Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems (2012). *Electronic Journal of Differential Equations*, Conference 21 (2014), pp. 77–86. ISSN: 1072-6691. http://ejde.math.txstate.edu, http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE, UNIQUENESS AND NUMERICAL APPROXIMATION OF SOLUTIONS TO A NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WHICH ARISES IN OPTION PRICING THEORY

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Abstract. This article studies the existence and uniqueness of solutions for a fully nonlinear Black-Scholes equation which arises in option pricing theory in connection with the jump and equilibrium model approach by using delta-hedging arguments. We prove existence and uniqueness for this nonlinear integro-differential equation by using a fixed point method. The convergence of the numerical scheme, which is based on finite differences, is also proved.

1. Introduction

The article by Black and Scholes [5] represents the foundation of modern option pricing theory. They were able to show that the option pricing problem is equivalent to a second-order final-value problem. In the following years there have been a lot of approaches to make the Black-Scholes model more realistic, in especially to relax the restrictive assumptions which were made. First developments considered the "smile effect" which was observed by using the Black-Scholes model in practice. These developments have led to *implied volatility models* and later, by assuming that the volatility is itself stochastic, to *stochastic volatility models* (cf. [2, 10, 14]). Other developments considered *transaction costs* (cf. [3, 17]), the *influence of large traders* (cf. [9, 11]) or studied the modeling of the underlying by Brownian motions and generalized it to Lévy processes, which yields to *jump models* (cf. [6, 16, 20]), or to fractional Brownian motions (cf. [18, 19]).

All these approaches lead to different types of partial differential equations. To be more precise, we consider final value problems of the form

$$\begin{gathered} V_t(t,S) + \mathcal{BS}\left(t,S,V,V_S,V_{SS}\right) = 0 \quad (t,S) \in [0,T) \times (0,\infty) \\ V(T,S) = g(S) \quad S \in (0,\infty), \end{gathered} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 35K15, 35K67, 91G80.
Key words and phrases. Option pricing; Black-Scholes equations; fully nonlinear equation, integro-differential equation.

Published February 10, 2014.

where $\mathcal{BS}$ is the nonlinear Black-Scholes operator which is given by

$$\mathcal{BS}\left(t, S, V, V_S, V_{SS}\right) \stackrel{\text{def}}{=} \frac{\tilde{\sigma}^2\left(t, S, V_S, V_{SS}\right)}{2} S^2 V_{SS} + (r(t) - q(t, S)) S V_S - r(t) V + \mathcal{I}\left[V\right],$$

where $V : [0, T] \times (0, \infty) \to \mathbb{R}$ is the pricing function, $g : (0, \infty) \to \mathbb{R}$ the payoff function, $r : [0, T] \to \mathbb{R}$ the risk-free interest rate, $q : [0, T] \times (0, \infty) \to \mathbb{R}$ the continuously paid dividend, $\mathcal{I}$ is a nonlocal integral term and $\tilde{\sigma} : [0, T] \times (0, \infty) \times \mathbb{R} \times \mathbb{R} \to [0, \infty)$ the modified volatility function which depends on the specific model. Because the different models describe different points, a natural question which arises in this context is: What happens if we combine these models with each other and can we still expect that there exists a unique solution?

2. Derivation of the equation

To answer that question, we want to combine the *jump model* with the *reaction-function equilibrium model*. The jump model approach criticizes the ansatz to use Brownian motions to simulate the development of the underlying because the increments of a Brownian motion are Gaussian and the observed stock price changes are fat-tailed (cf. [18]). Hence, the idea is to use an other stochastic process than a Brownian motion to simulate the stock price dynamics; i.e., instead of

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t),$$

we consider the following dynamics[1]

$$dS(t) = \mu S(t-)\, dt + dX(t), \tag{2.1}$$

where $X(t)$ is a Lévy process ($t \in [0, T]$). Lévy processes are more general than Brownian motions, but they are also more difficult to handle. The main difficulty lies in the fact that Lévy processes need not to be continuous which means that the stock price dynamics can also be discontinuous. Normally, it is assumed that $dX(t) = \sigma S(t-)\, dW(t) +$ an integral part (cf. [6, chapter 12]).

