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DIFFERENTIABILITY PROPERTIES OF p-TRIGONOMETRIC FUNCTIONS

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Abstract. p-trigonometric functions are generalizations of the trigonometric functions. They appear in context of nonlinear differential equations and also in analytical geometry of the p-circle in the plain. The most important p-trigonometric function is $\sin_p(x)$. For $p > 1$, this function is defined as the unique solution of the initial-value problem

$$(|u'(x)|^{p-2}u'(x))' = (p-1)|u(x)|^{p-2}u(x), \quad u(0) = 0,\ u'(0) = 1\,,$$

for any $x \in \mathbb{R}$. We prove that the n-th derivative of $\sin_p(x)$ can be expressed in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x)\,,$$

on $(0, \pi_p/2)$, where $\pi_p = \int_0^1 (1-s^p)^{-1/p}\mathrm{d}s$, and $\cos_p(x) = \sin_p'(x)$. Using this formula, we proved the order of differentiability of the function $\sin_p(x)$. The most surprising (least expected) result is that $\sin_p(x) \in C^\infty(-\pi_p/2, \pi_p/2)$ if p is an even integer. This result was essentially used in the proof of theorem, which says that the Maclaurin series of $\sin_p(x)$ converges on $(-\pi_p/2, \pi_p/2)$ if p is an even integer. This completes previous results that were known e.g. by Lindqvist and Peetre where this convergence was conjectured.

1. Introduction

In the previous two decades, p-trigonometric functions have attracted attention of many researchers; see, e.g., [1, 5, 6, 7, 10, 11, 12, 13, 15, 16, 25], and references therein. The p-trigonometric functions arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian. We assume $p > 1$ and say, that $\lambda \in \mathbb{R}$ is an eigenvalue of

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = 0 \quad \text{in } (0, \pi_p)\,, \\ u(0) = u(\pi_p) = 0\,, \end{gathered} \tag{1.1}$$

if there is a nonzero function $u \in W^{1,p}(0, \pi_p)$ that satisfy (1.1) in a weak sense. Here

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}\,. \tag{1.2}$$

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Let us note, that the problem can be considered on any bounded open interval, but the choice $(0, \pi_p)$ significantly simplifies the calculations. The discreetness of the spectrum of this eigenvalue problem was established already by Nečas [21]. This eigenvalue problem was later studied by means of the initial-value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = 0 \quad \text{in } (0,\infty)\,, \\ u(0) = 0, \quad u'(0) = 1\,; \end{gathered} \tag{1.3}$$

see Elbert [11] for initial work in this direction. Later it was independently studied by del Pino-Elgueta-Manasevich [8], Ôtani [22] and Lindqvist [14].

Let $\sin_p(x)$ denote the solution of (1.3) with $\lambda = (p-1)$. It follows from [11] that $\sin_p(x)$ is positive on $(0, \pi_p)$ and satisfies an identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \quad \forall x \in \mathbb{R}\,, \tag{1.4}$$

which for $p = 2$ becomes the familiar identity for sine and cosine. This suggest the definition $\cos_p(x) := \sin_p'(x)$ and justifies the notation $\sin_p(x)$ and $\cos_p(x)$. The identity (1.4) is called p-trigonometric identity. It also follows from [11] that the eigenvalues of (1.3) form a sequence $\lambda_k = k^p(p-1), k \in \mathbb{N}$ and corresponding eigenfunctions are functions $\sin_p(kx), k \in \mathbb{N}$. Thus all the eigenfunctions are determined by the function $\sin_p(x)$. It comes as no surprise that the properties of the function $\sin_p(x)$ were studied extensively in the previous 30 years. It was shown in [11] that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ (the p-trigonometric identity (1.4) can be thought of as the first integral of (1.3)) as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s\,, \quad x \in [0,1]\,, \tag{1.5}$$

which is extended to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it is extended to $\mathbb{R}$ as the $2\pi_p$-periodic function. The function $\operatorname{arcsin}_p(x)$ from (1.5) is extended to $[-1, 1]$ as an odd function. Then

$$\sin_p(\operatorname{arcsin}_p(x)) = x \quad \forall x \in [-1,1]\,. \tag{1.6}$$

Note that for $p = 2$, we obtain classical arcsine and sine from this definition. The (now familiar) notation $\sin_p$ appears in [8] for the first time, where the authors studied homotopic deformation along p to calculate the degree of trivial solutions of (1.1) in order to establish existence results for the nonlinear problem $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, $T > 0$. The homotopy result from [8] initiated development of bifurcation theory for quasilinear bifurcations.

As a historical remark, let us mention that generalizations of arcsine similar to (1.5) were studied in a very different context by Lundberg [17] in 1879. It is interesting to mention that the p-trigonometric functions satisfy certain relations to geometrical objects such as arclength and area of a circle in a noneuclidean metric; see Elbert [11], and Lindqvist [15]. The p-trigonometric functions also possesses some approximation properties in certain function spaces; see, e.g., Binding-Boulton-Čepička-Drábek-Girg [1], Lang-Edmunds [13] for theoretical research, and Boulton-Lord [6] for a very interesting computational application in evolutionary PDEs. In Wood [27], the particular case $p = 4$ was studied and "p-polar" coordinates in the xy-plane were proposed.

In this article we focus on the differentiability and analyticity properties of p-trigonometric functions. One can immediately see from (1.2), (1.5), and (1.6) that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ for all $p > 1$. From (1.4) and the definition of

$\cos_p(x)$, we obtain $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. It follows from the results in [11, 15, 22] that the possible differentiability issues are located at $x = 0$ and $x = \pi_p/2$. There are several results concerning differentiability and asymptotic behaviour of $\sin_p(x)$ at $x = 0$ and $x = \pi_p/2$ in Manásevich-Takáč [19] and Benedikt-Girg-Takáč [2]. In Peetre [25], generalized formal Maclaurin series for $\sin_p(x)$ were studied and their convergence was conjectured on $(-\pi_p/2, \pi_p/2)$. The local convergence of the generalized Taylor series (and/or the generalized Maclaurin series) for $\sin_p(x)$ follows from Paredes-Uchiyama [24]. Taking into account that the point $x = 0$ is often considered as the center for the Taylor (i.e. the Maclaurin) series or the generalized Taylor (i.e. the generalized Maclaurin) series for $\sin_p(x)$, we decided to provide detailed study of the convergence of these series towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. We were also motivated by work of Ôtani [23], where he studies properties of the solutions of

$$\begin{gathered}(|u'|^{p-2}u')' + |u|^{q-2}u = 0 \quad \text{in } (a,b)\,, \\ u(a) = u(b) = 0\,,\end{gathered} \tag{1.7}$$

for general exponents $p, q \in (1, +\infty)$ with $p \neq q$. Among other properties he proved that for $p = \frac{2m+2}{2m+1}, m \in \{0\} \cup \mathbb{N}$ and for q even, any solution of (1.7) belongs to $C^\infty(a, b)$. In our case, $p = q$ we find that $\sin_p(x)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ if and only if p is even. Let us also remark that local analytic solutions of the radial variant of (1.7) were studied in Bognár [4].

Though we are aware that our methods are elementary mathematics, we are sure that our results will help to better understand the behavior of $\sin_p(x)$ and its derivatives in the vicinity of 0. This behavior is crucial in establishing asymptotic estimates such as those in the proof of the Fredholm alternative for the p-Laplacian in the degenerate case Benedikt-Girg-Takáč [2, 3]. Moreover, knowledge of the convergence/nonconvergence of the Taylor and/or the Maclaurin series is very important in the development of numerical methods for calculating approximations of function values of p-trigonometric functions. Recently, Marichev [20] from the Wolfram Research, Inc., pointed out to the first author of this paper in a personal communication that Mathematica from version 8.0 has a capability to effectively compute coefficients for $\sin_p(x)$ for formal generalized Maclaurin power series by means of the Bell Polynomials. With few lines of Mathematica code one can obtain partial sums of generalized Maclaurin series for $\sin_p(x)$ of large order in a couple of minutes. Thus the question of the convergence of the partial sums of the Maclaurin series is becoming quite urgent. This was our main motivation to address this topic.

Our main result provides convergence of these partial sums. We treat two cases separately, $p > 2$ is an even integer and $p > 2$ is an odd integer. Namely, for the particular case $\sin_{2(m+1)}(x)$, $m \in \mathbb{N}$, $x \in (-\pi_p/2, \pi_p/2)$, we show that the Maclaurin series converges towards the values $\sin_{2(m+1)}(x)$ on the interval $(-\pi_p/2, \pi_p/2)$. On the other hand, we show that the Maclaurin series converge towards $\sin_{2m+1}(x)$, $m \in \mathbb{N}$, for $x \in (0, \pi_p/2)$ and does not for $x \in (-\pi_p/2, 0)$. More precisely, the Maclaurin series converges on $x \in (-\pi_p/2, \pi_p/2)$, but not towards values of $\sin_{2m+1}(x)$, $m \in \mathbb{N}$ for $x \in (-\pi_p/2, 0)$.

The article is organized as follows. In Section 2, we give a definition of the function $\sin_p(x)$ by means of a differential equation and also introduce other useful notation. In Section 3, we state and discuss our main results concerning differentiability and/or non-differentiability of $\sin_p(x)$ and convergence of Maclaurin series of

$\sin_p(x)$. In Section 4, we express higher derivatives of $\sin_p(x)$ by means of powers of $\sin_p(x)$ and $\cos_p(x)$. Finally, in Section 5, we prove our main results using formulas for higher derivatives of $\sin_p(x)$ from Section 4. In Section 6, we conclude with remarks and open problems.

2. Definitions of p-trigonometric functions

Proposition 2.1. *The initial-value problem*

$$\begin{gathered} -(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0 \\ u(0) = 0, \quad u'(0) = 1\,, \end{gathered} \tag{2.1}$$

has the unique local solution and moreover any local solution to (2.1) *can be continued to* $(-\infty, +\infty)$.

For uniqueness of the solution see [8, Sect. 3], and for the existence of global solutions see [9, Lemma A.1].

Definition 2.2. The function $\sin_p(x)$ is defined as the unique solution of the initial-value problem (2.1) on $\mathbb{R}$.

