CONVERGENCE OF A MOUNTAIN PASS TYPE ALGORITHM FOR STRONGLY INDEFINITE PROBLEMS AND SYSTEMS

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ABSTRACT. For a functional $\mathcal{E}$ and a peak selection that picks up a global maximum of $\mathcal{E}$ on varying cones, we study the convergence up to a subsequence to a critical point of the sequence generated by a mountain pass type algorithm. Moreover, by carefully choosing stepsizes, we establish the convergence of the whole sequence under a "localization" assumption on the critical point. We illustrate our results with two problems: an indefinite Schrödinger equation and a superlinear Schrödinger system.

1. INTRODUCTION

Let us consider $\mathcal{H}$ a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$, and a functional $\mathcal{E} \in \mathcal{C}^1(\mathcal{H}; \mathbb{R})$. In this work, we develop a provably convergent "general" mountain pass type algorithm to approximate saddle points of $\mathcal{E}$, with a Morse index possibly larger than one. The pioneer work in this direction is due to Y. S. Choi and P. J. McKenna [4] who proposed a constrained steepest descent method to compute saddle points with one "descent direction" (such as a mountain pass solution). A proof of convergence of a variant of that algorithm was later given by Y. Li and J. Zhou in [10, 11]. To briefly describe it, let us fix a closed subspace E of $\mathcal{H}$ and φ a continuous $E^\perp$-peak selection, i.e. $\varphi(u)$ is the location of a maximum of $\mathcal{E}$ on $E \oplus \mathbb{R}^+ u := \{e + ty \mid e \in E,\ t \geqslant 0\}$ for any $u \in \mathcal{H} \setminus E$ and φ is constant on $E \oplus \mathbb{R}^+ u$. As it will be convenient in the rest of the paper that φ is not solely defined on a unit sphere, we present here a slightly different version [22].

Algorithm 1.1 (Mountain Pass Algorithm)**.**

  (i) Choose $u_0 \in \operatorname{Ran} \varphi$, $\varepsilon > 0$ and $n \leftarrow 0$;
  (ii) if $\|\nabla \mathcal{E}(u_n)\| \leqslant \varepsilon$ then stop;
     else compute

$$u_{n+1} = \varphi\Big(u_n - s_n \frac{\nabla \mathcal{E}(u_n)}{\|\nabla \mathcal{E}(u_n)\|}\Big),$$

for some $s_n \in S(u_n) \subseteq (0, +\infty)$ where $S(u_n)$ is a set of "admissible step-sizes" chosen so that at least the following inequality (which is reminiscent of *Armijo condition* [17]) holds:

$$\mathcal{E}(u_{n+1}) - \mathcal{E}(u_n) < -\tfrac{1}{2} s_n \|\nabla \mathcal{E}(u_n)\|;$$

(iii) let $n \leftarrow n + 1$ and go to step 2.

Y. Li and J. Zhou proved that (u_n) converges to a nontrivial critical point of $\mathcal{E}$ up to a subsequence. The proof of convergence is performed in the space $\mathcal{H}$ to ensure that the rate of convergence for the discretized problem does not deteriorate when the approximating subspace becomes finer. The original goal of the authors for introducing E was to try to obtain multiple critical points by taking E as the linear subspace generated by previously found solutions which the algorithm must try to avoid. The proof is performed in two steps. First, they show that s_n exists and that $\mathcal{E}$ decreases along $(u_n)_{n \in \mathbb{N}}$. This step relies on the following Deformation Lemma.

Lemma 1.2. *If φ is continuous, $u_0 \in \operatorname{Ran} \varphi$, $u_0 \notin E$, $\nabla \mathcal{E}(u_0) \neq 0$, then there exists $s_0 > 0$ such that*

$$\forall s \in (0, s_0], \quad \mathcal{E}\big(\varphi(u_s)\big) - \mathcal{E}(u_0) < -\tfrac{1}{2} s \|\nabla \mathcal{E}(u_0)\|,$$

where

$$u_s := u_0 - s \frac{\nabla \mathcal{E}(u_0)}{\|\nabla \mathcal{E}(u_0)\|}.$$

The second step consists in proving, under some traditional assumptions on φ, that a subsequence of (u_n) converges. For this, it is essential to show that the stepsize s_n controls the distance between u_n and u_{n+1} and that s_n is chosen in such a way that it is close to 0 only when "mandated" by the functional. Let us remark that the choice of φ is very sensitive. Indeed, to seek sign-changing critical points, the modified Mountain Pass algorithm was introduced by J. M. Neuberger [14] (see also [7]). He considers Algorithm 1.1 above and only modifies the projection φ into a "sign-changing peak selection" φ_N which is a map defined from the set of sign-changing functions of $\mathcal{H} \setminus \{0\}$ to $\mathcal{H} \setminus \{0\}$ such that, for any u, $\mathcal{E}\big(\varphi_N(u)\big) > 0$ and $\varphi_N(u)$ is a maximum of $\mathcal{E}$ on $\mathbb{R}^+ u^+ \oplus \mathbb{R}^+ u^-$ where $u^+(x) := \max\{0, u(x)\}$ and $u^-(x) := \min\{0, u(x)\}$. Although in practice it appears to converge to a nontrivial sign-changing critical point, its convergence has yet to be formally proved.

In this paper, for any u, $\varphi(u)$ is allowed to be any maximum point of $\mathcal{E}$ inside an abstract cone C_u and we are interested in giving assumptions on C_u which imply the convergence of the mountain pass algorithm. This work is partly motivated by the article [18] where the authors define the notion of "natural constraints" to seek nontrivial critical points of functionals. To the best of the authors' knowledge, using peak selections along cones that vary from point to point in a "continuous" fashion (see (1.2)) is novel and allows us to propose a general, provably convergent, algorithm that subsumes into a unified setting several previous mountain-pass type algorithms. We will show how it can be applied both to strongly indefinite problems and to systems.