The reaction-function equilibrium model which we want to use, is the model of Sircar&Papanicolaou (cf. [23]). This type of model assumes that the stock prices in the economy are determined by the equilibrium condition of market clearing which depends on a fundamental (exogenous) value. This value is simulated by a second stochastic process Y with $dY(t) = \tilde{\mu}(t, Y)dt + \tilde{\eta}(t, Y)d\tilde{W}(t)$. In this and in other equilibrium models, it is often assumed that there are two different kinds of traders:

- *Reference traders* or *ordinary traders* are investors who sell and buy in such a way as they were the only agents in the economy. Their normalized stock demand is modelled as a function $D(t, Y(t), S(t))$.
- *Program traders* or *large traders* are investors who have an influence to the market equilibrium. Their normalized stock demand is written in the form $\rho\zeta(t)$, where the parameter ρ measures the size of the trader's position relative to the total supply and $\zeta(t)$ represents the cumulative trading strategy.

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[1]Because the stock price dynamics are not longer continuous, we have to use the left limit of S, in particular $S(t-) = \lim_{\tau \nearrow t} S(\tau)$. This is well-defined because a Lévy process is RCLL.

Consequently, the equilibrium price is determined by

$$G(t,Y,S) = D(t,Y,S) + \rho\zeta(t) = 1. \tag{2.2}$$

If we further assume that (2.2) admits a unique solution, we can express $S(t)$ as a function of $Y(t)$ and $\rho\zeta(t)$, i.e. $S(t) = \psi(Y(t), \rho\zeta(t))$. The function ψ is also called *reaction-function*. Following the argumentation in [23, Sections 2.2-2.5], one can show that the asset price dynamics are given by

$$dS(t) = \hat{\mu}(t,Y,S)\,dt + \hat{\sigma}(t,Y,S)\,dW(t), \tag{2.3}$$

where $\hat{\mu}$ is given by [23, equation (11)] and

$$\hat{\sigma} = -\frac{G_Y}{G_S}\tilde{\eta}(t,Y). \tag{2.4}$$

To join the equilibrium and jump model approach into one model, we suggest that the asset price dynamics are given by

$$dS(t) = \hat{\mu}(t,Y,S)\,dt + \hat{\sigma}(t,Y,S)\,dW(t) + \int_{\mathbb{R}\setminus\{0\}} S(t-)\left(e^z - 1\right)\tilde{m}\left(dt, dz\right), \tag{2.5}$$

where $\tilde{m}$ is a compensated Lévy measure. The additional integral term can be considered as deviation from the market equilibrium.

Theorem 2.1. *If the market is free of arbitrage, the stock price dynamics follow* (2.5) *and the large traders follow a delta-hedging strategy, then the pricing function V satisfies the following nonlinear integro-differential equation (for sake of readability, we write V and mean $V = V(t,S)$ with $(t,S) \in [0,T) \times (0,\infty)$)*

$$\begin{gathered} V_t + \frac{1}{2}\sigma^2(t,S)\Big(\frac{1-\rho V_S}{1-\rho V_S - \rho S V_{SS}}\Big)^2 S^2 V_{SS} + (r(t) - q(t,S))SV_S \\ -r(t)V + \int_{\mathbb{R}\setminus\{0\}} \nu(dz)\left[V\left(t, Se^z\right) - V(t,S) - S\left(e^z - 1\right)V_S(t,S)\right] = 0 \\ V(T,S) = g(S), \end{gathered} \tag{2.6}$$

where g is Lipschitz-continuous and ν is a compensated Lévy-measure which satisfies

$$\int_{|z|\geq 1} e^{2z}\nu(dz) < \infty.$$

Proof. We use no-arbitrage and delta-hedging arguments. Thus, let $(\gamma(t), \beta(t))$ be a self-financing strategy which replicates the portfolio, i.e. $V(t,S) = \gamma(t)S(t) + \beta(t)B(t)$. Using the extended Itô-formula (cf. [1, Theorem 4.4.10]), choosing $\gamma(t) = V_S(t,S)$ to avoid arbitrage with respect to the diffusion term and using that the volatility is given by (2.4), we obtain