For any $q > 1$ and $z \in \mathbb{R}$ we define

$$\varphi_q(z) = \begin{cases} |z|^{q-2}z & \text{for} z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases} \tag{2.2}$$

Note that $\varphi_{p'}(\varphi_p(z)) = \varphi_p(\varphi_{p'}(z)) = z$ provided $p > 1$ and $1/p + 1/p' = 1$. With this notation, we can rewrite the initial-value problem (2.1) as an equivalent first-order system

$$\begin{gathered} u'(x) = \varphi_{p'}(v(x))\,, \\ v'(x) = -(p-1)\varphi_p(u(x))\,, \\ u(0) = 0, \quad v(0) = 1\,. \end{gathered} \tag{2.3}$$

Clearly, from the definition of Carathéodory solution, it follows that $u(x) = \sin_p(x)$ and $v(x) = \varphi_p(\sin_p'(x))$ must be absolutely continuous on any compact interval $[-K, K]$, $K > 0$. Thus $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on any $[-K, K]$, $K > 0$, which entails that $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on $(-\infty, +\infty)$. Thus the following definition makes sense.

Definition 2.3. For $x \in \mathbb{R}$, we define $\cos_p(x) = \sin_p'(x)$.

Since $\cos_p(0) = \sin_p'(0) = 1$ and $\cos_p(x)$ is continuous, there exists an interval $(-c, c)$ such that $\cos_p(x) > 0$ on $(-c, c)$, $c > 0$. Moreover, since $\sin_p'(0) = 1$ and $\sin_p \in C^1(\mathbb{R})$, there exists an interval $[0, s)$, $s > 0$, such that $\sin_p(x) \geq 0$ on $[0, s)$.

Definition 2.4. For $p > 1$, let π_p denote

$$2\sup\{s > 0 : \forall x \in (0, s) \text{ holds } \sin_p(x) > 0 \wedge \cos_p(x) > 0\}\,.$$

It was shown in [11], that

$$\pi_p = 2\int_0^1 \frac{1}{(1 - x^p)^{1/p}}\,\mathrm{d}x = \frac{2\pi}{p \cdot \sin(\pi/p)}\,,$$

for $p > 1$. It was also shown in [11], that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s \quad x \in [0,1]\,, \tag{2.4}$$

and, moreover, it extends to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it extends to $\mathbb{R}$ as the $2\pi_p$-periodic function.

Remark 2.5. In the following text, formulas containing higher order derivatives and powers of $\sin_p(x)$ and $\cos_p(x)$ appear. We try to keep our notation as close as possible to the usual notation for classical trigonometric functions. Thus the derivatives are denoted by, e.g., $\sin_p'(x), \dots, \sin_p'''(x), \sin_p^{(\mathrm{iv})}(x)$ (primes and roman numerals) and/or, e.g., $\sin_p^{(n)}(x)$, $\sin_p^{(2n-1)}$ and $\sin_p^{(2n)}$ for $n \in \mathbb{N}$. On the other hand, the powers are denoted by $\sin_p^2(x), \sin_p^3(x), \sin_p^q(x)$, $q \in \mathbb{R}$. Where a confusion may happen, we denote the powers by, e.g., $(\sin_p(x))^m$, $m \in \mathbb{N}$, to distinguish them clearly from derivatives. For the convenience of the reader, we write the values of p as explicit as possible, with a few exceptions such as in the proofs of Theorems 3.3 and 3.4, where this approach would produce very lengthy formulas.

3. Main results

In the sequel, we study derivatives of $\sin_p(x)$ for $p \in \mathbb{N}$, $p > 2$ on the interval $x \in (-\pi_p/2, \pi_p/2)$. We distinguish two cases p is even, i.e., $p = 2(m+1)$ and $m \in \mathbb{N}$, and p is odd; i.e., $p = 2m+1$ and $m \in \mathbb{N}$. In the first case $p = 2(m+1)$, the p-trigonometric identity (1.4) takes form

$$(\sin_{2(m+1)}(x))^{2(m+1)} + (\cos_{2(m+1)(x)})^{2(m+1)} = 1\,, \tag{3.1}$$

which is valid for any $x \in \mathbb{R}$ and hence on $(-\pi_p/2, \pi_p/2)$. Note that there is no absolute value, since there are even powers.

In the second case $p = 2k+1$, we have to distinguish two subcases. For $0 < x < \frac{\pi_p}{2}$, the p-trigonometric identity takes form

$$(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.2}$$

On the other hand, for $-\pi_p/2 < x < 0$, the p-trigonometric identity takes form

$$-(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.3}$$

Since there is only one identity (3.1) for $p = 2(m+1)$, this case has nice smoothness properties on $(-\pi_p/2, \pi_p/2)$ and we obtain a rather surprising result concerning smoothness of function $\sin_p(x)$ for even p.

Theorem 3.1. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then*

$$\sin_{2(m+1)}(x) \in C^\infty\Big(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\Big)\,.$$

On the other hand, for $p = 2m+1$, we have to distinguish two subcases (3.2) and (3.3), which has damaging effect on the differentiability of $\sin_p(x)$. Thus the smoothness is lost when p is odd. The smoothness is also lost if p is not an integer.

Theorem 3.2. *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then*

$$\sin_p(x) \in C^{\lceil p \rceil}(-\pi_p/2, \pi_p/2)\,,$$

but

$$\sin_p(x) \notin C^{\lceil p \rceil + 1}(-\pi_p/2, \pi_p/2)\,.$$

Here $\lceil p\rceil := \min\{k \in \mathbb{N} : k \geq p\}$.

Our last result gives an explicit radius of convergence of the Maclaurin series for even $p > 2$. To the best of our knowledge, all previous results concerning convergence of series for $\sin_p(x)$ were only local; see, e.g., [24].

Theorem 3.3. *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then the Maclaurin series of $\sin_{2(m+1)}(x)$ converges on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.*

Theorem 3.4. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then the formal Maclaurin series of $\sin_{2m+1}(x)$ converges on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_{2m+1}(x)$ on $[0, \frac{\pi_{2m+1}}{2})$, but does not converge towards $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, 0)$.*

The proofs of Theorems 3.1–3.4 are postponed to Section 5.

4. Derivatives of $\sin_p(x)$

The following lemma summarizes basic properties of $\sin_p(x)$ and $\cos_p(x)$.

Lemma 4.1. *Let $p \in \mathbb{R}, p > 1$. Functions $\sin_p(x)$ and $\cos_p(x)$ have the following basic properties.*

1. *$\sin_p(x) > 0$ on $(0, \pi_p)$, $\sin_p(0) = 0$, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\frac{\pi_p}{2}, \pi_p)$, and $\sin_p(x) = -\sin_p(-x)$ on $(-\pi_p, 0)$. The function $\sin_p(x)$ extends to $\mathbb{R}$ as $2\pi_p$-periodic function.*
2. *$\sin_p(x)$ is strictly increasing on $(-\pi_p/2, \pi_p/2)$.*
3. *$\cos_p(x) > 0$ on $(-\pi_p/2, \pi_p/2)$, $\cos_p(-\frac{\pi_p}{2}) = \cos_p(\frac{\pi_p}{2}) = 0$ and $\cos_p(x) < 0$ on $\big[-\pi_p, -\frac{\pi_p}{2}) \cup (\frac{\pi_p}{2}, \pi_p\big]$.*
4. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n-1)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is even function on $(-\pi_p/2, \pi_p/2)$.*
5. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is odd function on $(-\pi_p/2, \pi_p/2)$.*

Statements 1–3 follows from [11]. Statements 4, and 5 are trivial consequence of statement 1.

Lemma 4.2. *For all $p \in \mathbb{R}, p > 1$*

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,, \tag{4.1}$$

$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (-\pi_p/2, 0)\,. \tag{4.2}$$

Proof. The identity (4.1) is obtained by a straightforward calculation; see, e.g., [13]. For $x \in (-\pi_p/2, 0)$, we obtain from Lemma 4.1 statement 1 and 3 and the identity (1.4)

$$\sin_p^p(-x) + \cos_p^p(x) = |-\sin_p(-x)|^p + |\cos_p(x)|^p = |\sin_p(x)|^p + |\cos_p(x)|^p = 1. \tag{4.3}$$

Taking

$$\sin_p^p(-x) + \cos_p^p(x) = 1 \tag{4.4}$$

into derivative we obtain

$$-p \cdot \sin_p^{p-1}(-x) \cdot \cos_p(-x) + p \cdot \cos_p^{p-1}(x) \cdot \sin_p''(x) = 0\,. \tag{4.5}$$

From Lemma 4.1, statements 3 and 4, we obtain

$$\sin_p^{p-1}(-x) \cdot \cos_p(x) = \cos_p^{p-1}(x) \cdot \sin_p''(x)$$

which yields
$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x)\,.$$
□

Lemma 4.3. *Let $p \in \mathbb{R} \setminus \{2\}$ such that $p > 1$.*

(1) *If $p > 2$, then the function $\sin_p(x) \in C^1(\mathbb{R})$ and $\sin_p(x) \notin C^2(\mathbb{R})$.*
(2) *If $p \in (1,2)$, then the function $\sin_p(x) \in C^2(\mathbb{R})$ and $\sin_p(x) \notin C^3(\mathbb{R})$.*

Proof. By the definition of $\cos_p(x)$, $\sin_p'(x) = \cos_p(x)$. The function $\cos_p(x) \in C(\mathbb{R})$, for all $p > 1$. Thus $\sin_p(x) \in C^1(\mathbb{R})$. By Lemma 4.2,
$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,.$$
Taking into account that
$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p^{p-1}(x) = 1 \quad \text{and} \quad \lim_{x \to \frac{\pi_p}{2}-} \cos_p^{2-p}(x) = +\infty \quad \text{for } p > 2\,,$$
we find that
$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p''(x) = -\infty\,.$$
Thus the continuity of $\sin_p''(x)$ fails at $x = \pi_p/2$ for $p > 2$ and the statement 1 of Lemma 4.3 follows.