Let us first make precise the peak selection φ that we use. We write $\operatorname{int} C$ for the interior of C relative to $\overline{\operatorname{span}}\, C$, the smallest closed subspace containing C, for the topology induced by $\mathcal{H}$.

Definition 1.3. Let $\mathcal{A}$ be an open subset of $\mathcal{H}$. We say that φ is a *peak selection for* $(C_u)_{u\in\mathcal{A}}$ if φ is a map from $\mathcal{A}$ to $\mathcal{A}$ such that, for all $u \in \mathcal{A}$,

(i) C_u is a closed cone pointed at 0;
(ii) $\varphi(u) \in \operatorname{int} C_u$;
(iii) for any $v \in \operatorname{int} C_u$, $\varphi(v) = \varphi(u)$;
(iv) $\varphi(u)$ is a global maximum point of $\mathcal{E}$ on C_u.

Note that properties (ii) and (iii) imply that $\varphi(\varphi(u)) = \varphi(u)$. We write that $d \perp C_u$ if and only if $d \perp \overline{\operatorname{span}}\, C_u$ for the inner product $\langle\cdot|\cdot\rangle$. In Section 2, we assume that φ is continuous and that (C_u) verifies the following conditions:

$$\forall u \in \mathcal{A}, \ u \in C_u \tag{1.1}$$

and

$$\begin{aligned}&\exists\gamma \in \left(0, \tfrac{\pi}{2}\right),\ \exists\delta \in (0,1),\ \forall u_0 \in \operatorname{Ran}\varphi,\ \exists r > 0,\ \forall\tilde{u}_0 \in \operatorname{Ran}\varphi \cap B(u_0, r),\\&\forall d \in B(0,r),\ d \perp C_{\tilde{u}_0},\quad C_{\tilde{u}_0+d} \cap B(u_0,r) \subseteq C_{\tilde{u}_0} + [1-\delta, 1+\delta]\, A_\gamma(d),\end{aligned} \tag{1.2}$$

where $A_\gamma(d) := \{d' \mid \|d'\| = \|d\|$ and $\angle(d', d) \leqslant \gamma\}$ and $\angle(d, d') := \arccos\left(\frac{\langle d|d'\rangle}{\|d\|\|d'\|}\right)$ denotes the angle between two non-zero vectors d and d' (we set $A_\gamma(0) := \{0\}$). This assumption, which speaks about the behavior of the cones under small deformations, is essential to prove a Deformation Lemma in this generalized setting (see Lemma 2.1). This lemma ensures the non-emptiness of the set $S(u_0)$ of admissible stepsizes at u_0 which we now define. For any $u_0 \in \operatorname{Ran}\varphi$ such that $\nabla\mathcal{E}(u_0) \neq 0$, we set

$$S^*(u_0)$$
$$:= \left\{s > 0 : u_s := u_0 - s\frac{\nabla\mathcal{E}(u_0)}{\|\nabla\mathcal{E}(u_0)\|} \in \mathcal{A} \text{ and } \mathcal{E}\big(\varphi(u_s)\big) - \mathcal{E}(u_0) < -\alpha s\|\nabla\mathcal{E}(u_0)\|\right\}$$

for some value $\alpha > 0$ given by Lemma 2.1 and we require that $s \in S(u_0) := S^*(u_0) \cap \left[\frac{1}{2}\sup S^*(u_0), +\infty\right)$. Other definitions of admissible stepsizes are possible provided they imply a local uniformity in the sense that $s_n \in S(u_n)$ forces the stepsize s_n not to be small when the gradient is not (see Lemma 2.4). The definition given above draws its inspiration from a paper [22] written by N. Tacheny and C. Troestler.

To obtain the convergence of (u_n) up to a subsequence (see Theorem 2.14), we unfortunately need to replace (1.2) with the following stronger assumption:

there exist a closed subspace $E \subseteq \mathcal{H}$ (possibly infinite dimensional) and a family of $\mathcal{C}^1$-vector fields $\xi_i : \mathcal{A} \to E^\perp$, $i = 1, \ldots, k$, for some $k \in \mathbb{N}$, such that for all $u \in \mathcal{A}$ and $i \in \{1, \ldots, k\}$,

  (i) the family $\big(\xi_i(u)\big)_{i=1}^k$ is orthonormal;
  (ii) $\forall v \in V_u$, $\xi_i'(u)[v] \in V_u$, where $V_u := \overline{\operatorname{span}}\{\xi_1(u), \ldots, \xi_k(u)\}$;
  (iii) $\forall v \in \operatorname{int} C_u \cap \mathcal{A}$, $\xi_i(v) = \xi_i(u)$;
  (iv) $\langle u|\xi_i(u)\rangle \neq 0$;
  (v) $\exists r > 0$, ξ_i' is bounded on $\{u \mid \operatorname{dist}(u, \operatorname{Ran}\varphi) < r\} \cap \mathcal{A}$.

For $u \in \mathcal{A}$, the cone C_u is defined as $C_u := E \oplus \{\sum_i t_i\xi_i(u) \mid t_i \geqslant 0 \text{ for all } i\}$.

$$(1.3)$$

Here, the notation $\operatorname{dist}(u, \partial\mathcal{A})$ stands for $\inf\{\|u - v\| \mid v \in \partial\mathcal{A}\}$. Let us remark that conditions (1.3) (i), (ii), (iv) and (v) are already present (albeit somehow

implicitly for (iv)) in [18] in the context of trivial $\mathcal{C}^1$-subbundles intead of cones. The additional condition (iii) is equivalent to $\forall v \in \operatorname{int} C_u \cap \mathcal{A}$, $C_v = C_u$. This condition is rather natural to require in view of property (iii) of the definition of peak selection. This "finite presentation" of the cones is used in Lemma 2.8 to ensure that the stepsize s_n controls the distance between u_{n+1} and u_n.

