □

Proof of Theorem 3.24. Thanks to the above Lemma and Theorem 3.23 on Hardy inequality, we have $\delta^{-a} \in W^{-1}L^{\frac{1}{a-1},\infty}(\Omega)$, thus
$$Lu \in W^{-1}L^{\frac{1}{a-1},\infty}(\Omega).$$
Indeed, for $\psi \in W_0^1 L^{n_a,1}(\Omega),\ n_a = \frac{a-1}{2-a}$, we have
$$\Big|\int_\Omega Lu\psi dx\Big| \leqslant c\int_\Omega \delta^{-a}|\psi|dx \leqslant c\int_\Omega |\nabla\psi|\delta^{1-a} \leqslant c|\nabla\psi|_{L^{n_a,1}}|\delta^{1-a}|_{L^{\frac{1}{a-1},\infty}}.$$
Applying well known regularity (see Theorem 3.8) we have $|\nabla u| \in L^{\frac{1}{a-1},\infty}(\Omega)$. □

For the case $a = 1$, we have the following regularity.

Theorem 3.27. *Let $u \in W_0^{1,1}(\Omega) \cap H^2_{\rm loc}(\Omega)$ be a solution of $0 \leqslant Lu \leqslant c\delta^{-1}$ in Ω for some constant $c > 0$. Then,*
$$u \in \cap_{p<+\infty} W_0^{1,p}(\Omega) \hookrightarrow C^{0,\nu}(\overline{\Omega}),\ quad\forall\nu \in [0,1[.$$
Moreover, if $L = -\Delta$ (for simplicity) then
$$|\nabla u(x)| \leqslant c_p\delta^{-N/p}(x) \quad \forall p > N,\ \forall x \in \Omega.$$

Remark 3.28. Compared to recent results [15, 16, 17, 20], our result here make precise the behavior of the gradient.

We recall now the well known Hardy inequality in $W_0^{1,p}(\Omega)$, $1 < p \leqslant +\infty$ (see [19]).

Theorem 3.29. *Let Ω be an open bounded Lipschitz domain, $1 < p \leqslant +\infty$, there exists a constant $c_\Omega > 0$ such that*
$$\Big(\int_\Omega \Big|\frac{\varphi(x)}{\delta(x)}\Big|^p dx\Big)^{1/p} \leqslant c_\Omega \frac{p}{p-1}\Big(\int_\Omega |\nabla\varphi|^p dx\Big)^{1/p}, \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

Proof of Theorem 3.27. Since $\delta^{-1} \in W^{-1,p}(\Omega)$, $\forall p, 1 < p < +\infty$, we deduce that
$$Lu \in W^{-1,p}(\Omega),\ u \in W_0^{1,1}(\Omega).$$
Thus from Theorem 3.8, $\nabla u \in L^p(\Omega)^N$ for all $p < +\infty$. While for the second statement, we have
$$-\Delta\Big(\frac{u^2}{2}\Big) + |\nabla u|^2 \in L^p(\Omega), \quad \forall p < +\infty.$$

Then
$$\frac{u^2}{2} \in W^{2,p}(\Omega), \quad \forall p < +\infty.$$
In particular, $\frac{u^2}{2} \in C^{1,1-\frac{N}{p}}(\overline{\Omega})$. Since $u(x) \geqslant c\delta(x)$, one deduces that
$$|\nabla u(x)| \leqslant c_p \delta(x)^{-N/p} \quad \forall p < +\infty,\ p > N.$$
□

Here is an example of an application: the following problem $(\mathcal{L})$ was considered by various authors [6, 7, 15, 20] and it was shown that for $\alpha > 1$, and $p \in L^\infty_+(\Omega)$, there exists $c_1 > 0$ such that
$$u(x) \geqslant c_1 \delta^{\frac{2}{1+\alpha}}$$
and
$$\begin{gathered} -\Delta u = \frac{p(x)}{u^\alpha(x)} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.18}$$
But none of the previous article studied the behaviour of the gradient when $\alpha \geqslant 1$. Our main result is as follows.