$$\begin{aligned} 0 &= \mathfrak{A}(t)dt + d\mathfrak{M}(t) \\ &:= \Big(V_t + (r(t) - q(t,S))SV_S + \frac{1}{2}\tilde{\eta}^2\Big(\frac{G_Y}{G_S}\Big)^2 S^2 V_{SS} - r(t)V \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \nu(dz)[V\left(t, Se^z\right) - V(t,S) - S\left(e^z - 1\right)V_S(t,S)]\Big)dt \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} [V(t, Se^z) - V(t,S) - S\left(e^z - 1\right)V_S(t,S)]\;\tilde{m}(dt,dz). \end{aligned}$$

Next, we use the fact that we can represent V also as

$$V(t,S) = e^{-\int_t^T r(\tau)\,d\tau}\mathbb{E}\left(g\left(Se^{\int_t^T r(\tau)d\tau + X(T-t)}\right)\right),$$

where $X(t)$ is the non-drift part of the right hand side of the risk neutral version of (2.5) (note that X is a Lévy process). Following the argumentation in [8, Proof of Proposition 2], one can prove that the remaining stochastic term $\mathfrak{M}$, which is a compensated Poisson integral, is a square-integrable martingale. Furthermore, $-\mathfrak{M}(t) = \int_0^t \mathfrak{A}(s)\,ds$ is also a continuous process with finite variation.[2] It follows that $\mathfrak{A}(t) = 0$ almost surely with respect to the equivalent martingale measure.[3] Furthermore, using that $\frac{G_Y}{G_S} = \frac{D_Y}{D_S + \rho\frac{\partial\zeta}{\partial S}}$ and setting $D(t,Y,S) = \frac{Y^{\frac{\sigma}{\tilde\eta}}}{S} = 1 - \rho V_S$, $\tilde\eta(t,Y) = \tilde\eta \cdot Y$, one can derive $\tilde\sigma^2 = \sigma^2[\frac{1-\rho V_S}{1-\rho V_S - \rho S V_{SS}}]^2$. □

3. Existence and uniqueness

In this section, we want to show that there exists a unique classical solution of (2.6). To do this, we rewrite the diffusion term of (2.6) as

$$\frac{1-\rho V_S}{1-\rho V_S - \rho S V_{SS}} = \frac{1}{1 - \rho\Xi\left(t,S,V_S,V_{SS}\right)},$$

where $\Xi = \frac{SV_{SS}}{1-\rho V_S}$. Of course, the main difficulty is to control the behavior of $\rho(t,S)\Xi(t,S,V_S,V_{SS})$ and to assure that there exists an $\epsilon > 0$ such that $1 - \rho(t,S)\Xi(t,S,V_S,V_{SS}) > \epsilon$ for all $(t,S) \in [0,T]\times(0,\infty)$.

Theorem 3.1 (Existence and uniqueness)**.** *Assume that*

(1) $x \mapsto g(e^x)$ *is in* $C^{2+\alpha}(\mathbb{R})$*;*
(2) $r \in C^{\frac{\alpha}{2}}([0,T])$*,* $(\tau,x) \mapsto \sigma^2(\tau,e^x), q(\tau,e^x)$ *are in* $C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R})$*;*
(3) ν *is a compensated Lévy measure with* $\int_{|z|\ge 1} e^{2z}\nu(dz) < \infty$*.*

Then there exists a positive constant $\epsilon_0 > 0$ *such that for any*

$$\hat\rho(\tau,e^x) = e^{-x}\rho(T-\tau,e^x) \in C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R})$$

with

$$\|\hat\rho\|_{C^{\frac{\alpha}{2},\alpha}([0,T]\times\mathbb{R})} \le \epsilon_0, \tag{3.1}$$

problem (2.6) *possesses a unique classical solution.*

Proof. The core of the proof is a fixed point procedure which is based on the fixed point theorem of Banach and on [12, Theorem 3.1]. At first, we perform the Euler ($x = \ln(S)$) together with the backward time transformation ($\tau = T - t$). Doing that, we obtain