As a particular case of (1.3), let us mention that we can work with a family of continuous linear projectors (see Proposition 2.12). This case is an abstract formulation of the setting of [2, 3] where the convergence (up to a subsequence) of a mountain pass type algorithm for systems has been announced.

In Section 2.3, we are interested in the convergence of the whole sequence generated by the Mountain Pass Algorithm. To that aim, we need to refine the definition of S^* in order to control $\mathcal{E}\big(\varphi(u_n - s\frac{\nabla u_n}{\|\nabla u_n\|})\big)$ for any $0 < s < s_n$.

In Section 3, we illustrate our method with two semi-linear problems. The first application takes its inspiration from a paper due to A. Szulkin and T. Weth [21] in which the authors study the following Schrödinger problem

$$-\Delta u(x) + V(x)u(x) = |u(x)|^{p-2}u(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where $V : \Omega \to \mathbb{R}$ is such that 0 is in a spectral gap of $-\Delta + V$ and $2 < p < 2^* := \frac{2N}{N-2}$ ($+\infty$ when $N = 2$). They are interested in the existence of non-zero solutions on an open bounded domain $\Omega \subseteq \mathbb{R}^N$ or on $\Omega = \mathbb{R}^N$ (in the latter case, V is assumed to be 1-periodic in each x_i, $i = 1, \dots, N$). Solutions to this equation are critical points of the indefinite functional

$$\mathcal{E} : \mathcal{H} \to \mathbb{R} : u \mapsto \frac{1}{2}\int_\Omega \big(|\nabla u(x)|^2 + V(x)u(x)^2\big)\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u(x)|^p\,\mathrm{d}x, \tag{1.5}$$

where $\mathcal{H} = H_0^1(\Omega)$. The first proof of the existence of non-zero critical points for $\mathcal{E}$ when $-\Delta + V$ is not positive definite and Ω is an open bounded domain is due to P. H. Rabinowitz [19]. Recently, A. Szulkin and T. Weth proposed an alternative method [21] that also makes easier to deal with the lack of compactness that occurs when $\Omega = \mathbb{R}^N$. Denoting E the negative eigenspace of $-\Delta + V$, they introduce the following nonlinear map

$$\varphi : \mathcal{H} \setminus E \to \mathcal{H} : u \mapsto \varphi(u)$$

where $\varphi(u)$ is the point at which $\mathcal{E}$ reaches its maximum value on $E \oplus \mathbb{R}^+ u$. They prove that minimizing $\mathcal{E}$ on $\operatorname{Ran}\varphi = \big\{ u \in \mathcal{H} \setminus E \mid \partial\mathcal{E}(u)[v] = 0$ for $v = u$ and any $v \in E \big\}$ yields a non-zero solution with least energy. Notice that, here, E is used to deal with the indefiniteness of the problem and not to compute multiple critical points as in the papers of J. Zhou & al. [10, 11]. We will prove that our algorithm converges for this problem. The numerical solutions that we obtain lead to some conjectures on the symmetries of ground state solutions.

The second application is based on a paper by B. Noris and G. Verzini [18]. The authors study the superlinear Schrödinger system

$$-\Delta u_i(x) = \partial_i F\big(u_1(x), \dots, u_k(x)\big), \quad x \in \Omega,$$
$$u_i(x) = 0, \quad x \in \partial\Omega, \qquad i = 1, \dots, k, \tag{1.6}$$

where $k \in \mathbb{N}$. They require that $\Omega \subseteq \mathbb{R}^N$ is a bounded smooth domain and $F \in \mathcal{C}^2(\mathbb{R}^k; \mathbb{R})$. Note that the system $-\Delta u_i = \mu_i u_i^3 + u_i \sum_{j \neq i} \beta_{i,j} u_j^2$ where $\mu_i > 0$ and $\beta_{i,j} = \beta_{j,i}$ is a particular case of (1.6). Such type of nonlinearities have

been studied due to their applications to nonlinear optics and to Bose-Einstein condensation (see [6, 5, 8, 23]). Solutions to (1.6) are critical points of the functional

$$\mathcal{E} : \mathcal{H} \to \mathbb{R} : u = (u_1, \ldots, u_k) \mapsto \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x - \int_\Omega F(u) \, \mathrm{d}x, \qquad (1.7)$$

where $\mathcal{H} = H_0^1(\Omega; \mathbb{R}^k)$. As already mentioned, B. Noris and G. Verzini [18] propose a general method of "natural constraints". Applied to the above problem, it goes as follows. Denote $\mathcal{A} := \{u \in \mathcal{H} : u_i \neq 0 \text{ for every } i\}$. To find a solution $u = (u_1, \ldots, u_k)$ of (1.6) with $u_i \neq 0$ for all $i = 1, \ldots, k$, they minimize $\mathcal{E}$ on the constraint $\mathcal{N} := \{u \in \mathcal{A} \mid \forall i = 1, \ldots, k, \ \int_\Omega |\nabla u_i|^2 \, \mathrm{d}x = \int_\Omega \partial_i F(u) u_i \, \mathrm{d}x\}$. We will show that, under their assumptions, $\mathcal{N} = \mathrm{Ran}\, \varphi$ with φ being the peak selection

$$\varphi : \mathcal{A} \to \mathcal{A} : u \mapsto \mathrm{argmax}\{\mathcal{E}(t_1 u_1, \ldots, t_k u_k) : t_1 > 0, \ldots, t_k > 0\}.$$

Again, we prove that our algorithm converges for this problem and some numerical experiments are performed.