Theorem 3.30. *Any solution u of* (3.18) *satisfies*
$$|\nabla u| \in L^{\frac{\alpha+1}{\alpha-1},\infty}(\Omega).$$

Proof. With the growth of u one has $0 \leqslant -\Delta u \leqslant c\delta^{-2\alpha/(1+\alpha)}$. We apply Theorem 3.24 with $a = \frac{2\alpha}{1+\alpha}$. □

Theorem 3.27 can be applied also to the following equation considered by Gui-Lin[17] when $0 \leqslant p \leqslant c\delta(x)^\beta$,
$$\begin{gathered} -\Delta u = \frac{p(x)}{u^\alpha(x)} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$
They showed that $u \in C^{0,\nu}(\overline{\Omega})$ for all $0 < \nu < 1$ if $\alpha - \beta = 1$. In fact, the growth of the solution u implies
$$0 \leqslant -\Delta u \leqslant c\delta(x)^{\beta-\alpha} = c\delta^1(x).$$
So our Theorem 3.27 implies, in particular, their results thanks to Sobolev imbedding. The result seems to be optimal since in this case Gui-Lin [17] showed that
$$u \notin C^{0,1}(\overline{\Omega}).$$

3.2. **A new existence result for a singular semilinear equation with a general right hand side.** We may apply Theorem 3.24 to solve the following equation for $a \in]1, 2[$,
$$\begin{gathered} -\Delta u = \frac{\widehat{a}(x)}{u^a(1 + \log_+ u)^m + K(x, u)} = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.19}$$
when we assume that $\widehat{a}(x) \in L^\infty(\Omega)$, $\inf \widehat{a} = \operatorname{ess\,inf} \widehat{a} > 0$, K is a Caratheodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$, nondecreasing with respect to the second variable for almost all x on $\mathbb{R}_+$; i.e. if $0 \leqslant \sigma \leqslant t$, then $K(x, \sigma) \leqslant K(x, t)$ and $K(x, 0) = 0$ for a.e. x.

Theorem 3.31. *Let u be a non negative solution of* (3.19). *Then*

(1) *If $u \geqslant c\delta$ for some $c > 0$ then $u \in W_0^1 L^{\frac{1}{a-1},\infty}(\Omega)$.*
(2) *If $m \geqslant 0$, then $u \in L^\infty(\Omega)$.*

Proof. (1) If $u \geqslant c\delta$ for some $c > 0$, since $K(x, u(x)) \geqslant 0$, we then have

$$0 \leqslant f(x) \leqslant c_0 \frac{1}{u^a(1+\log_+ u)^m} \leqslant \frac{1}{c^a \delta^a}.$$

Therefore, $0 \leqslant -\Delta u \leqslant c^{-a}\delta^{-a}$, thus $u \in W_0^1 L^{\frac{1}{a-1},\infty}(\Omega)$ according to Theorem 3.24.

(2) For the second statement, we use standard method for the boundedness of the solution (see [9, 25]).

Since $u \geqslant 0$, for $s \in [0, |u > \frac{1}{2}|]$, we have $u_*(s) \geqslant \frac{1}{2}$ and we choose $(u - u_*(s))_+$ as a test function for the variational equation, where s is fixed. Then

$$\begin{aligned} \int_{u>u_*(s)} |\nabla u|^2 dx &= \int_\Omega \frac{\widehat{a}(x)}{u^\beta(1+|Ln_+ u|)^m + K(x,u)}(u-u_*(s))_+ \\ &= \int_\Omega \widetilde{f}(x)(u-u_*(s))_+(x)dx, \end{aligned} \tag{3.20}$$

with $\widetilde{f}(x) = 0$ on $\{u \leqslant 1/2\}$. We note that

$$|\widetilde{f}(x)| \leqslant M_0 = 2^a|\widehat{a}|_\infty < +\infty \quad \text{on } \{x : (u > u_*(s))_+(x)\} \subset \{u > \frac{1}{2}\}.$$

Differentiating (3.20) with respect to s one has using the notion of relative rearrangement, [22, 26],

$$(|\nabla u|^2_{*u})(s) = \Big(\int_{u>u_*(s)} \widetilde{f}dx\Big)(-u_*'(s)) \leqslant c_N s^{\frac{1}{N}-1}|\nabla u|_{*u}(s)\int_0^s \widetilde{f}_* dt. \tag{3.21}$$

By the properties of the relative rearrangement one has

$$-u_*'(s) \leqslant c_N s^{\frac{1}{N}-1}|\nabla u|_{*u}(s), \tag{3.22}$$

$$(|\nabla u|^2_{*u}) \geqslant (|\nabla u|_{*u})^2. \tag{3.23}$$

From the equation (3.21) to (3.23) with the estimate of $\widetilde{f}$, we have

$$|\nabla u|_{*u}(s) \leqslant c_N s^{\frac{1}{N}-1} M_0 s = c_N M_0 s^{1/N}.$$

Therefore relation (3.22) implies

$$-u_*'(s) \leqslant M_0 \widetilde{c}_N s^{\frac{2}{N}-1}$$

An integration of this inequality leads to

$$u_*(0) \leqslant \frac{1}{2} + \widetilde{c}_N M_0 \int_0^{|u>1/2|} t^{\frac{2}{N}-1} dt = \frac{1}{2} + \widetilde{c}_N M_0 |u > \frac{1}{2}|^{2/N};$$

that is,

$$|u|_\infty \leqslant \frac{1}{2} + \widetilde{c}_N 2^a \|\widehat{a}\|_\infty |\Omega|^{2/N}.$$