From (2.3), we find that the function $v'(x) = -(p-1)\varphi_p(\sin_p(x))$ is continuous on $\mathbb{R}$ as $\sin_p(x)$ is continuous on $\mathbb{R}$. We also find that $\cos_p(x) = \varphi_{p'}(v(x))$ from (2.3). Taking into account that $\varphi_{p'} \in C^1(\mathbb{R})$ for $p \in (1,2)$ (observe that $p' = \frac{p}{p-1} > 2$ in this case), we infer that $\cos_p'(x) = \varphi_{p'}'(v(x)) \cdot v'(x)$ is continuous on $\mathbb{R}$. Thus $\sin_p(x)$ is two times continuously differentiable on $\mathbb{R}$ for $p \in (1,2)$. On the other hand, taking
$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{on } (0, \frac{\pi_p}{2})$$
into derivative, we obtain
$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) - (2-p) \cdot \sin_p^{p-1}(x) \cdot \cos_p^{1-p}(x) \cdot \sin_p''(x)\,.$$
Substituting for $\sin_p''(x)$ from the later equation into the former, we have
$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$
Since $\lim_{x\to 0+} \sin_p(x) = 0$ and $\lim_{x \to 0+} \cos_p(x) = 1$, we obtain
$$\lim_{x \to 0+} \sin_p'''(x) = -\infty$$
for $p \in (1,2)$. This concludes the proof of statement 2 of Lemma 4.3. □

Let us define the following 'symbolic' operators (rewriting rules) defined on expressions of the form
$$a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \quad \text{with } a, q \in \mathbb{R} \tag{4.6}$$
as follows
$$\mathrm{D}_s\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} a \cdot q \cdot \sin_p^{q-1}(x) \cdot \cos_p^{1-(q-1)}(x) & q \neq 0\,, \\ 0 & q = 0\,. \end{cases} \tag{4.7}$$
$$\mathrm{D}_c\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} -a \cdot (1-q) \cdot \sin_p^{q+p-1}(x) \cdot \cos_p^{1-(q+p-1)}(x) & q \neq 1\,, \\ 0 & q = 1\,. \end{cases} \tag{4.8}$$

Let us observe that the results of application D_s and D_c have the form (4.6). Hence they are also in the domain of definition of D_s and D_c. Thus we can consider compositions of D_c and D_s of arbitrary length. We will show that the first derivative of $\sin_p^q(x)\cdot\cos_p^{1-q}(x)$ (here $a=1$) can be written using these symbolic operators as follows

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=\mathrm{D}_s\sin_p^q(x)\cdot\cos_p^{1-q}(x)+\mathrm{D}_c\sin_p^q(x)\cdot\cos_p^{1-q}(x)\,.\end{aligned}$$

To show this, we have to distinguish three cases $q\in\mathbb{R}\setminus\{0,1\}$, $q=1$, and $q=0$.

Case $q\in\mathbb{R}\setminus\{0,1\}$. Here

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=q\sin_p^{q-1}(x)\cdot\cos_p^{1-(q-1)}(x)-(1-q)\sin_p^{q+p-1}(x)\cdot\cos_p^{1-(q+p-1)}(x)\\&=\mathrm{D}_s\sin_p^q(x)\cdot\cos_p^{1-q}(x)+\mathrm{D}_c\sin_p^q(x)\cdot\cos_p^{1-q}(x)\,.\end{aligned}$$

Note that the distance between the exponents of $\sin_p(x)$ in the resulting terms, i.e., $\sin_p^{q_0-1}(x)\cdot\cos_p^{2-q_0}(x)$ and $\sin_p^{q_0+p-1}\cdot\cos_p^{2-p-q_0}(x)$, is exactly p. This is crucial in the sequel of the paper, because in a sum of the type

$$c_0\sin_p^{q_0}(x)\cdot\cos_p^{1-q_0}(x)+c_1\sin_p^{q_0+p}(x)\cdot\cos_p^{1-(q_0+p)}(x)$$

the terms combine together as in the diagram depicted on Fig. 1

Case $q=1$. In this case the term $\sin_p^q(x)\cdot\cos_p^{1-q}(x)=\sin_p(x)$. Thus the derivative of this term is the *single* term $\cos_p(x)$. By the definitions of $\mathrm{D}_s,\mathrm{D}_c$, we find that $\mathrm{D}_s\sin_p(x)=\cos_p(x)$ and $\mathrm{D}_c\sin_p(x)=0$. Thus $\frac{\mathrm{d}}{\mathrm{d}x}\sin_p(x)=\mathrm{D}_s\sin_p(x)+\mathrm{D}_c\sin_p(x)$. The fact $\mathrm{D}_c\sin_p(x)=0$ will be reflected in our diagrams by omitting 'right-down' edge departing from this node, see Figure 2.

Case $q=0$. This case corresponds to $\sin_p^q(x)\cdot\cos_p^{1-q}(x)=\cos_p(x)$. Thus the derivative of this term is the *single* term $-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)$. By the definitions of $\mathrm{D}_s,\mathrm{D}_c$, we find that $\mathrm{D}_s\cos_p(x)=0$ and

$$\mathrm{D}_c\cos_p(x)=-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)\,.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x}\cos_p(x)=\mathrm{D}_s\cos_p(x)+\mathrm{D}_c\cos_p(x)$. The fact $\mathrm{D}_s\cos_p(x)=0$ will be reflected in our diagrams by omitting 'left-down' edge departing from this node, see Figure 3. Note that since in our diagrams we write powers only, the node corresponding to $-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)$ is labeled by $s_p^{p-1}\,c_p^{1-(p-1)}$.

In the same way, we can express higher order derivatives, thus, e.g., the second derivative of $\sin_p^q(x)\cdot\cos_p^{1-q}(x)$ (here $a=1$) can be written as

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=(\mathrm{D}_s\circ\mathrm{D}_s)\sin_p^q(x)\cdot\cos_p^{1-q}(x)+(\mathrm{D}_c\circ\mathrm{D}_s)\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&\quad+(\mathrm{D}_s\circ\mathrm{D}_c)\sin_p^q(x)\cdot\cos_p^{1-q}(x)+(\mathrm{D}_c\circ\mathrm{D}_c)\sin_p^q(x)\cdot\cos_p^{1-q}(x).\end{aligned}$$

To better understand our methods of proof, it is good to have in mind the diagrams Figures 1–3.

The way how the term in the n-th derivative on the k-th position was derived from $\sin_p''(x)$ can be recovered from n and k as follows. First let us recall some notation from formal languages.

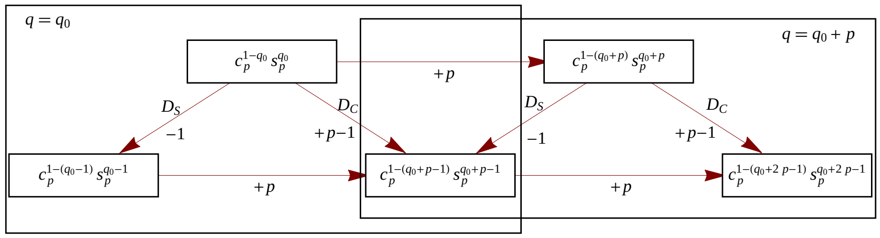

FIGURE 1. Rewriting diagram of the first derivative of $c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$. For the lack of space, we *do not* write the coefficients standing in front of these terms and *use short-cuts*, i.e., we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$

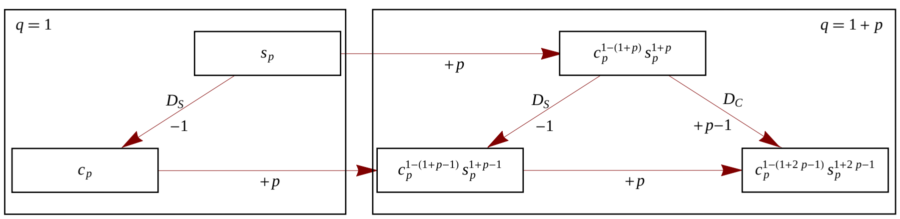

FIGURE 2. Rewriting diagram of the case $q = 1$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

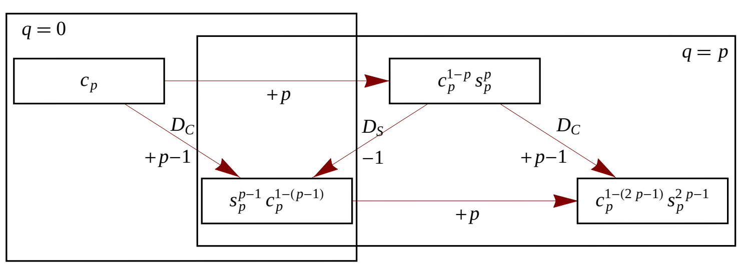

FIGURE 3. Rewriting diagram of the case $q = 0$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

Definition 4.4. (Salomaa-Soittola [26, I.2, p. 4,], and/or Manna [18, p. 2–3, p. 47, p. 78]) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \{0\} \cup \mathbb{N}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [18, p. 47, p. 78]).

For our purposes here, we consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$\text{“}\mathtt{0}\text{”}, \text{“}\mathtt{1}\text{”}, \text{“}\mathtt{01}\text{”}, \text{“}\mathtt{10}\text{”}, \text{“}\mathtt{11}\text{”} \dots .$$

For instance, $\mathrm{cat}(\text{“}\mathtt{1110}\text{”}, \text{“}\mathtt{011}\text{”}) = \text{“}\mathtt{1110011}\text{”}$, and

$$\mathrm{rev}(\text{“}\mathtt{010011000}\text{”}) = \text{“}\mathtt{000110010}\text{”}\,,$$
$$\mathrm{len}(\text{“}\mathtt{010011000}\text{”}) = 9\,.$$

Let $n \in \mathbb{N}$, $k \in \{0\} \cup \mathbb{N}, 0 \le k \le 2^{n-2} - 1$ and $(k)_{2,n-2}$ be the string of bits of the length $n-2$ which represents binary expansion of k (it means, e.g., for $k = 3$ and $n = 5$, $(3)_{2,5-2} = \text{“}\mathtt{011}\text{”}$). Now we are ready to define $D_{k,n}$ in two steps as follows.