$$\begin{gathered} u_\tau - A_u u_{xx} - B_u u_x - C_u u - \mathcal{I}u = 0 \\ u(0,x) = \hat g(x), \end{gathered} \tag{3.2}$$

where we have used the abbreviations

$$A_u = \frac{\hat\sigma^2}{2(1-\hat\rho\hat\Xi)^2}, \quad B_u = \Big(-\frac{\hat\sigma^2}{2(1-\hat\rho\hat\Xi)^2} + (\hat r - \hat q) - \int_{\mathbb{R}\setminus\{0\}}(e^z-1)\,\nu(dz)\Big),$$

$$C_u = -\hat r, \quad \mathcal{I}u = \int_{\mathbb{R}\setminus\{0\}}[u(\tau,x+z) - u(\tau,x)]\,\nu(dz), \quad \hat\Xi = \frac{(u_{xx}-u_x)}{1-\hat\rho u_x},$$

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[2]Here, we have used that every continuous differentiable function has bounded variation.

[3]Here, we have used the result that every square-integrable martingale with paths of finite variation is constant with probability one. This result can be found in [15, Proposition I.4.50].

$$\hat{\rho} = e^{-x}\rho(T-\tau, e^x), \quad \hat{q} = q\left(T-\tau, e^x\right), \quad \hat{\sigma} = \sigma\left(T-\tau, e^x\right).$$

Next, we estimate $\hat{\Xi}$ as follows

$$\begin{aligned} \|\hat{\Xi}\|_{C^{\frac{\alpha}{2},\alpha}} &\le \|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}} \|\frac{1}{1-\hat{\rho}u_x}\|_{C^{\frac{\alpha}{2},\alpha}} \\ &\le, \frac{\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}} + \|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}} \|u\|^2_{C^{1+\frac{\alpha}{2},2+\alpha}}}{\left(1-\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}}\right)^2} =: \Theta\,. \end{aligned} \tag{3.3}$$

Here, we have used the inequality

$$\|\frac{1}{f}\|_{C^{\frac{\alpha}{2},\alpha}} \le \frac{\|f\|_{C^{\frac{\alpha}{2},\alpha}}}{\min_{(\tau,x)\in[0,T]\times\mathbb{R}}(f)^2}.$$

Moreover, we have

$$\|A_u\|_{C^{\frac{\alpha}{2},\alpha}} \overset{(3.3)}{\le} \frac{1}{2}\|\hat{\sigma}^2\|_{C^{\frac{\alpha}{2},\alpha}} \frac{\left(1+\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\Theta\right)^2}{\left(1-\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\Theta\right)^4}. \tag{3.4}$$

Because the $C^{\frac{\alpha}{2},\alpha}$-norms of $\hat{r}$ and $\hat{q}$ are also bounded, we can estimate the other two coefficients B_u and C_u in an analogous way. Furthermore, a lower bound is given by

$$\frac{\hat{\sigma}^2}{2\left(1-\hat{\rho}\hat{\Xi}\right)^2} \ge \frac{\hat{\sigma}^2}{2\left(1+\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\Theta\right)^2}. \tag{3.5}$$

The next step is to consider only such $u \in C^{1+\frac{\alpha}{2},2+\alpha}([0,T]\times\mathbb{R})$ which satisfies

$$\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|u\|_{C^{1+\frac{\alpha}{2},2+\alpha}} \le \xi, \tag{3.6}$$

with a positive constant $\xi \in (0, 1/3)$. Hence, by using (3.4) and (3.5), we conclude that (3.2) is uniformly parabolic as long as (3.6) holds. Now, we have proven all requirements of [12, Theorem 3.1] and thus there exists a solution $\breve{u}$ of the linearized problem

$$\begin{aligned} \breve{u}_\tau - A_u\breve{u}_{xx} - B_u\breve{u}_x - C_u\breve{u} - \mathcal{I}\breve{u} &= 0 \\ \breve{u}(0,x) &= \hat{g}(x). \end{aligned}$$

Furthermore, [12, Theorem 3.1] yields that there exists a positive constant $K_1 = K_1\left(A, B, C\right)$ such that

$$\|\breve{u}\|_{C^{1+\frac{\alpha}{2},2+\alpha}} \le K_1\|\hat{g}(x)\|_{C^{2+\alpha}} = K_1\|g(e^x)\|_{C^{2+\alpha}} < \infty. \tag{3.7}$$

Hence, there exists a map $M : u \mapsto M(u) = \breve{u}$ from $C^{1+\frac{\alpha}{2},2+\alpha}\left([0,T]\times\mathbb{R}\right)$ into itself.