2. Steepest descent method on varying cones

2.1. Uniform Deformation Lemma.
Let $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ be a $\mathcal{C}^1$-functional defined on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ an open subset of $\mathcal{H}$. The following lemma is instrumental in proving the convergence of the algorithm.

Lemma 2.1 (Uniform Deformation Lemma). *Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a peak selection for $(C_u)_{u \in \mathcal{A}}$ and $u_0 \in \mathrm{Ran}\, \varphi$ be such that $\nabla \mathcal{E}(u_0) \neq 0$. Assume that φ is continuous at u_0 and that (1.1)–(1.2) hold. Then there exist $s_0 > 0$ and $r_0 > 0$ such that, for any $s \in (0, s_0]$ and $\tilde{u}_0 \in B(u_0, r_0) \cap \mathrm{Ran}\, \varphi$, one has*

- *$\nabla \mathcal{E}(\tilde{u}_0) \neq 0$,*
- *$\tilde{u}_s \in \mathcal{A}$ where $\tilde{u}_s := \tilde{u}_0 - s \frac{\nabla \mathcal{E}(\tilde{u}_0)}{\|\nabla \mathcal{E}(\tilde{u}_0)\|}$ and*
- *there exists some $\alpha > 0$ solely depending on γ and δ given in assumption (1.2) such that*

$$\mathcal{E}\big(\varphi(\tilde{u}_s)\big) - \mathcal{E}(\tilde{u}_0) < -\alpha s \|\nabla \mathcal{E}(\tilde{u}_0)\|. \qquad (2.1)$$

Proof. Let $u_0 \in \mathrm{Ran}\, \varphi \subseteq \mathcal{A}$ and let us consider γ, δ and r given by the assumption (1.2) for u_0. Since $\mathcal{A}$ is open, there exists $\varepsilon_1 > 0$ such that for any $u \in B(u_0, \varepsilon_1)$ and $v \in B(u, \varepsilon_1)$, one has $u, v \in \mathcal{A}$, $\nabla \mathcal{E}(u) \neq 0$, $\nabla \mathcal{E}(v) \neq 0$ and $u, v \in B(u_0, r)$.

For any $u \in B(u_0, \varepsilon_1)$, let $d_u := -\nabla \mathcal{E}(u)/\|\nabla \mathcal{E}(u)\|$. Then

$$\forall u \in B(u_0, \varepsilon_1), \ \forall d \in A_\gamma(d_u), \quad \langle \nabla \mathcal{E}(u) | d \rangle \leqslant -\cos \gamma \, \|\nabla \mathcal{E}(u)\|.$$

Let $\tilde{\gamma} := \frac{1}{2} \cos \gamma > 0$. Taking ε_1 smaller if necessary, we may assume that

$$\forall u, v \in B(u_0, \varepsilon_1), \ \forall d \in A_\gamma(d_u), \quad \langle \nabla \mathcal{E}(v) | d \rangle < -\tilde{\gamma} \|\nabla \mathcal{E}(u)\|.$$

Thus, on one hand, there exists $\varepsilon_2 > 0$ such that, for any $u \in B(u_0, \varepsilon_2)$, $v \in B(u, \varepsilon_2)$, $d \in A_\gamma(d_u)$ and $\sigma \in (0, \varepsilon_2)$,

$$\langle \nabla \mathcal{E}(v + \sigma d) | d \rangle < -\tilde{\gamma} \|\nabla \mathcal{E}(u)\|.$$

For any $\tilde{u}_0 \in B(u_0, \varepsilon_2) \cap \mathrm{Ran}\, \varphi$, $v \in C_{\tilde{u}_0} \cap B(\tilde{u}_0, \varepsilon_2)$, $d \in A_\gamma(d_{\tilde{u}_0})$ and $\sigma < \varepsilon_2$, the Mean Value Theorem implies there exists a $\tilde{\sigma} \in (0, \sigma)$ such that

$$\mathcal{E}(v + \sigma d) - \mathcal{E}(\tilde{u}_0) \leqslant \mathcal{E}(v + \sigma d) - \mathcal{E}(v) \qquad (2.2)$$
$$= \big\langle \nabla \mathcal{E}(v + \tilde{\sigma} d) \, \big| \, \sigma d \big\rangle$$
$$< -\tilde{\gamma} \sigma \|\nabla \mathcal{E}(\tilde{u}_0)\|, \qquad (2.3)$$

where the first inequality results from $v \in C_{\tilde{u}_0}$ and $\tilde{u}_0 = \varphi(\tilde{u}_0)$ is a global maximum of $\mathcal{E}$ on $C_{\tilde{u}_0}$.

On the other hand, by the continuity of φ at u_0, there exist $s_0 \in (0, r)$ and $\varepsilon_3 \in \left(0, \min\{r, \frac{1}{3}\varepsilon_2\}\right)$ such that, for any $\tilde{u}_0 \in B(u_0, \varepsilon_3)$ and $s \in [0, s_0]$, one has $\varphi(\tilde{u}_0 + sd_{\tilde{u}_0}) \in B(u_0, \min\{r, \frac{1}{3}\varepsilon_2\})$. Let $\tilde{u}_s := \tilde{u}_0 + sd_{\tilde{u}_0}$. If in addition $\tilde{u}_0 \in \operatorname{Ran}\varphi$, one has $d_{\tilde{u}_0} \perp \overline{\operatorname{span}}\, C_{\tilde{u}_0}$ (because $\tilde{u}_0 = \varphi(\tilde{u}_0) \in \operatorname{int} C_{\tilde{u}_0}$ is a local maximum) and therefore one deduces from assumption (1.2) that

$$\varphi(\tilde{u}_s) \in C_{\tilde{u}_s} \cap B(u_0, r) \subseteq C_{\tilde{u}_0} + [1 - \delta, 1 + \delta] A_\gamma(sd_{\tilde{u}_0}).$$