Step 1 We create an ordered $n-2$-tuple $d_{k,n-2} \in \{\mathrm{D}_s, \mathrm{D}_c\}^{n-2}$ (cartesian product of sets $\{\mathrm{D}_s, \mathrm{D}_c\}$ of length $n-2$) from $\mathrm{rev}((k)_{2,n-2})$ such that for $1 \le i \le n-2$, $d_{k,n-2}$ contains D_s on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “0” on the i-th position, and $d_{k,n}$ contains D_c on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “1” on the i-th position (it means, e.g., for $k = 3$, and $n = 5$, we obtain $d_{3,5-2} = (\mathrm{D}_c, \mathrm{D}_c, \mathrm{D}_s)$).

Step 2 We define $D_{k,n}$ as the composition of operators $\mathrm{D}_s, \mathrm{D}_c$ in the order they appear in the ordered n-tuple $d_{k,n-2}$ (it means, e.g., for $k = 3$, and $n = 5$, we obtain $D_{3,5} = (\mathrm{D}_c \circ \mathrm{D}_c \circ \mathrm{D}_s)$).

The following Lemma implies that

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} D_{k,n} \sin_p''(x) \tag{4.9}$$

for all $x \in (0, \pi_p/2)$.

Lemma 4.5. *Let $p \in \mathbb{R}$, $p > 1$, $n \in \mathbb{N}$. Then $\sin_p^{(n)}(x)$ exists on $(0, \pi_p/2)$ and it is continuous. Moreover,*

$$\text{for } n = 1: \quad \sin_p'(x) = \cos_p(x)\,, \tag{4.10}$$

$$\text{for } n = 2: \quad \sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,, \tag{4.11}$$

and for $n = 3, 4, 5, \dots$, $k = 0, 1, 2, 3, \dots, 2^{n-2} - 1$ there exists $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p''(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.12}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1 - p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.13}$$

Moreover, let $j(k) \in \{0\} \cup \mathbb{N}$ *be the digit sum of the binary expansion of* $k = 0, 1, 2, \dots, 2^{n-2} - 1$ *(thus* $j(k)$ *is the number of occurrences of* D_c *in* $\mathrm{D}_{k,n}$*) and let* $\mathrm{D}_{k,n} \sin''_p(x) \not\equiv 0$*. Then, for* $k = 0, 1, 2, \dots, 2^{n-2} - 1$*, the exponents*

$$q_{k,n} := p \cdot l_{k,n} + m_{k,n} \tag{4.14}$$

satisfy

$$q_{k,n} = j(k)(p-1) + (n-2-j(k))(-1) + p - 1\,. \tag{4.15}$$

Proof. The cases $n = 1$ and $n = 2$ follows immediately from the definition of $\cos_p(x)$ and from Lemma 4.2.

We proceed by induction to prove the validity of the statement for $n = 3, 4, 5, \dots$.
Step 1. Taking (4.11) into derivative, we obtain

$$\sin'''_p(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$

For $k = 0, 1$ we obtain $a_{0,3} = -(p-1)$, $a_{1,3} = (2-p)$, $l_{0,3} = 1$, $l_{1,3} = 2$ $m_{0,3} = -2$, and $m_{1,3} = -2$. Hence

$$\sin'''_p(x) = \sum_{k=0}^{1} a_{k,3} \cdot \sin_p^{p \cdot l_{k,3} + m_{k,3}}(x) \cdot \cos_p^{1-p\cdot l_{k,3} - m_{k,3}}(x)\,.$$

Since we assume $p > 1$ we obtain $p - 1 \neq 0$ and thus by the definition of D_s and $\mathrm{D}_{k,n}$

$$\begin{aligned}
\mathrm{D}_{0,3} \sin''_p(x) &= \mathrm{D}_s(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\
&= -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) \\
&= a_{0,3} \cdot \sin_p^{p\cdot l_{0,3} + m_{0,3}}(x) \cdot \cos_p^{1-p\cdot l_{0,3} - m_{0,3}}(x).
\end{aligned}$$

Analogously, by the definition of D_c and $\mathrm{D}_{k,n}$ for $p \neq 2$, we find

$$\begin{aligned}
\mathrm{D}_{1,3} \sin''_p(x) &= \mathrm{D}_c(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\
&= -(-1) \cdot (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \\
&= a_{1,3} \cdot \sin_p^{p\cdot l_{1,3} + m_{1,3}}(x) \cdot \cos_p^{1-p\cdot l_{1,3} - m_{1,3}}(x)\,,
\end{aligned}$$

and for $p = 2$, we obtain

$$\mathrm{D}_{1,3} \sin''_p(x) = \mathrm{D}_c(-\sin_2(x) \cdot \cos_2^0(x)) = 0\,.$$

Hence,

$$\begin{aligned}
\sin'''_p(x) &= \mathrm{D}_s \sin''_p(x) + \mathrm{D}_c \sin''_p(x) \\
&= \mathrm{D}_{0,3} \sin''_p(x) + \mathrm{D}_{1,3} \sin''_p(x) \\
&= \sum_{k=0}^{1} \mathrm{D}_{k,3} \sin''_p(x)\,.
\end{aligned}$$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ exists, it is continuous on $(0, \pi_p/2)$, and for all $k = 0, 1, 2, \dots, 2^{n-2} - 1$ there exist $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p^{(n)}(x) = a_{k,n} \cdot \sin_p^{p\cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.16}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p\cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.17}$$

By the additivity rule of the derivative, we find that

$$\begin{aligned} \sin_p^{(n+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x) \\ &= \sum_{k=0}^{2^{n-2}-1} \frac{\mathrm{d}}{\mathrm{d}x} \big(a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\big) . \end{aligned} \tag{4.18}$$

For all $k = 0, 1, 2, \dots, 2^{n-2} - 1$, we find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\big(a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\big) \\ &= a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x) \\ &\quad + a_{k,n}(1 - p \cdot l_{k,n} - m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}-1}(x) \sin_p''(x) \\ &= a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x) \\ &\quad - a_{k,n}(1 - p \cdot l_{k,n} - m_{k,n}) \cdot \sin_p^{p\cdot (l_{k,n}+1)+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot(l_{k,n}+1)+m_{k,n}-1)}(x). \end{aligned} \tag{4.19}$$

For $k = 0, 1, 2, \dots, 2^{n-2} - 1$, we denote

$$a_{2k,n+1} := a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) , \tag{4.20}$$

$$a_{2k+1,n+1} := -a_{k,n} \cdot (1 - p \cdot l_{k,n} - m_{k,n}) , \tag{4.21}$$

$$l_{2k,n+1} := l_{k,n} , \tag{4.22}$$

$$m_{2k,n+1} := m_{k,n} - 1 , \tag{4.23}$$

$$l_{2k+1,n+1} := l_{k,n} + 1 , \tag{4.24}$$

$$m_{2k+1,n+1} := m_{k,n} - 1 . \tag{4.25}$$

Hence from (4.18), (4.19), and (4.20)–(4.25) we obtain

$$\sin_p^{(n+1)}(x) = \sum_{k'=0}^{2^{n-1}-1} a_{k',n+1} \cdot \sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x) \cdot \cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x). \tag{4.26}$$

Note that $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for $x \in (0, \pi_p/2)$ by Lemma 4.1, statements 1 and 3, and continuous by Lemma 4.3. Moreover, the function $z \mapsto z^q$, defined for $z > 0$ and $q \in \mathbb{R}$ belongs to $C^\infty(0, +\infty)$. Thus the function on the right-hand side of (4.26) is continuous for $x \in (0, \pi_p/2)$ which implies the continuity of $\sin_p^{(n+1)}(x)$ for $x \in (0, \pi_p/2)$.

Now, we show that for all $k' = 0, 1, 2, \dots, 2^{n-2}-1$: $a_{k',n+1} \in \mathbb{R}$, $l_{k',n+1}, m_{k',n+1} \in \mathbb{Z}$ and, moreover,

$$\mathrm{D}_{k',n+1} \sin_p^{(n)}(x) = a_{k',n+1} \cdot \sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x) \cdot \cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x) . \tag{4.27}$$

Let us set

$$\mathrm{D}_{2k,n+1} := \mathrm{D}_s \circ \mathrm{D}_{k,n} , \tag{4.28}$$

$$\mathrm{D}_{2k+1,n+1} := \mathrm{D}_c \circ \mathrm{D}_{k,n} . \tag{4.29}$$

Then it follows easily from corresponding binary expansion of k and $2k$ that

$$(2k)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“0”}),$$

$$(2k+1)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“1”})$$

and also that (4.28), (4.29) cover all 2^{n-1} of $k' = 0, 1, \dots 2^{n-1} - 1$. Thus our definitions (4.28) and (4.29) conform the relation between binary expansion of $k' = 2k$ and/or $k' = 2k+1$ and order of compositions of $\mathrm{D}_s, \mathrm{D}_c$ in $\mathrm{D}_{k',n+1}$.

For $k' = 0, 2, 4, \dots, 2^{n-1} - 2$ even,

$$\mathrm{D}_{k',n+1} \sin_p''(x) = \mathrm{D}_{2k,n+1} \sin_p''(x) = \mathrm{D}_s \circ \mathrm{D}_{k,n} \sin_p''(x)\,. \tag{4.30}$$

From the induction assumption (4.16), the definition of D_s (4.7) and (4.20), (4.22), (4.23), we find

$$\begin{aligned}
&\mathrm{D}_s(\mathrm{D}_{k,n} \sin_p''(x)) \\
&= \mathrm{D}_s(a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)) \\
&= a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x) \\
&= a_{2k,n+1} \cdot \sin_p^{p\cdot l_{2k,n+1}+m_{2k,n+1}}(x) \cdot \cos_p^{1-p\cdot l_{2k,n+1}-m_{2k,n+1}}(x).
\end{aligned}$$

We can treat $k' = 1, 3, 5, \dots, 2^{n-1} - 1$ in the same way using D_c instead of D_s and (4.8) and (4.21), (4.24), (4.25). This concludes the proof by induction.