The final step is to prove that M admits a unique fixed point. To establish that result, we consider two functions $v, w \in C^{1+\frac{\alpha}{2},2+\alpha}\left([0,T]\times\mathbb{R}\right)$ which satisfy assumption (3.6). In analogy to the foregoing, we define

$$\breve{v} = M(v), \quad \breve{w} = M(w), \quad \breve{\delta} = \left(\breve{v} - \breve{w}\right), \quad \delta = (v - w), \quad \hat{\Upsilon} = \hat{\Xi}(\tau, x, w_x, w_{xx}).$$

From these definitions, it follows immediately that $\breve{\delta}$ is a solution of

$$\begin{aligned} \breve{\delta}_\tau = A_v\breve{\delta}_{xx} + B_v\breve{\delta}_x + C_v\breve{\delta} + \mathcal{I}\breve{\delta} + [A_v - A_w][\breve{w}_{xx} - \breve{w}_x] \\ \breve{\delta}(0,x) = 0. \end{aligned} \tag{3.8}$$

We can estimate $\|A_v - A_w\|_{C^{\frac{\alpha}{2},\alpha}}$ by

$$C(\hat{\sigma})\,\frac{\|2-\hat{\rho}(\hat{\Xi}+\hat{\Upsilon})\|_{C^{\frac{\alpha}{2},\alpha}}\|\hat{\rho}(\hat{\Upsilon}-\hat{\Xi})\|_{C^{\frac{\alpha}{2},\alpha}}\|(1-\hat{\rho}\hat{\Upsilon})^2(1-\hat{\rho}\hat{\Xi})^2\|_{C^{\frac{\alpha}{2},\alpha}}}{(1-\|\hat{\rho}\hat{\Xi}\|_{C^{\frac{\alpha}{2},\alpha}})^4(1-\|\hat{\rho}\hat{\Upsilon}\|_{C^{\frac{\alpha}{2},\alpha}})^4}.$$

Next, we estimate the enumerator from above by $C_1(\xi)\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|v-w\|_{C^{1+\frac{\alpha}{2},2+\alpha}}$ and the denominator from below by $C_2(\xi)$, where $C_i(\xi)$ are suitable constants which depend on $\xi \in (0,1/3)$ $(i=1,2)$. Plugging these estimates together yields

$$\|A_v - A_w\|_{C^{\frac{\alpha}{2},\alpha}} \le C(\xi)\|\hat{\sigma}^2\|_{C^{\frac{\alpha}{2},\alpha}}\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|\delta\|_{C^{1+\frac{\alpha}{2},2+\alpha}}.$$

Next, we apply [12, Theorem 3.1] to the solution $\breve{\delta}$ of (3.8) which yields that there exist positive constants $C_i < \infty$ $(i = 3,\dots,6)$ such that

$$\begin{aligned}
\|\breve{\delta}\|_{C^{1+\frac{\alpha}{2},2+\alpha}} &\le C_3\|\,[A_v - A_w]\,[\breve{w}_{xx} - \breve{w}_x]\,\|_{C^{\frac{\alpha}{2},\alpha}}\\
&\le C_4\|\breve{w}\|_{C^{1+\frac{\alpha}{2},2+\alpha}}\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|\delta\|_{C^{1+\frac{\alpha}{2},2+\alpha}}\\
&\le C_5\|g(e^x)\|_{C^{2+\alpha}}\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|\delta\|_{C^{1+\frac{\alpha}{2},2+\alpha}}\\
&\le C_6\,\|\hat{\rho}\|_{C^{\frac{\alpha}{2},\alpha}}\|\delta\|_{C^{1+\frac{\alpha}{2},2+\alpha}},
\end{aligned}$$

where we have used (3.7) in the third step. Finally, using (3.1), we can choose $\epsilon_0 = \frac{1}{2(C_6+1)}$ and obtain

$$\|M(v) - M(w)\|_{C^{1+\frac{\alpha}{2},2+\alpha}} \le \frac{1}{2}\|v-w\|_{C^{1+\frac{\alpha}{2},2+\alpha}}.$$