Thus, $\varphi(\tilde{u}_s) = v_s + K_s s d_s^*$ for some $v_s \in C_{\tilde{u}_0}$, $K_s \in [1 - \delta, 1 + \delta]$ and $d_s^* \in A_\gamma(d_{\tilde{u}_0})$. So, possibly taking s_0 smaller, we get that $K_s s < \frac{1}{3}\varepsilon_2$ and $v_s = \varphi(\tilde{u}_s) - K_s s d_s^* \in B(\tilde{u}_0, \varepsilon_2)$. Using Equation (2.3), we conclude that

$$\mathcal{E}\big(\varphi(\tilde{u}_s)\big) - \mathcal{E}(\tilde{u}_0) = \mathcal{E}(v_s + K_s s d_s^*) - \mathcal{E}(\tilde{u}_0) \leqslant -\tilde{\gamma}\,(1 - \delta)s\|\nabla\mathcal{E}(\tilde{u}_0)\|$$

for any $\tilde{u}_0 \in B(u_0, \varepsilon_3) \cap \operatorname{Ran}\varphi$ and $s \in (0, s_0]$. $\square$

Remark 2.2.

- Equation (2.2) is the unique place we use that $\varphi(u)$ is a global maximum of $\mathcal{E}$ on C_u. This assumption can be weakened by only requiring that the neighborhood on which $\varphi(u)$ achieves the maximum of $\mathcal{E}$ is locally uniform w.r.t. u:

$$\forall u_0 \in \operatorname{Ran}\varphi, \ \exists \rho > 0, \ \forall u \in \operatorname{Ran}\varphi \cap B(u_0, \rho), \quad \mathcal{E}\big(\varphi(u)\big) = \max_{v \in C_u \cap B(u, \rho)} \mathcal{E}(v). \quad (2.4)$$

  This assumption allows the existence of multiple maximum points in C_u. It was not used in Definition 1.3 for simplicity but also because, in the examples of Section 3, $\varphi(u)$ is a maximum on the whole C_u.
- Let us also note that, if we are just interested in the inequality (2.1) at u_0 (and not for all $\tilde{u}_0$ in a neighborhood of u_0), we only need to require that $\varphi(u)$ is a local maximum of $\mathcal{E}$ on C_u.
- A careful reader may notice that we did not really use the fact that C_u is a cone pointed at 0. However, if (C_u) was just a family of sets satisfying (1.1), (1.2), (2.4) and the fact that $\varphi(u) \in \operatorname{int} C_u$ in a locally uniform way:

$$\forall u_0 \in \operatorname{Ran}\varphi, \ \exists \rho > 0, \ \forall u \in \operatorname{Ran}\varphi \cap B(u_0, \rho), \quad B(u, \rho) \cap \overline{\operatorname{span}}\, C_u \subseteq C_u, \quad (2.5)$$

  then the cone $\hat{C}_u$, defined as the closure of $\{tv \mid t \geqslant 0 \text{ and } v \in C_u\}$, also satisfies (1.1), (1.2), (2.4) and $\varphi(u) \in \operatorname{int} \hat{C}_u$. So very little is gained by not using cones, especially because they are the natural structures encountered in our examples.
- As a consequence of the above Deformation Lemma, one can interpret $\operatorname{Ran}\varphi$ as somewhat a natural constraint for $\mathcal{E}$ in the sense of [18]. More precisely, it implies that if $u_0 \in \operatorname{Ran}\varphi$ is a local minimum of $\mathcal{E}$ on $\operatorname{Ran}\varphi$ then u_0 is a critical point of $\mathcal{E}$ on the whole space $\mathcal{H}$.

2.2. **Convergence up to a subsequence.** In this section, we first remark that it is possible to construct a sequence of stepsizes s_n such that the energy $\mathcal{E}$ decreases along the sequence $(u_n)_{n \in \mathbb{N}}$ generated by Algorithm 1.1. In the following, without loss of generality, we can assume that $\nabla\mathcal{E}(u_n) \neq 0$ for any $n \in \mathbb{N}$ (otherwise the algorithm finds a critical point in a finite number of steps).

Proposition 2.3. *If $s_n > 0$ verifies inequality (2.1) given in Lemma 2.1 for any $n \in \mathbb{N}$, then the functional $\mathcal{E}$ decreases along the sequence $(u_n)_{n \in \mathbb{N}}$.*

Proof. As $\nabla \mathcal{E}(u_n) \neq 0$, s_n is well-defined by Lemma 2.1. By construction, we have

$$\mathcal{E}(u_{n+1}) - \mathcal{E}(u_n) = \mathcal{E}\Big(\varphi\big(u_n - s_n \frac{\nabla \mathcal{E}(u_n)}{\|\nabla \mathcal{E}(u_n)\|}\big)\Big) - \mathcal{E}(u_n) < -\alpha s_n \|\nabla \mathcal{E}(u_n)\| < 0.$$

So, $\mathcal{E}(u_{n+1}) < \mathcal{E}(u_n)$. □

As explained in the introduction, we now consider the sets $S^*(u_0)$ and $S(u_0)$. The set $S^*(u_0)$ is not empty as soon as u_0 is not a critical point of $\mathcal{E}$ (thanks to the Deformation Lemma). Concerning the set $S(u_0)$, it is not-empty once $\mathcal{E}$ is bounded from below on $\operatorname{Ran}\varphi$, an assumption that we will later make (see Theorem 2.10).