It remains to show (4.15). In fact, from the definition (4.8) of D_c, each occurrence of the symbolic operator D_c in $\mathrm{D}_{k,n}$ increases the exponent q of $\sin_p^q(x)$ by $p-1$. Analogously, from the definition of (4.7) of D_s, each occurrence of the symbolic operator D_s in $\mathrm{D}_{k,n}$ decreases the exponent q of $\sin_p^q(x)$ by 1. Taking into account these facts and also that $q_{1,2} = p-1$, the formula (4.15) follows. This concludes the proof of Lemma 4.5. □

Lemma 4.6. *Let $p \in \mathbb{N}$, $p > 1$, and for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x)\,. \tag{4.31}$$

Then for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$

$$q_{k,n} \in \{0\} \cup \mathbb{N}\,. \tag{4.32}$$

Proof. From the definitions (4.7) and (4.8),

$$\begin{aligned}
q_{2k,n+1} &= q_{k,n} - 1 \quad \text{(we applied } \mathrm{D}_S \text{ on the expression)} \\
q_{2k+1,n+1} &= q_{k,n} + p - 1 \quad \text{(we applied } \mathrm{D}_\mathrm{C} \text{ on the expression)}
\end{aligned} \tag{4.33}$$

The proof proceeds by induction in n.

Step 1. From Lemma 4.2, for $\sin_p''(x)$ on $(0, \pi_p/2)$ we obtain the formula

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,.$$

Thus $q_{1,2} = p-1$. By assumption $p \in \mathbb{N}$, $p > 1$ we find $q_{1,2} \in \mathbb{N}$.

Step 2. We distinguish two cases, $q_{k,n} \in \mathbb{N}$ and $q_{k,n} = 0$. Let $q_{k,n} \in \mathbb{N}$. Then from (4.33), $p \in \mathbb{N}$, $p > 1$, we obtain

$$\begin{aligned}
q_{2k,n+1} &= q_{k,n} - 1 \in \{0\} \cup \mathbb{N}, \\
q_{2k+1,n+1} &= q_{k,n} + p - 1 \in \mathbb{N},
\end{aligned}$$

which satisfies (4.32). Let $q_{k,n} = 0$. Then the corresponding term in (4.31) has form

$$a_{k,n} \cdot \cos_p(x)\,, \tag{4.34}$$

since $\sin_p^0(x) \equiv 1$ for $x \in (0, \pi_p/2)$. Taking (4.34) into derivative, we find

$$a_{k,n} \cdot \cos_p'(x) = -a_{k,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)$$

and $q_{2k+1,n+1} = p - 1 \in \mathbb{N}$, because $p \in \mathbb{N}$, $p > 1$. This concludes the proof by induction. □

Lemma 4.7. *Let $p \in \mathbb{N}$, $p \geq 3$. Then for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) \leq 0 \quad \text{on} \quad (0, \frac{\pi_p}{2})\,.$$

Proof. By Lemma 4.5 and substitution (4.14), we have

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x)\,. \tag{4.35}$$

Let Q_n denote the set of all values of $q_{k,n}$ attained in the previous expression (this is to handle possible multiplicities), i.e.,

$$Q_n = \{q_{k,n} : k = 0, \dots, 2^{n-2} - 1\}\,. \tag{4.36}$$

By Lemma 4.6 for all $n \geq 2$ and for all $k \leq 2^{n-2} - 1$, we have $q_{k,n} \in \{0\} \cup \mathbb{N}$. Clearly, $Q_n \subset \{0\} \cup \mathbb{N}$ has at most 2^{n-2} elements and thus there exists $i_0 \in \mathbb{N} : 0 < i_0 \leq 2^{n-2} - 1$ and bijective mapping

$$\overline{q}_n : \{0, 1, 2, \dots i_0\} = Q_n \tag{4.37}$$

satisfying the order condition

$$\forall i, j = 0, 1, \dots, i_0 : i < j \Rightarrow \overline{q}_i < \overline{q}_j\,. \tag{4.38}$$

In the sequel, $\overline{q}_{i,n}$ stands for $\overline{q}_n(i)$. With this at hand, we add together the coefficients in (4.35) corresponding to the same value of powers $q_{k,n}$ and for any $i = 0, 1, \dots, i_0$ define

$$c_{i,n} := \sum_{\substack{k=0,1,\dots 2^{n-2}-1 \\ q_{k,n}=\overline{q}_{i,n}}} a_{k,n}\,. \tag{4.39}$$

Now, we rewrite (4.35) using coefficients $c_{i,n}$:

$$\sin_p^{(n)}(x) = \sum_{i=0}^{i_0} c_{i,n} \cdot \sin_p^{\overline{q}_{i,n}}(x) \cdot \cos_p^{1-\overline{q}_{i,n}}(x)\,. \tag{4.40}$$

Later, we will prove by induction that

$$\forall i = 0, 1, \dots, i_0 : c_{i,n} \leq 0\,. \tag{4.41}$$

By Lemma 4.1 statements 1 and 3, $\sin_p(x) > 0$ and $\cos_p(x) > 0$ on $(0, \frac{\pi_p}{2})$, which implies that for all $q, r \in \{0\} \cup \mathbb{N}$ and $x \in (0, \pi_p/2)$

$$\sin_p^q(x) \cdot \cos_p^r(x) > 0\,. \tag{4.42}$$

Thus from (4.40)–(4.42) the statement of Lemma 4.7 follows.

Now it remains to prove by induction in n that (4.41) holds.

Step 1. By Lemma 4.2 we find that

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.43}$$

for all $x \in (0, \pi_p/2)$ and so $c_{0,2} = -1 < 0$.

Taking the derivative of (4.43) (and after some straightforward rearrangements),

$$\sin_p'''(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \tag{4.44}$$

for $x \in (0, \pi_p/2)$. Since $p \geq 3$, we have $c_{0,3} = -(p-1) \leq -2 \leq 0$ and $c_{1,3} = (2-p) \leq -1 \leq 0$ as desired. Taking the derivative (4.44),

$$\begin{aligned}
\sin_p^{(\text{iv})} &= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x)+\\
&\quad + (p-1)\cdot(3-p)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x)\\
&\quad + (2-p)\cdot(2p-2)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x)\\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x)\\
&= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x)\\
&\quad + ((p-1)\cdot(3-p)+(2-p)\cdot(2p-2))\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x)\\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x)
\end{aligned} \tag{4.45}$$

for all $x \in (0, \pi_p/2)$. Since $p \geq 3$ we have $c_{0,4} = -(p-1)\cdot(p-2) \leq -2 \leq 0$, $c_{1,4} = (p-1)\cdot(3-p)+(2-p)\cdot(2p-2) \leq -4 \leq 0$, and $c_{2,4} = -(2-p)\cdot(3-2p) \leq -3 \leq 0$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ for $n \geq 4$ can be written in the form (4.40) and

$$\forall i \leq i_0 \,:\, c_{i,n} \leq 0\,. \tag{4.46}$$

The proof falls naturally into two parts.

Case 1. If

$$\overline{q}_{i,n} \geq 1\,, \tag{4.47}$$

then taking the i-th term of (4.40), which is

$$c_{i,n}\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}}(x)\,, \tag{4.48}$$

into derivative we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{1-\overline{q}_{i,n}+1}(x)\\
&+ c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}-1}(x)\cdot\sin_p''(x)\,.
\end{aligned}$$

Substituting (4.43) for $\sin_p''(x)$ into the previous expression, we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{2-\overline{q}_{i,n}}(x)\\
&- c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}+p-1}(x)\cdot\cos_p^{-\overline{q}_{i,n}-p+2}(x)\,.
\end{aligned}$$

Let us denote

$$\begin{aligned}
a'_{2i-1,n+1} &:= c_{i,n}\cdot\overline{q}_{i,n}\,,\\
a'_{2i,n+1} &:= c_{i,n}\cdot(\overline{q}_{i,n}-1)\,.
\end{aligned}$$

By the induction assumption (4.46) and assumption of Case 1 (4.47), we have $a'_{2i-1,n+1}, a'_{2i,n+1} \leq 0$.

Case 2. If $\overline{q}_{i,n} = 0$, then $i = 0$ (by the ordering) and the corresponding term of (4.40) is

$$c_{0,n}\cdot\sin_p^0(x)\cdot\cos_p^1(x). \tag{4.49}$$

Taking derivatives in (4.49) we find

$$-\,c_{0,n}\cdot\sin_p^{p-1}(x)\cdot\cos_p^{2-p}(x)\,. \tag{4.50}$$

Denote $a'_{1,n+1} := -c_{0,n}$ which is clearly nonnegative by the induction assumption (4.46). We consider the second term of (4.40) ($i = 1$) and take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}c_{1,n}\cdot\sin_p^{\overline{q}_{1,n}}(x)\cdot\cos_p^{1-\overline{q}_{1,n}}(x)$$

$$= \mathrm{D}_s\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x) + \mathrm{D}_c\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x)\,.$$

Since $\overline{q}_{0,n} = 0$, $\overline{q}_{1,n} = p$ (see Figure 4). Note that the right-hand side of

$$\mathrm{D}_s\, c_{1,n} \sin_p^{p}(x) \cdot \cos_p^{1-p}(x) = p \cdot c_{1,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.51}$$

has the same exponent $q = p-1$ as (4.50) has. It remains to prove that $p \cdot c_{1,n} - c_{0,n} \le 0$. Using $(n-2)$-th derivative of $\sin_p(x)$ we obtain (4.50),

$$\begin{aligned}(\mathrm{D}_c \circ \mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) &= (\mathrm{D}_c \circ \mathrm{D}_s) 2 \cdot c_{0,n-2} \cdot \sin_p(x) \\ &= \mathrm{D}_c\, 2 \cdot c_{0,n-2} \cdot \cos_p(x) \\ &= -2 \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cos_p^{2-p}(x)\end{aligned} \tag{4.52}$$

and (4.51),

$$\begin{aligned}&(\mathrm{D}_s \circ \mathrm{D}_s \circ \mathrm{D}_c) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) \\ &= (\mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^{1+p}(x) \cos_p^{-p}(x) \\ &= \mathrm{D}_s (1+p) \cdot c_{0,n-2} \cdot \sin_p^{p}(x) \cdot \cos_p^{1-p}(x) \\ &= p \cdot (1+p) \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,.\end{aligned} \tag{4.53}$$

Comparing (4.52) with (4.50), we find that

$$-c_{0,n} = -2 \cdot c_{0,n-2}\,.$$

In addition, comparing (4.51) and (4.53), we find

$$p \cdot c_{1,n} = p \cdot (p+1) \cdot c_{0,n-2}\,.$$

From the induction assumption, $c_{0,n-2} \le 0$ and for $p \ge 3$, we easily find

$$p \cdot c_{1,n} - c_{0,n} = (p \cdot (p+1) - 2) c_{0,n-2} \le 0$$

by adding the previous two identities.