$\square$

Remark 3.32. We may replace 1/2 by any k_0 to obtain

$$|u|_\infty \leqslant k_0 + \widetilde{c}_N M_0 |u > k_0|^{1/N} \quad k_0 > 0.$$

The L^∞ estimate will imply the existence of a constance $c > 0$ such that $u \geqslant c\delta$. The operator $-\Delta$ by L in all of this section provided that the coefficients are Lipschitz that is $\gamma = 1$.

The existence result for (3.19) follows from the following theorem.

Theorem 3.33. *Assume that K is a Caratheodory function, with $K(x,0) = 0$, for any constant k, $K(.,k)$ is a bounded function in Ω. Then (3.19) admits a solution $u \in W_0^1 L^{\frac{1}{a-1},\infty}(\Omega)$ whenever $a \in]1,2[$.*

Proof. Let $0 < \varepsilon < 1$, then there exists a function $u_\varepsilon \in W^2 L^{N,1}(\Omega) \cap H_0^1(\Omega)$

$$-\Delta u_\varepsilon = \frac{\widehat{a}(x)}{D(u_\varepsilon + \varepsilon)},\ u_\varepsilon \geqslant 0$$

where we have set $D(\sigma) = \sigma^a(1 + \log_+ \sigma)^m + K(x,\sigma)$. Since

$$D(u_\varepsilon + \varepsilon) \geqslant D(u_\varepsilon) \geqslant u_\varepsilon^a(1 + \log_+ u_\varepsilon)^m,$$

we have

$$0 \leqslant -\Delta u_\varepsilon \leqslant \frac{\widehat{a}(x)}{u_\varepsilon^a(1 + \log_+ u_\varepsilon)^m}.$$

Arguing as in the second statement, of theorem 3.31 we deduce that

$$\|u_\varepsilon\|_\infty \leqslant \frac{1}{2} + \widetilde{c}_N 2^a \|\widehat{a}\|_\infty |\Omega|^{2/N} \doteq M_0.$$

Let $\eta > 0$ such that

$$-\lambda_1 \eta \varphi_1 D(M_0 + 1) \leqslant \inf \widehat{a},$$

where λ_1 is the first eigenvalue associated to the Dirichlet problem and φ_1 the first eigenfunction. Then

$$-\lambda_1 \eta \varphi_1 \leqslant \frac{\inf \widehat{a}}{D(u_\varepsilon + \varepsilon)};$$

therefore,

$$-\Delta(\eta\varphi_1) \leqslant -\Delta u_\varepsilon.$$

By the maximum principle we deduce $u_\varepsilon \geqslant \eta\varphi_1$. Thus

$$0 \leqslant -\Delta u_\varepsilon \leqslant c\varphi_1^{-a}.$$

Applying Theorem 3.24 and knowing that φ_1 is equivalent to the distance function δ we deduce that u_ε belongs to a bounded set of $W_0^1 L^{\frac{1}{a-1},\infty}(\Omega)$. By usual argument we can pass to the limit as $\varepsilon \to 0$,

$$-\Delta u = \frac{\widehat{a}(x)}{D(u)} \text{ in } \mathcal{D}'(\Omega).$$

□

Remark 3.34. In Merker-Rakotoson [21], we extend some of those results to Neumann problems. A more general version of Theorem 3.4 is also presented.

Acknowledgments. A major part of this article was presented at the Colloquium "Variational and Topological methods" June 2012, in Flagstaff, Northern Arizona University, by the second author. He is grateful to organizers for their invitation and hospitality. J. I. Díaz was supported by projects MTM 2011-26119 of the DG-ISPI (Spain) and the ITN FIRST of the European Community (Grant Agreement 238702).

References

[1] F. Abergel, J. M. Rakotoson; Gradient blow-up in Zygmund spaces for the very weak solution of a linear elliptic equation. *Discrete and Continuous Dynamical Systems* **33**, 5 (2013) 1809-1818. (doi:10.3934/dcds.2013.33).

[2] C. Bennett, R. Sharpley; Interpolation of Operators, *Academic Press*, London, (1983).

[3] H. Brezis; Personal communication with J.I. Díaz.

[4] H. Brezis, W. A. Strauss; Semilinear second order elliptic equations in L^1, *J.Math. Soc . Japan*, **25** (1973) 565-590.