Therefore, M admits a unique fixed point in the set of all functions u which satisfy (3.6). Furthermore, one can see that $\|u_\tau\|_{C^{\frac{\alpha}{2},\alpha}} < \infty$. $\square$

Remark 3.2. The fixed point method, which we have used above, can be used to obtain further regularity results. This can be seen by differentiating (3.2) again and repeating the fixed point procedure of Theorem 3.1. However, by doing that, condition (3.6) has to be strengthened such that ϵ_0 has to be chosen smaller. Hence, to prove C^∞-regularity with this method, ϵ_0 would converge to 0 and we would be back in the linear case.

Remark 3.3. The condition with respect to ρ and the initial condition can be relaxed in the nonintegral case by considering the problem in a weighted Hölder-space (for example by taking the weight function $e^{-\kappa\sqrt{1+x^2}}$ with $\kappa > 1$). Unfortunately, considering the integro case in a weighted Hölder-space is not straightforward since the integral is nonlocal.

4. Numerical implementation

To obtain an initial value problem and for sake of readability, we perform the transformation $x = \ln(S), \tau = T - t, u(\tau,x) = V(t,S)$ and obtain

$$\begin{gathered}
u_\tau(\tau,x) = \frac{1}{2}\hat{\hat{\sigma}}^2(\tau,x,u_x,\mathfrak{X})\mathfrak{X} + \Big(\hat{r}(\tau) - \hat{q}(\tau,x) - \int_{\mathbb{R}\setminus\{0\}}(e^y-1)\nu(dy)\Big)u_x(\tau,x)\\
- \hat{r}(\tau)u(\tau,x) + \int_{\mathbb{R}\setminus\{0\}}[u(\tau,x+y) - u(\tau,x)]\nu(dy)\\
u(0,x) = \hat{g}(x),
\end{gathered} \tag{4.1}$$

where $(\tau, x) \in [0,T]\times[a,b]$ and $\mathfrak{X} := u_{xx} - u_x$. To solve the problem numerically, we have to introduce boundary conditions, although the problem is unbounded. Here, one goes back to the financial setup and considers the observed option prices. The choice of the individual boundary conditions depends on the option type (cf. [22]). To keep the analysis uniform, we use respectively u_{beh}^{left} and u_{beh}^{right} for the behaviors at the boundaries. Next, we introduce a uniform grid on $[0,T]\times[a,b]$ by setting

$$\tau_n = n\Delta\tau,\ n = 0,\dots,m_\tau, \quad x_i = a + i\Delta x,\ i = 0,\dots,m_x,$$

where (m_x+1), $(m_\tau+1)$ denote the respective number of grid points in space and time dimension and $\Delta\tau = \frac{T}{m_\tau}$, $\Delta x = \frac{b-a}{m_x}$. We denote the values of u on this grid by u_i^n. We approximate the derivatives by the following finite differences

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(\tau_n, x_i) &\approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = \partial_{xx} u_i^n, \\ \frac{\partial u}{\partial x}(\tau_n, x_i) &\approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = \partial_x u_i^n, \\ \frac{\partial u}{\partial \tau}(\tau_n, x_i) &\approx \frac{u_i^{n+1} - u_i^n}{\Delta\tau} = \partial_\tau u_i^n. \end{aligned} \tag{4.2}$$

It remains to discretize the integral. Because the integral is nonlocal, we have to determine constants $a_I < b_I$, where we want to evaluate it $\int_{-\infty}^{\infty} \nu(dy) \approx \int_{a_I}^{b_I} \nu(dy)$.

Remark 4.1. If the conditions of Theorem 2.1 are satisfied, one can show that the error, which is implied by this truncation, is of exponential decay. For more information, we refer the reader to [7, Proposition 6].