In the definition (4.39) of $c_{i,n}$, we are adding coefficients

$$a'_{k,n}\,, \quad k = 0, 1, \dots, 2(i_0 + 1)$$

corresponding to the same value of exponent $\overline{q}$. From the both cases, we obtain $c_{i,n+1} \le 0$ for all $i \in \mathbb{N}$, $i \le i_1$, $0 < i_1 \le 2^{n-1} - 1$. This concludes the proof by induction. □

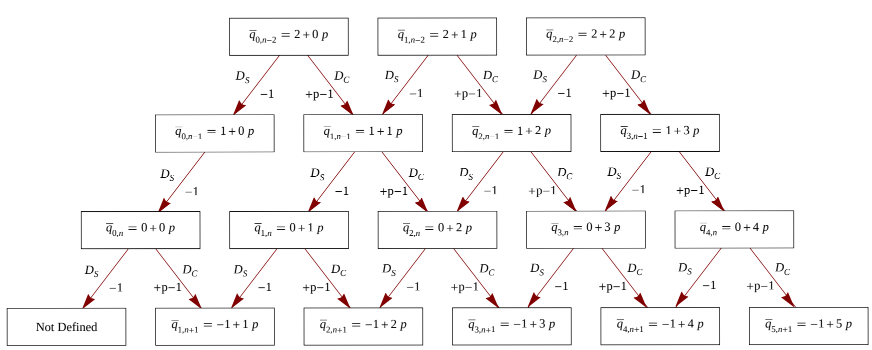

FIGURE 4. Rewriting diagram - starting with $\overline{q}_{0,n-2}, \overline{q}_{1,n-2}, \overline{q}_{2,n-2}$

5. Proofs of main results

Proof of Theorem 3.1. By Lemma 4.5 and substitution (4.14), we can write

$$\sin_{2(m+1)}^{(n)}(x)=\sum_{k=0}^{2^{n-2}-1} a_{k,n}\cdot \sin_{2(m+1)}^{q_{k,n}}(x)\cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,,$$

where

$$q_{k,n}=(2(m+1)-1)\cdot j(k)+(n-j(k)-2)+2(m+1)-1\,,$$

and $j(k)$ has the same meaning as in Lemma 4.5. Thus $a_{k,n}\in\mathbb{Z}$.

From Lemma 4.1, statement 4 and 5, we also know that $\sin_{2(m+1)}^{(n)}(x)$ is even function for n odd and $\sin_{2(m+1)}^{(n)}(x)$ is odd function for n even. It follows that for $x\in(-\frac{\pi_{2(m+1)}}{2},0)$

$$\sin_{2(m+1)}^{(n)}(x)=\begin{cases}-\sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ even}\,,\\ \sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ odd}\,.\end{cases}\tag{5.1}$$

Now we assume $p=2(m+1)$, $m\in\mathbb{N}$, and

$$\begin{aligned}q_{k,n}&=(2(m+1)-1)j(k)+(n-j(k)-2)+2(m+1)-1\\&=(2(m+1)-1)(j(k)+1)+j(k)+2-n\\&=2(m+1)(j(k)+1)-n+1\end{aligned}$$

which implies $q_{k,n}$ is *odd* for n even. Thus we obtain

$$\begin{aligned}-\sin_{2(m+1)}^{(n)}(-x)&=-\sum_{k=0}^{2^{n-2}-1} a_{k,n}\sin_{2(m+1)}^{q_{k,n}}(-x)\cdot\cos_{2(m+1)}^{1-q_{k,n}}(-x)\\&=\sum_{k=0}^{2^{n-2}-1} a_{k,n}\sin_{2(m+1)}^{q_{k,n}}(x)\cdot\cos_{2(m+1)}^{1-q_{k,n}}(x)\,.\end{aligned}\tag{5.2}$$

Analogously, $q_{k,n}$ is *even* for n odd and

$$\begin{aligned}\sin_{2(m+1)}^{(n)}(-x)&=\sum_{k=0}^{2^{n-2}-1} a_{k,n}\sin_{2(m+1)}^{q_{k,n}}(-x)\cdot\cos_{2(m+1)}^{1-q_{k,n}}(-x)\\&=\sum_{k=0}^{2^{n-2}-1} a_{k,n}\sin_{2(m+1)}^{q_{k,n}}(x)\cdot\cos_{2(m+1)}^{1-q_{k,n}}(x)\,.\end{aligned}\tag{5.3}$$

Hence from (5.2), (5.3), we obtain

$$\sin_{2(m+1)}^{(n)}(x)=\sum_{k=0}^{2^{n-2}-1} a_{k,n}\sin_{2(m+1)}^{q_{k,n}}(x)\cdot\cos_{2(m+1)}^{1-q_{k,n}}(x)\tag{5.4}$$

for all $x\in(-\frac{\pi_{2(m+1)}}{2},\frac{\pi_{2(m+1)}}{2})\setminus\{0\}$.

Now, we prove the continuity of $\sin_{2(m+1)}^{(n)}(x)$ for all $x\in(-\frac{\pi_{2(m+1)}}{2},\frac{\pi_{2(m+1)}}{2})$ by induction in n.

Step 1. For $x\in(-\frac{\pi_{2(m+1)}}{2},\frac{\pi_{2(m+1)}}{2})$ the function

$$v(x)=\varphi_{2(m+1)}(\cos_{2(m+1)}(x))>0$$

and so we can take the first equation in (2.3) into its derivative and obtain

$$u''(x) = \varphi'_{p'}(v(x))v'(x)\,, \text{ where } p' = \frac{2m+1}{2m}\,.$$

Since v' is continuous and $\varphi_{p'} \in C^1(0,+\infty)$ ($\varphi_{p'}(z) = z^{p-1}$ for $z > 0$), we obtain continuity of $\sin_{2(m+1)}^{(n)}(x)$ for $n = 2$.

Step 2. Let us assume that $\sin_{2(m+1)}^{(n)}(x)$ is continuous on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. From Lemma 4.5 we know that $\sin_{2(m+1)}^{(n+1)}(x)$ is continuous on $(0, \frac{\pi_{2(m+1)}}{2})$. Now we distinguish two cases: $n+1$ is odd then $\sin_{2(m+1)}^{(n+1)}(x)$ is even by Lemma 4.1, statement 4, and $n+1$ is even then $\sin_{2(m+1)}^{(n+1)}(x)$ is odd by Lemma 4.1, statement 5. In both cases, $\sin_{2(m+1)}^{(n+1)}(x) \in C(0, \frac{\pi_{2(m+1)}}{2})$ implies $\sin_{2(m+1)}^{(n+1)}(x) \in C(-\frac{\pi_{2(m+1)}}{2}, 0)$. It remains to prove the continuity at $x = 0$. From (5.4) we know that

$$\lim_{x\to 0-} \sin_{2(m+1)}^{(n+1)}(x) = \lim_{x\to 0+} \sin_{2(m+1)}^{(n+1)}(x)\,. \tag{5.5}$$

At the end we compute the derivative of $\sin_{2(m+1)}^{(n)}(0)$ from its definition:

$$\sin_{2(m+1)}^{(n+1)}(0) = \lim_{h\to 0} \frac{\sin_{2(m+1)}^{(n)}(h) - \sin_{2(m+1)}^{(n)}(0)}{h}\,.$$

It is a limit of the type "0/0". Since the limit $\lim_{h\to 0} \sin_{2(m+1)}^{(n+1)}(h)$ exists, we obtain $\sin_{2(m+1)}^{(n+1)}(0) = \lim_{h\to 0} \sin_{2(m+1)}^{(n+1)}(h)$ by L'Hôspital's rule. Note that by Lemma 4.6, $q_{k,n} \geq 0$ for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$, these limits are finite and we obtain continuity. This proves the continuity of $\sin_{2(m+1)}^{(n+1)}(x)$ for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. □

Proof of Theorem 3.2. By Lemma 4.5 and substitution (4.14), we have

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) \quad \text{on } (0, \frac{\pi_p}{2})\,.$$

Moreover, by Lemma 4.1, statement 4 and 5, we obtain

$$\sin_p^{(n)}(x) = \begin{cases} -\sin_p^{(n)}(-x) & \text{for } n \text{ even}\,,\\ \sin_p^{(n)}(-x) & \text{for } n \text{ odd}\,, \end{cases} \tag{5.6}$$

for $x \in (-\pi_p/2, 0)$. Since $\sin_p^{(n)}(x)$ is continuous for $x \in (0, \pi_p/2)$, it is also continuous on $x \in (-\pi_p/2, 0)$ by (5.6). Thanks to (5.6) it is enough to study the behavior of $\sin_p(x)$ in the right neighborhood of 0. From Lemma 4.5, we have that

$$q_{k,n} = j(k)\cdot(p-1) + (-1)\cdot(n-2-j) + p - 1 = p\cdot(j(k)+1) + 1 - n\,. \tag{5.7}$$

for all $n \in \mathbb{N}$, $n \geq 2$ and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$. Since $j(k) \in \{0\} \cup \mathbb{N}$ we find that

$$q_{k,n} \geq p + 1 - n\,.$$

Then, for $n < p+1$, we have $q_{k,n} > 0$ for all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$. And so using the theorem of the algebra of the limits from any classical analysis textbook, we find that

$$\lim_{x\to 0+} \sin_p^{(n)}(x) = 0\,.$$

From (5.6),

$$\lim_{x\to 0-}\sin_p^{(n)}(x)=\begin{cases}-\lim_{x\to 0+}\sin_p^{(n)}(x)=0 & \text{for } n \text{ even}\,,\\ \lim_{x\to 0+}\sin_p^{(n)}(x)=0 & \text{for } n \text{ odd}\,.\end{cases}$$

The continuity at $x=0$ follows from L'Hôspital's rule used recurrently from $n=2$ to $n=\lceil p\rceil$.