Lemma 2.4. *If $u_0 \in \operatorname{Ran}\varphi$, $\nabla \mathcal{E}(u_0) \neq 0$ and φ is continuous at u_0, then there exists an open neighborhood V of u_0 and $s^* > 0$ such that $S(u) \subseteq [s^*, +\infty)$ for any $u \in V \cap \operatorname{Ran}\varphi$.*

Proof. By the Uniform Deformation Lemma 2.1, there exists $s_0 > 0$ and $r_0 > 0$ such that, for any $0 < s \leqslant s_0$ and $u \in B(u_0, r_0) \cap \operatorname{Ran}\varphi$, we have $u_s := u - s\frac{\nabla \mathcal{E}(u)}{\|\nabla \mathcal{E}(u)\|} \in \mathcal{A}$, $\nabla \mathcal{E}(u) \neq 0$ and

$$\mathcal{E}\big(\varphi(u_s)\big) - \mathcal{E}(u) < -\alpha s \|\nabla \mathcal{E}(u)\|. \tag{2.6}$$

In particular, for any $u \in B(u_0, r_0) \cap \operatorname{Ran}\varphi$, $s_0 \in S^*(u)$. It follows that $S(u) \subseteq [\frac{s_0}{2}, +\infty)$. It suffices to take $s^* \leqslant s_0/2$. □

Remark 2.5. To prove Lemma 2.4, let us remark that we could only use inequality (2.6) at $u = u_0$ for $s = s_0$ fixed. Indeed, by continuity, it directly implies that $s_0 \in S(u)$ for u close to u_0. However, to obtain Lemma 2.4 for $\tilde{S}(u)$ (see Section 2.3) instead of $S(u)$, the full strength of the Deformation Lemma is needed.

From now on, we have to require condition (1.3). Let us first show it subsumes (1.2).

Lemma 2.6. *Let $(\xi_i)_{i=1}^k$ be the family of vector fields given by (1.3) and assume (1.1) holds. Then*

$$\forall u \in \mathcal{A}, \ \forall d \perp C_u, \quad \sum_{i=1}^k \langle u \mid \xi_i(u) \rangle \, \xi_i'(u)[d] = d - \sum_{i=1}^k \langle u \mid \xi_i'(u)[d] \rangle \, \xi_i(u).$$

Proof. For any $u \in \mathcal{A}$, the fact that $u \in C_u \subseteq V_u$ and that $(\xi_i)_{i=1}^k$ is an orthonormal basis of V_u imply $u = \sum \langle u | \xi_i(u) \rangle \xi_i(u)$. Differentiating in a direction $d \in \mathcal{H}$, yields

$$d = \sum_{i=1}^k \langle d | \xi_i(u) \rangle \, \xi_i(u) + \sum_{i=1}^k \langle u | \xi_i'(u)[d] \rangle \, \xi_i(u) + \sum_{i=1}^k \langle u | \xi_i(u) \rangle \, \xi_i'(u)[d].$$

If $d \perp \overline{\operatorname{span}} \, C_u$, the first term vanishes. This completes the proof. □

Proposition 2.7. *Properties (1.1) and (1.3) imply (1.2).*

Proof. Let $\delta \in (0,1)$ (property (1.2) will be satisfied whatever value is chosen). Simple geometrical considerations show that there exists a $\gamma \in (0, \pi/2)$ such that

$$B(d, \delta\|d\|) \subseteq [1 - \delta, 1 + \delta] A_\gamma(d).$$

Let $u_0 \in \operatorname{Ran} \varphi$. As $u_0 \in \operatorname{int} C_{u_0}$, there exist $\alpha > 0$ such that $\langle u_0 | \xi_i(u_0) \rangle > \alpha$ for all i. Using the continuity of ξ_i and ξ_i' at u_0, we can choose r sufficiently small and a $M > 0$ (depending only on u_0) so that, for all $u \in B(u_0, r)$ and all $i = 1, \dots, k$,

$$\langle u | \xi_i(u) \rangle > \alpha, \quad \| \xi_i(u) - \xi_i(\tilde{u}_0) \| \leqslant \varepsilon, \quad \| \xi_i'(u) \| \leqslant M, \quad \text{and} \quad \| \xi_i'(u) - \xi_i'(\tilde{u}_0) \| \leqslant \varepsilon,$$

where $\varepsilon > 0$ is a constant depending only on δ and u_0 (to be chosen later).

Let $\tilde{u}_0 \in B(u_0, r)$ and $d \in B(0, r)$ such that $d \perp C_{\tilde{u}_0}$. Let $w \in C_{\tilde{u}_0 + d} \cap B(u_0, r)$. One can write $w = e + \sum t_i \xi_i(\tilde{u}_0 + d)$ for some $e \in E$ and $t_i \geqslant 0$. Let us start by noticing that $t_i = \langle w | \xi_i(\tilde{u}_0 + d) \rangle$. Therefore, recalling that $\| \xi_i \| = 1$, one deduces

$$\begin{aligned}
\left| t_i - \langle \tilde{u}_0 | \xi_i(\tilde{u}_0) \rangle \right| &\leqslant \left| \langle w - \tilde{u}_0 \, | \, \xi_i(\tilde{u}_0 + d) \rangle \right| + \left| \langle \tilde{u}_0 \, | \, \xi_i(\tilde{u}_0 + d) - \xi_i(\tilde{u}_0) \rangle \right| \\
&\leqslant \| w - \tilde{u}_0 \| + \| \tilde{u}_0 \| \| \xi_i(\tilde{u}_0 + d) - \xi_i(\tilde{u}_0) \| \\
&\leqslant 2r + (\| u_0 \| + r)\varepsilon. \quad\quad (2.7)
\end{aligned}$$

Provided that ε and r are chosen small enough, one can assume that $2r + (\| u_0 \| + r)\varepsilon \leqslant \alpha/3$. In particular, this implies $t_i > 2\alpha/3 > 0$.