The next step is to approximate the integral terms. Because we only have the values of u at the points x_i, it is appropriate to choose a quadrature rule which uses only these points. We set

$$\int_{a_I}^{b_I} \left[u(\tau, x_i + y) - u(\tau, x_i)\right] \nu(dy) \approx \sum_{j=a_I^d}^{b_I^d} \nu_j \left(u_{i+j} - u_i\right), \quad \nu_j = \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} \nu(dy),$$

where a_I^d and b_I^d are integers such that $[a_I, b_I] \subset \left[\left(a_I^d - \frac{1}{2}\right)\Delta x, \left(b_I^d + \frac{1}{2}\right)\Delta x\right]$. If we cannot evaluate the integral term ν_j directly, we have to use a quadrature rule. By using all these considerations, we can construct a difference scheme to solve the problem. Hence, for $\lambda \in [0,1]$ we obtain

$$\mathcal{BS}(\tau_n, x_i, u, D_x u, D_{xx} u) \approx (1-\lambda)\left(D_\Delta^{n+1} + I_\Delta\right) u(\tau_{n+1}, x_i) + \lambda\left(D_\Delta^n + I_\Delta\right) u(\tau_n, x_i),$$

where

$$\begin{aligned} D_\Delta^{\tilde n} u(\tau_n, x_i) &= \frac{1}{2}\hat{\tilde\sigma}^2\left(\tau_{\tilde n}, x_i, \partial_x u_i^{\tilde n}, \mathfrak{X}_i^{\tilde n}\right)\mathfrak{X}_i^n \\ &\quad + \Big(\hat r(\tau_{\tilde n}) - \hat q(\tau_{\tilde n}, x_i) - \sum_{j=a_I^d}^{b_I^d} \nu_j (e^{j\Delta x} - 1)\Big)\partial_x u_i^n - \hat r(\tau_{\tilde n}) u_i^n \\ I_\Delta u(\tau_n, x_i) &= \sum_{j=a_I^d}^{b_I^d} \nu_j \left(u_{i+j}^n - u_i^n\right). \end{aligned}$$

Considering the integral approximation $I_\Delta u_{i+j}^n$ in more detail, we see that in the nonexplicit case, this term would change $(b_I^d - a_I^d)$ entries in the evaluation matrix.

This would not only destroy the tridiagonal property of the matrix, but even much worse, it would destroy the sparse property. Unfortunately, the complexity of the numerical algorithms, which solve the system of equations, depends on the fact that the matrices are sparse. That is why we only consider the integral approximation for the explicit case. Moreover, we observe that the index $(i+j)$ can be negative or can exceed the given indices. Therefore, we must define values of u outside the given range. Here, we use the values of the boundary conditions. To sum it up, we obtain the following explicit-implicit scheme

Initialization: $u_i^0 = \hat{g}(x_i)$

For n=0,...,m$_\tau$-1:

$$
\begin{aligned}
u_i^{n+1} &\leftarrow F_i\left(u_i^{n+1}, u_{i-1}^{n+1}, u_{i+1}^{n+1}, u_i^n, u_{i-1}^n, u_{i+1}^n\right) = 0, && \text{if } i \in \{0,\dots,m_x\}\\
u_i^{n+1} &\leftarrow u_{beh}^{left}(\tau_{n+1}, x_i)\cdot \mathbf{1}_{\{i<0\}} + u_{beh}^{right}(\tau_{n+1}, x_i)\cdot \mathbf{1}_{\{i>m_x\}}, && \text{if } i \notin \{0,\dots,m_x\},
\end{aligned}
\tag{4.3}
$$

where

$$
\begin{aligned}
&F_i\left(u_i^{n+1}, u_{i-1}^{n+1}, u_{i+1}^{n+1}, u_i^n, u_{i-1}^n, u_{i+1}^n\right)\\
&= -u_i^{n+1} + (1-\lambda)\Delta\tau D_\Delta^{n+1} u_i^{n+1} + u_i^n + \lambda\Delta\tau D_\Delta^n u_i^n + \Delta\tau I_\Delta u_i^n
\end{aligned}
$$