[5] H. Brezis, M. Marcus; Hardy's inequality revisited, *Ann. Scuola Norm. Sup. Pisa*, **cl. sci. 4** (1997) 217-237.

[6] Sun-Sig Byun; Parabolic equations with BMO coefficients in Lipschitz domains, *Journal of differential Equations*, **209** (2005) 229-295.

[7] M. G. Crandall, P. H. Rabinowitz, L. Tartar; On a Dirichlet problem with a singular nonlinearity, *Comm. Part. Diff. Eq.* **2**, (1977) 193-222.

[8] S. Campanato; Equazioni ellittiche dell IIe ordine e spacazi $\mathcal{L}^{2,\lambda}$, *Annali di Matematica*, (1965) 41-370.

[9] J. I. Díaz; *Nonlinear partial differential equations and free boundaries*, (1985) Pitman, London.

[10] M. Del Pino; A global estimate for the gradient in a singular elliptic boundary value problem. *Proc. Roy. Soc.* Edinburgh Sect A, 122, (1992), 341-352

[11] J. I. Díaz, J. M. Rakotoson; On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary. *J. Functional Analysis* doi:10.1016/j.jfa.2009.03.002, **257**, (2009), 807-831.

[12] J. I. Díaz, J. M. Rakotoson; On very weak solution of semi-linear elliptic equation in the framework of weighted spaces with respect to the distance to the boundary. *Discrete and Continuous Dynamical Systems* , **27** (3), (2010), 1037-1085.

[13] J. I. Díaz, J. Hernández, J. M. Rakotoson; On very weak positive solutions to some semilinear elliptic problems with simultaneous singular nonlinear and spacial dependence terms, *Milan J. Math*,**79** (2011) 233-245.

[14] P. Hajlasz; Pointwise Hardy inequalities, *Proc. Amer. Math. Soc.*, **127** 2, (1999), 417-423.

[15] J. Hernández, F. Mancebo; *Singular elliptic and parabolic equations* in Handbook of differential equations Stationary partial differential equations, vol **3** 317-400.

[16] M. Ghergu; Lane-Emden systems with negative exponents, *J. Functional Analysis.* **258**(2010) 3295-3318.

[17] C. Gui, F. Hua Lin; Regularity of an elliptic problem with a singular nonlinearity, *Proc. Roy. Soc. Edimburg Sect.* A **123** (1993) 1021-1029.

[18] F. John, L. Niremberg; On functions of bounded mean oscillation, *Comm. Pure Appl. Math.*, 14,(1961) 415-426.

[19] A. Kufner; *Weighted Sobolev spaces*, John Wiley & Sons, New-York, 1985.

[20] A. C. Lazer, P. J. Mc Kenna; *On a singular nonlinear elliptic boundary value problem*, Proc. Amerr. Math. Soc. **111** (1991), 721-730.

[21] J. Merker, J. M. Rakotoson; Very weak solutions of Poisson's equation with singular data under Neumann boundary conditions. to appear.

[22] J. Mossino, R. Temam; Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in plasma physics, *Duke Make J.* **48** (1981) 475-495.

[23] J. M. Rakotoson; Regularity of a very weak solution for parabolic equations and applications, *Adv. Diff. Equa.* **16** 9-10 (2011), 867-894.

[24] J. M. Rakotoson; New Hardy inequalities and behaviour of linear elliptic equations, *J. Funct. Ann.* **263** (2012), 2893-2920.
[25] J. M. Rakotoson; A few natural extension of the regularity of a very weak solution, *Diff. Int. Eq.* **24** (11-12) (2011), 1125-1140.
[26] J. M. Rakotoson; *Réarrangement Relatif: un instrument d'estimation dans les problèmes aux limites*, (2008), Springer Verlag, Berlin.
[27] G. Stampacchia; Some limit case of L^p-estimates for solutions of second order elliptic equations, *Comm. Pure Appli. Math.* 16 (1963) 505-510.
[28] A. Torchinsky; *Real-Variable Methods in Harmonic Analysis*, *Academic Press*, Orlando, 1986.

Jesús Idelfonso Díaz
Instituto de Matemática Interdisciplinar and Departamento de Matemática Aplicada, Universidad Complutense de Mmadrid, Plaza de las Ciencias No. 3, 28040 Madrid, Spain
E-mail address: diaz.racefyn@insde.es

Jean-Michel Rakotoson
Laboratoire de Mathématiques et Applications, Université de Poitiers, Boulevard Marie et Pierre Curie, Teleport 2, BP 30179, 86962 Futuroscope Chasseneuil Cedex, France
E-mail address: rako@math.univ-poitiers.fr