Using the integral form of the Mean Value Theorem, we get

$$w = e + \sum_{i=1}^{k} t_i \xi_i(\tilde{u}_0 + d) = e + \sum_{i=1}^{k} t_i \xi_i(\tilde{u}_0) + \int_0^1 \sum_{i=1}^{k} t_i \, \xi_i'(\tilde{u}_0 + sd)[d] \, \mathrm{d}s. \quad (2.8)$$

The third term can be rewritten as follows:

$$\sum_{i=1}^{k} \langle \tilde{u}_0 | \xi_i(\tilde{u}_0) \rangle \, \xi_i'(\tilde{u}_0)[d] + \sum_{i=1}^{k} \left(t_i - \langle \tilde{u}_0 | \xi_i(\tilde{u}_0) \rangle \right) \xi_i'(\tilde{u}_0)[d]$$

$$+ \int_0^1 \sum_{i=1}^{k} t_i \left(\xi_i'(\tilde{u}_0 + sd)[d] - \xi_i'(\tilde{u}_0)[d] \right) \mathrm{d}s =: d_1 + d_2 + d_3.$$

Using Lemma 2.6 on d_1, one can write equation (2.8) as

$$w = e + \sum_{i=1}^{k} \left(t_i - \langle \tilde{u}_0 | \xi_i'(\tilde{u}_0)[d] \rangle \right) \xi_i(\tilde{u}_0) + d + d_2 + d_3.$$

Since

$$\left| \langle \tilde{u}_0 | \xi_i'(\tilde{u}_0)[d] \rangle \right| \leqslant \| \tilde{u}_0 \| \| \xi_i'(\tilde{u}_0) \| \| d \| \leqslant (\| u_0 \| + r) M r,$$

we can assume r was chosen small enough so that this is smaller that $\alpha/3$. Recalling that $t_i > 2\alpha/3$, one sees that the coefficients of $\xi_i(\tilde{u}_0)$ are positive and therefore $e + \sum \left(t_i - \langle \tilde{u}_0 | \xi_i'(\tilde{u}_0)[d] \rangle \right) \xi_i(\tilde{u}_0) \in C_{\tilde{u}_0}$.

The proof is complete if we show that $d + d_2 + d_3 \in B(d, \delta \| d \|)$. Using (2.7), we deduce $|t_i| \leqslant \left| t_i - \langle \tilde{u}_0 | \xi_i(\tilde{u}_0) \rangle \right| + \| \tilde{u}_0 \| \leqslant 2r + (\| u_0 \| + r)(\varepsilon + 1)$. Thus the following estimates

$$\| d_2 \| \leqslant \sum_{i=1}^{k} \left| t_i - \langle \tilde{u}_0 | \xi_i(\tilde{u}_0) \rangle \right| \| \xi_i'(\tilde{u}_0) \| \| d \| \leqslant k \left(2r + (\| u_0 \| + r)\varepsilon \right) M \| d \|,$$

$$\| d_3 \| \leqslant \sum_{i=1}^{k} |t_i| \sup_{s \in [0,1]} \| \xi_i'(\tilde{u}_0 + sd) - \xi_i'(\tilde{u}_0) \| \| d \| \leqslant k \left(2r + (\| u_0 \| + r)(\varepsilon + 1) \right) \varepsilon \| d \|,$$

show that $\| d_i \| \leqslant \frac{1}{2} \delta \| d \|$, $i = 2, 3$, provided that the constants ε and r were chosen small enough. $\qquad\square$

Lemma 2.8 is the second key element to prove the convergence up to a subsequence.

Lemma 2.8. *Let φ be a peak selection for $(C_u)_{u \in \mathcal{A}}$ which verifies conditions (1.1) and (1.3). Let $(u_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ be given by the MPA (Algorithm 1.1) with $s_n \in S(u_n)$ for all n. Let us assume that φ is continuous, $\overline{\mathrm{Ran}\,\varphi} \subseteq \mathcal{A}$, and either*

$$\exists \tau_1, \ldots, \tau_k \in (0, +\infty), \ \mathrm{dist}\Big(\Big\{\sum_{i=1}^{k} \tau_i \xi_i(u) : u \in \mathrm{Ran}\,\varphi\Big\}, \partial \mathcal{A}\Big) > 0, \qquad (2.9a)$$

$$or \ \begin{cases} \forall (v_n) \subseteq \mathrm{Ran}\,\varphi, \ \big(\mathcal{E}(v_n)\big) \text{ is bounded from above} \Rightarrow (v_n) \text{ is bounded} \\ and \ \dim E < \infty. \end{cases} \qquad (2.9b)$$

If $\sum_{n=0}^{+\infty} s_n < +\infty$ then $(u_n)_{n \in \mathbb{N}}$ converges in $\mathcal{A}$.

Proof. Let k be given by the assumption (1.3). For $i = 1, \ldots, k$, set $v_{i,n} := \xi_i(u_n)$, and $d_n := -\frac{\nabla \mathcal{E}(u_n)}{\|\nabla \mathcal{E}(u_n)\|}$. Let r be given by assumption (1.3) (v) and K_i be a bound for ξ_i'. Denote $K := \max_{i=1,\ldots,k} K_i$.