By Lemma 4.5, $\sin_{2(m+1)}^{(2m+2)}(x)$ satisfies

$$\sin_p^{(\lceil p\rceil+1)}(x)=\sum_{k=0}^{2^{\lceil p\rceil-1}-1} D_{k,\lceil p\rceil+1}\sin_p''(x)\quad \text{on } (0,\frac{\pi_p}{2})\,.$$

Since $q_{k,n}>0$ for all $n<\lceil p\rceil$ and all $k\in\{0\}\cup\mathbb{N}$, $k<2^{\lceil p\rceil}-1$, the function $\mathrm{D}_S\, a_{k,n}\cdot\sin_p^{q_{k,n}}(x)\cdot\cos_p^{1-q_{k,n}}(x)$ does not vanish identically. Thus $a_{0,\lceil p\rceil+1}\neq 0$. Since $a_{0,\lceil p\rceil+1}\neq 0$, we can apply (5.7) for $j(0)=0$ which gives

$$q_{0,\lceil p\rceil+1}=p-\lceil p\rceil\le 0\,.$$

From the fact that $j(k)>j(0)$ for all $k\in\{0\}\cup\mathbb{N}$, $k\le 2^{\lceil p\rceil-1}-1$ and from (5.7) we know that

$$q_{k,\lceil p\rceil+1}>q_{0,\lceil p\rceil+1}\,.$$

Moreover from (5.7),

$$q_{k,\lceil p\rceil+1}=(j(k)+1)\cdot p+1-(\lceil p\rceil+1)=(j(k)+1)\cdot p-\lceil p\rceil>0$$

for $j(k)\ge 1$ and $p>1$. Since, for all $q_{k,n}>0$,

$$\lim_{x\to 0} a_{k,n}\cdot\sin_p^{q_{k,n}}(x)\cdot\cos_p^{1-q_{k,n}}(x)=0\,,$$

we obtain

$$\begin{aligned}\lim_{x\to 0+}\sin_p^{(\lceil p\rceil+1)}(x)&=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)\\ &\quad+\sum_{k=1}^{2^{\lceil p\rceil-1}-1} a_{k,\lceil p\rceil+1}\cdot\sin_p^{q_{k,\lceil p\rceil+1}}(x)\cdot\cos_p^{1-q_{k,\lceil p\rceil+1}}(x)\\ &=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)\end{aligned}\tag{5.8}$$

by the theorem of the algebra of the limits.

Now the proof falls into two cases, $p=2m+1$ and $p\in\mathbb{R}\setminus\mathbb{N}$, $p>1$.

Case 1. For $p=2m+1$, we have by (5.8)

$$\lim_{x\to 0+}\sin_{2m+1}^{(2m+2)}(x)=\lim_{x\to 0+} a_{0,2m+2}\cdot\cos_p(x)=a_{0,2m+2}\neq 0\,.$$

Since $2m+2$ is even, $\sin_{2m+1}^{(2m+2)}(x)$ is odd function by Lemma 4.1, statement 5. Thus

$$\lim_{x\to 0-}\sin_{2m+1}^{(2m+2)}(x)=-a_{0,2m+2}\,.$$

Hence $\sin_{2m+1}^{(2m+2)}(x)$ is not continuous at $x=0$.

Case 2. Since for $p\in\mathbb{R}\setminus\mathbb{N}$, $p>1$, we have

$$\lim_{x\to 0+}\sin_p^{(\lceil p\rceil+1)}(x)=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)=+\infty$$

from (5.8). Hence $\sin_p^{\lceil p\rceil+1}(x)$ is discontinuous at $x=0$. This concludes the proof. $\square$

Proof of Theorem 3.3. It follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that there exists a unique analytic function $F(z)$ near origin such that the unique solution $u(x) = \sin_p(x)$ of the initial value problem (2.1); i.e.,

$$-(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0$$
$$u(0) = 0, \quad u'(0) = 1\,,$$

takes the form $\sin_p(x) = u(x) = x \cdot F(|x|^p)$. Note that for $p = 2(m+1)$ and $m \in \mathbb{N}$,

$$\sin_p(x) = x \cdot F(|x|^p) = x \cdot F(x^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot p+1}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l z^l\,,$$

which is also an analytic function in a neighborhood of $x = 0$. In the sequel of this proof $p = 2(m+1)$, $m \in \mathbb{N}$. By the uniqueness of the Maclaurin series of analytic function, we see that

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot p+1} = \sum_{l=0}^{+\infty} \frac{\sin_p^{(l\cdot p+1)}(0)}{(l\cdot p+1)!} \cdot x^{l\cdot p+1}\,,$$

where the right-hand side also converges to $\sin_p(x)$ on some neigbourhood of $x = 0$. Note that $\sin_p^{(k)}(0) = 0$ for any $k \in \mathbb{N}$ such that

$$\forall l \in \{0\} \cup \mathbb{N} : k \neq l \cdot p + 1$$

as it follows from Lemma 4.5 and Lemma 4.6.

Since the restriction of $\sin_p(x)$ to $[-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$ is the inverse function of $\operatorname{arcsin}_p(x)$, by the identity (1.6); i.e.,

$$\forall x \in [-1, 1] : \sin_p(\operatorname{arcsin}_p(x)) = x\,.$$

It is well known see, e.g., [13] that

$$\begin{aligned}\operatorname{arcsin}_p(x) &= \int_0^x (1 - s^p)^{-\frac{1}{p}}\,\mathrm{d}s \\ &= \frac{s \cdot {}_2F_1(1, \frac{1}{p}; 1 + \frac{1}{p}; s^p)}{p} \\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(n + \frac{1}{p})}{\Gamma(\frac{1}{p})(n\cdot p+1)} \cdot \frac{1}{n!} \cdot x^{n\cdot p+1}\end{aligned} \tag{5.9}$$

for $x \in (0, 1)$. Observe that for our special case $p = 2(m+1)$ with $m \in \mathbb{N}$, this formula is valid on $[-1, 1]$. Note also that in our special case, (5.9) is in fact the Maclaurin series for $\operatorname{arcsin}_p(x)$ and, moreover, all coefficients are nonnegative (the explicitly written coefficients are positive, the other ones are zero).

To apply the formula for composite formal power series, we need to consider series for $\sin_p(x)$ and $\operatorname{arcsin}_p(x)$ including the zero terms. For this reason, we define for all $j \in \mathbb{N}$

$$\alpha'_j := \sin_p^{(j)}(0)/j! = \begin{cases} \alpha_i & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

and

$$\beta'_j := \begin{cases} \frac{\Gamma(n+\frac{1}{p})}{\Gamma(\frac{1}{p})(n\cdot p+1)} \cdot \frac{1}{n!} & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.11}$$

Thus by well-known composite formal power series formula

$$\sin_p(\arcsin_p(x)) = \sum_{n=1}^{+\infty} c_n x^n \,, \tag{5.12}$$

where

$$c_n = \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}\,. \tag{5.13}$$

Since both functions $\sin_p(x)$ and $\arcsin_p(x)$ are analytic in some neighborhood of $x = 0$, the series from (5.12) with coefficients given by (5.13) is convergent towards the identity $x \mapsto x$ on some neighborhood of $x = 0$. From this fact, we infer that $c_1 = 1$ and $c_n = 0$ for all $n \in \mathbb{N}, n \geq 2$. Thus for any $x \in \mathbb{R}$

$$x = \sum_{n=1}^{+\infty} x^n \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k} \tag{5.14}$$

and in particular

$$1 = \sum_{n=1}^{+\infty} \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} \alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}\,. \tag{5.15}$$

Now we show that also

$$\sum_{n=1}^{+\infty} \sum_{\substack{k\in\mathbb{N},\ j_1,j_2,\dots,j_k\in\mathbb{N}\\ j_1+j_2+\dots+j_k=n}} |\alpha'_k\cdot\beta'_{j_1}\cdot\beta'_{j_2}\cdot\dots\cdot\beta'_{j_k}| \tag{5.16}$$

is convergent. By Lemma 4.7 and (5.10) we see that $\alpha'_j \leq 0$ for all $j \in \mathbb{N}, j \geq 2$ and $\alpha'_1 = \cos_p(0) = 1$. Moreover, from (5.11) it follows that $\beta'_j \geq 0$ for all $j \in \mathbb{N}$. Thus the product $\alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k}$ is positive if and only if $k = 1$. All positive terms can be written as $\alpha'_1 \cdot \beta'_n = \beta'_n$ for $n \in \mathbb{N}$ (if $k = 1$ then $j_1 = n$ is the only decomposition of n). Since the sum of all positive terms in (5.15) is $\sum_{n=1}^{+\infty} \beta'_n = \arcsin_p(1) = \frac{\pi_p}{2} < +\infty$, the sum of all negative terms must be finite too and equals $1 - \frac{\pi_p}{2}$. Thus (5.16) converges. This means that the series (5.15) converges absolutely to 1 and any rearrangement of this series must converge. Also any subseries of any rearrangement of this series must converge absolutely. Let $s_M = \sum_{m=1}^{M} \beta'_m$. Then the series $\sum_{k=1}^{+\infty} \alpha'_k \cdot (s_M)^k$ is a subseries of one of the rearrangements of (5.15) and it is convergent. Observe that s_M is nondecreasing and converging to $\sum_{m=1}^{+\infty} \beta'_m = \pi_p/2$ as $M \to +\infty$. Thus the Maclaurin series for $\sin_p(x) = \sum_{k=1}^{+\infty} \alpha'_k \cdot x^k$ is convergent for any $x \in (-\pi_p/2, \pi_p/2)$ to some analytic function.