If we consider the explicit-implicit scheme in more detail, we observe that we have to solve a nonlinear system of equations in each time step. In this case it makes sense to compute the generalized Jacobian

$$
D\mathbf{F}(\mathbf{u}^{n+1}) = \Big(\frac{\partial F_i}{\partial u_k^{n+1}}\Big)_{0\le i,k\le m_x}
$$

and use the Newton's method. (Although the equations are nonlinear, the generalized Jacobian can be specified explicitly, see for example [24].) To avoid this problem one could consider from the start the discretization of the space derivative in more detail. So we have distinguished between the discretization of the linear part and the nonlinear part. If we simplify the nonlinear part by using the result from the time step before, we obtain a linear system of equations, which shortens the computations significantly. Next, we have to check that the scheme converges to the desired solution. Following Barles [4], every monotone, stable and consistent scheme converges to the unique (viscosity) solution.

Theorem 4.2. *Let $\hat{\tilde{\sigma}}^2(\tau, x, u_x, \mathfrak{X})\mathfrak{X}$ be continuous and monotonously nondecreasing in $\mathfrak{X}$. Furthermore, we assume for the fully implicit case ($\lambda = 0$) that*

(1) *there exist nonnegative constants $c_i < \infty$ $(i = 1, 2)$ such that for every $\mathfrak{X} \in \mathbb{R}$ and $\epsilon > 0$,*

$$
\hat{\tilde{\sigma}}^2(\tau, x, u_x \pm \xi, \mathfrak{X}+\epsilon)(\mathfrak{X}+\epsilon) \ge \hat{\tilde{\sigma}}^2(\tau, x, u_x, \mathfrak{X})\,\mathfrak{X} + c_1\epsilon - c_2\xi,
$$

$$
\Big(c_1(2\mp\Delta x) - c_2\Delta x \pm \Delta x\Big(\hat{r}(\tau) - \hat{q}(\tau,x) - \sum_{j=a_I^d}^{b_I^d} \nu_j(e^{j\Delta x}-1)\Big)\Big) \ge 0;
$$

(2) *I_Δ is monotonously nondecreasing in u, i.e. for every $\epsilon > 0$*

$$
I_\Delta(u+\epsilon) \ge I_\Delta u;
$$

and for the nonimplicit case ($\lambda \in (0, 1]$), we assume additionally that

(3) *there exist positive constants* $c_i < \infty$ $(i = 1, 2)$ *such that for every* $\mathfrak{X} \in \mathbb{R}$ *and* $\epsilon > 0$

$$\hat{\hat{\sigma}}^2 (\tau, x, u_x \pm \xi, \mathfrak{X} - \epsilon) (\mathfrak{X} - \epsilon) \geq \hat{\hat{\sigma}}^2 (\tau, x, u_x, \mathfrak{X}) \mathfrak{X} - c_1 \epsilon - c_2 \xi,$$

$$\Big(1 - c_1 \frac{2\lambda \Delta\tau}{(\Delta x)^2}\Big) \geq 0.$$

Then, the explicit-implicit scheme (4.3) *converges to the unique (viscosity) solution of* (4.1).

Proof. To use the result of Barles, we have to prove monotonicity, stability and consistency.

(1) **Monotonicity:** The proof of the monotonicity can be proven in analogy to the proof of [13, Theorems 3.3 and 3.4] with the only difference, that one has to use additionally the assumption about the integral term.
(2) **Stability:** To prove stability, we use Theorem 3.1 which already says that the problem stays parabolic as long as ρ is small enough. Using that, stability can be proven in analogy to [21, Lemma 2].
(3) **Consistency:** The consistency of the differential part of the equation follows from the respective Taylor expansions of the individual finite differences (4.2). The consistency of the integral part follows from the used quadrature rule, which is the trapezoid rule, and Remark 4.1. A more detailed investigation of consistency can be found in [7, chapter 4], where the respective Taylor expansions and quadrature rules have been written down.

□

Remark 4.3. Using the result of Theorem 3.1, one can show that, for ρ sufficiently small, (2.6) satisfies Theorem 4.2 for the implicit case.

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