Now it remains to show that it converges towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. This last step follows from the formal identity (5.14), which on the established range of convergence holds also analytically and the fact that the function $\sin_p(x)$ is the only function that satisfies the identity (1.6). $\square$

Proof of Theorem 3.4. From [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] it follows that, for any $p > 1$, there exists a unique analytic function $F(z)$ near origin such that $\sin_p(x) = x \cdot F(|x|^p)$; thus we have

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l \cdot p} \quad , \text{ where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l \,.$$

Note that for $p = 2m + 1$, $m \in \mathbb{N}$, the series

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} \tag{5.17}$$

defines an analytic function $G(x)$ in a neighborhood of $x = 0$ and also that

$$\sin_p(x) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} = G(x) \quad \text{for } x > 0 \tag{5.18}$$

on a neighborhood of 0. Our aim is to show that the radius of convergence of (5.17) is $\pi_p/2$ for $p = 2m + 1$, $m \in \mathbb{N}$. By (5.18), the following derivatives are equal

$$\sin_p^{(n)}(x) = G^{(n)}(x) = \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}$$

for $x > 0$ on the neighborhood of 0 where the series converges. Now take a one-sided limit from the right in the previous equation

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \lim_{x \to 0+} \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}.$$

For $j := \frac{n-1}{p} \in \{0\} \cup \mathbb{N}$, we obtain

$$\lim_{x \to 0+} \sum_{l=j}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1} = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!} \,.$$

Thus

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!}$$

for $j \in \{0\} \cup \mathbb{N}$. By Lemma 4.7, $\lim_{x \to 0+} \sin_p^{(n)}(x) \le 0$ for $n \ge 2$, $p \in \mathbb{N}$ and $p \ge 3$. Thus $\alpha_j \le 0$ for $j \in \mathbb{N}, j > 1$.

The rest of the proof of the theorem is identical to the proof of Theorem 3.3 and we find that the convergence radius of the series (5.17) is $\frac{\pi_p}{2}$ for $p = 2m + 1$, $m \in \mathbb{N}$. The only difference against the proof of Theorem 3.3 is that the series (5.17) converges towards $\sin_p(x)$ only on $(0, \pi_p/2)$ for $p = 2m + 1$, $m \in \mathbb{N}$. Note that the series is still convergent on $(-\pi_p/2, 0)$ towards $G(x) \neq \sin_p(x)$ for $x < 0$. The changes in the proof are obvious and are left to the reader. $\square$

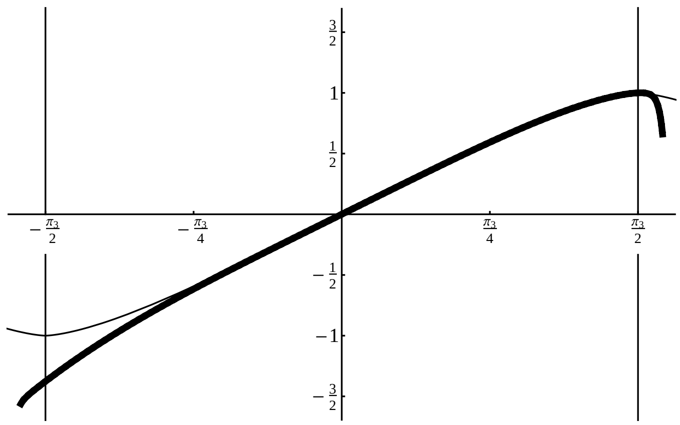

FIGURE 5. Graph of $\sin_3(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_3(x)$ up to the power x^{100} (thick line). Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \frac{\pi_3}{2}$

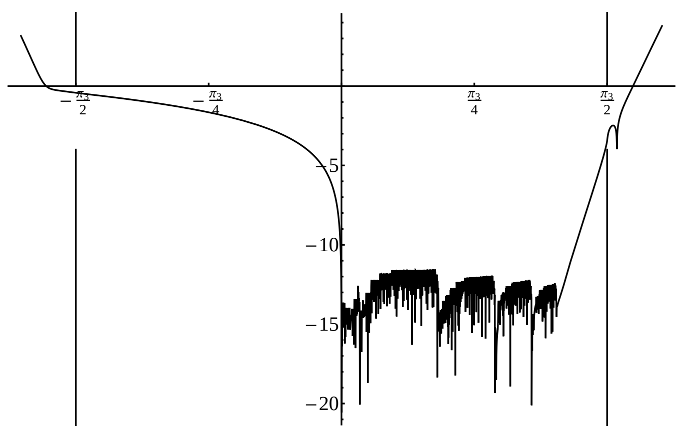

FIGURE 6. Graph of the function $\log_{10} |\sin_3(x) - \sum_{n=1}^{100} \alpha_n' x^n|$ where $\sum_{n=1}^{100} \alpha_n' x^n$ is the partial sum of the Maclaurin series of $\sin_3(x)$. The values of $\sin_3(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \pi_3/2$

6. Concluding remarks and open problems

As it was mentioned in the proofs of Theorems 3.3 and 3.4, it follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that, for any $p > 1$, there exists a unique

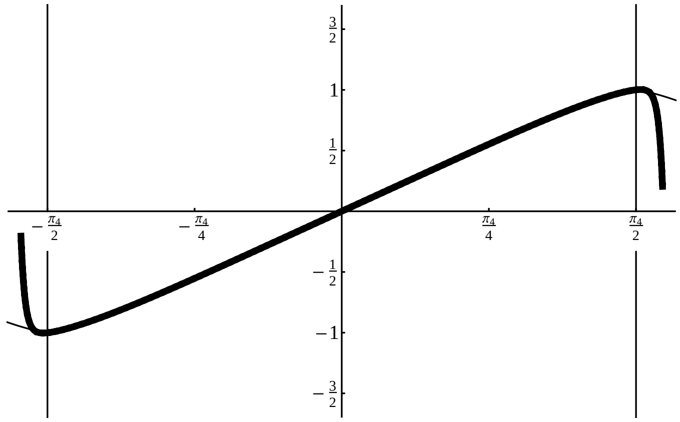

FIGURE 7. Graph of $\sin_4(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_4(x)$ up to the power x^{100} (thick line)

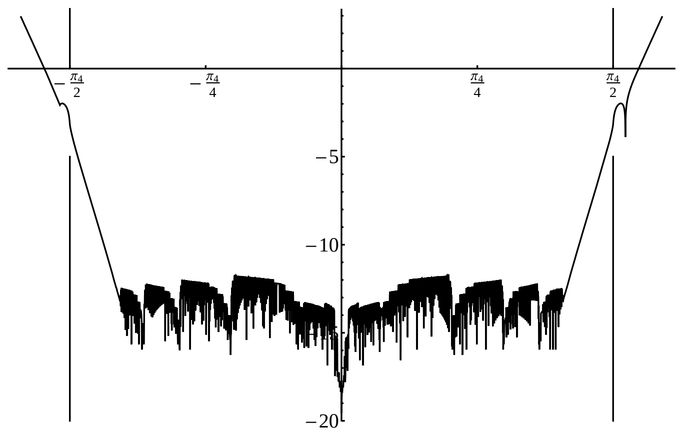

FIGURE 8. Graph of the function $\log_{10}|\sin_4(x) - \sum_{n=1}^{100}\alpha'_n x^n|$ where $\sum_{n=1}^{100}\alpha'_n x^n$ is the partial sum of the Maclaurin series of $\sin_4(x)$. The values of $\sin_4(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_4(x)$ for $|x| > \pi_4/2$

analytic function $F(z)$ near origin such that

$$\sin_p(x) = x \cdot F(|x|^p)\,.$$

Thus the function $\sin_p(x)$ can be expanded into generalized Maclaurin series near the origin:

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l \cdot p}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l\,.$$

Remark 6.1. (Convergence of generalized Maclaurin series) Let $p = 2m+1$ for $m \in \mathbb{N}$. It follows from the symmetry of the function $\sin_{2m+1}(x)$ with respect to the origin and from the proof of Theorem 3.4 that the generalized Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l\cdot(2m+1)}$ converges towards the values of $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$.

Remark 6.2 (Complex argument for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. It follows from the proof of Theorem 3.3 that the Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot 2(m+1)+1}$ converges towards the values of $\sin_{2(m+1)}(x)$ on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ absolutely. This enables us to extend the range of definition of the function $\sin_{2(m+1)}(x)$ to the complex open disc

$$B_m = \{z \in \mathbb{C} : |z| < \frac{\pi_{2(m+1)}}{2}\}$$

by setting $\sin_{2(m+1)}(z) := \sum_{l=0}^{+\infty} \alpha_l \cdot z^{l\cdot 2(m+1)+1}$. Since all the powers of z are of positive-integer order $l \cdot 2(m+1)+1$, the function $\sin_{2(m+1)}(z)$ is an analytic complex function on B_m and thus is single-valued. Unfortunately, this easy approach works only for $p = 2(m+1)$ with $m \in \mathbb{N}$; cf [15].

Our methods for proving convergence of the Maclaurin or generalized Maclaurin series are based on the fact that p is an integer. Thus a natural question appears.

Open Problem 6.3 (Convergence for $p > 1$ not integer)**.** Consider $p > 1$, $p \notin \mathbb{N}$. Prove (or find a counterexample) that the generalized Maclaurin series corresponding to $\sin_p(x)$ 'suggests the convergence' on $(-\pi_p/2, \pi_p/2)$ towards the values of $\sin_p(x)$.

For the sake of completeness, we remark that [15] claims the convergence of the generalized Maclaurin series on $(-\pi_p/2, \pi_p/2)$ for any $p > 1$, but there is no proof nor any indication for the proof of this claim.

Moreover, we are not able to decide about the convergence at the endpoints. This is another open question.

Open Problem 6.4 (Endpoints of the interval)**.** Consider $p > 1$. Prove (or find a counterexample) that the generalized Maclaurin series of $\sin_p(x)$ converge at $-\frac{\pi_p}{2}$ and/or $\frac{\pi_p}{2}$.

Remark 6.5 (Function $\cos_p$ for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Since $\cos_p(x) = \sin_p'(x)$ by definition, the Maclaurin series for $\cos_{2(m+1)}(x)$ can be obtained by taking into derivative the Maclaurin series for $\sin_{2(m+1)}(x)$ term by term. The Maclaurin series for $\cos_{2(m+1)}(x)$ then converges towards the value $\cos_{2(m+1)}(x)$ for any $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.

Remark 6.6 (Function $\cos_p$ for p odd)**.** Let $p = 2m+1$ for $m \in \mathbb{N}$. In this case the Maclaurin series for $\cos_{2m+1}(x)$ can also be obtained by taking into derivative the Maclaurin series for $\sin_{2m+1}(x)$ term by term. This Maclaurin series then converges for $x \in (-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. However, the Maclaurin series for $\cos_{2m+1}(x)$ converges towards the value $\cos_{2m+1}(x)$ for $x \in [0, \frac{\pi_{2m+1}}{2})$, but it does not converge towards the value $\cos_{2m+1}(x)$ for any $x \in (-\frac{\pi_{2m+1}}{2}, 0)$.

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