By assumption (1.3) and as $\varphi(u_n + s_n d_n) \in \mathrm{int}\, C_{u_n + s_n d_n}$, we have

$$v_{i,n+1} = \xi_i\big(\varphi(u_n + s_n d_n)\big) = \xi_i(u_n + s_n d_n)$$

for any $n \in \mathbb{N}$. Let n^* be large enough so that for all $n \geqslant n^*$, $s_n < r$. Thus, for all $n \geqslant n^*$,

$$\|v_{i,n+1} - v_{i,n}\| \leqslant K\|s_n d_n\| = K s_n. \qquad (2.10)$$

Since $\sum s_n < +\infty$, it follows that for any $i = 1, \ldots, k$, $(v_{i,n})_{n \in \mathbb{N}}$ is a Cauchy sequence and therefore converges to, say, $v_{i,\infty}$.

Let us assume (2.9a) holds. Consider $\tilde{v}_n := \sum_{i=1}^{k} \tau_i v_{i,n} = \sum_{i=1}^{k} \tau_i \xi_i(u_n)$. The sequence converges and its limit belongs to $\mathcal{A}$. Since $\varphi(\tilde{v}_n) = \varphi(u_n) = u_n$ and φ is continuous, $(u_n)_{n \in \mathbb{N}}$ converges. Its limit lies in $\overline{\mathrm{Ran}\,\varphi}$ and thus in $\mathcal{A}$.

If on the other hand (2.9b) holds, the fact that the sequence $(\mathcal{E}(u_n))$ is decreasing implies that (u_n) is bounded. Let (u_n') be a subsequence of (u_n). Since $u_n' \in C_{u_n'}$, one can write $u_n' = e_n' + \sum_{i=1}^{k} t_{i,n}' \xi_i(u_n')$ for some $e_n' \in E$ and $t_{i,n}' \in (0, +\infty)$. As (u_n') is bounded, so are (e_n') and $|t_{i,n}'| = |\langle u_n' | \xi_i(u_n')\rangle| \leqslant \|u_n'\|$. So, up to subsequences, $(e_n')_{n \in \mathbb{N}}$ and $(t_{i,n}')_{n \in \mathbb{N}}$ converge to, say, e_∞' and $t_{i,\infty}'$. Thus $u_n' \to u_\infty' := e_\infty' + \sum t_{i,\infty}' v_{i,\infty}$. Thanks to $\overline{\mathrm{Ran}\,\varphi} \subseteq \mathcal{A}$, $u_\infty' \in \mathcal{A}$. But then the continuity of ξ_i and φ imply

$$v_{i,\infty} = \xi_i(u_\infty') \qquad \text{and} \qquad u_n' = \varphi(u_n') \to \varphi(u_\infty'). \qquad (2.11)$$

If the same reasoning is performed with another subsequence (u_n''), (2.11) implies that $\xi_i(u_\infty') = \xi_i(u_\infty'')$ and therefore, in view of Definition 1.3 (1.3), $\varphi(u_\infty') = \varphi(u_\infty'')$. As the limit does not depend on the subsequence, the whole sequence (u_n) converges in $\mathcal{A}$. $\qquad \square$

Remark 2.9. If we wanted to seek sign-changing solutions using the cones $C_u := \mathbb{R}^+ u^+ \oplus \mathbb{R}^+ u^-$ (as explained in the Introduction), then we would not be able to write (2.10) in the above computation because the map $H_0^1 \to H_0^1 : u \mapsto u^+$ is not Lipschitz. This sheds some light on the difficulty of proving the convergence of the Modified Mountain Pass Algorithm [14].

Theorem 2.10. *Assume $\varphi : \mathcal{A} \to \mathcal{A}$ is a continuous peak selection s.t. $\overline{\operatorname{Ran}\varphi} \subseteq \mathcal{A}$ and the cones $(C_u)_{u \in \mathcal{A}}$ verify the conditions (1.1), (1.3) and (2.9a) or (2.9b). Suppose moreover that $\mathcal{E} \in \mathcal{C}^1(\mathcal{H}; \mathbb{R})$ satisfies the Palais-Smale condition in $\operatorname{Ran}\varphi$ and that $\inf_{u \in \operatorname{Ran}\varphi} \mathcal{E}(u) > -\infty$. Then the sequence $(u_n)_{n \in \mathbb{N}}$ given by the Mountain Pass Algorithm 1.1 possesses a subsequence converging to a critical point of $\mathcal{E}$ in $\operatorname{Ran}\varphi$. In addition, all limit points of $(u_n)_{n \in \mathbb{N}}$ are critical points of $\mathcal{E}$.*

Proof. Let us start by showing that $\big(\nabla \mathcal{E}(u_n)\big)_{n \in \mathbb{N}}$ converges to zero up to a subsequence. If not, we could assume there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that, for any $n \geqslant n_0$, $\|\nabla \mathcal{E}(u_n)\| > \delta$. Then, for any $n \geqslant n_0$, the Deformation Lemma 2.1 implies

$$\mathcal{E}(u_{n+1}) - \mathcal{E}(u_n) \leqslant -\alpha s_n \delta.$$

Thus, summing up,

$$\lim_{n \to +\infty} \mathcal{E}(u_n) - \mathcal{E}(u_{n_0}) = \sum_{n=n_0}^{+\infty} \mathcal{E}(u_{n+1}) - \mathcal{E}(u_n) \leqslant -\delta \alpha \sum_{n=n_0}^{+\infty} s_n.$$

As the left-hand side is a real number ($\mathcal{E}$ is bounded from below on $\operatorname{Ran}\varphi$ and decreasing along $(u_n)_{n \in \mathbb{N}}$), we have $\sum_{n=0}^{+\infty} s_n < +\infty$. So, by Lemma 2.8, $u_n \to u^* \in \mathcal{A}$ and $\|\nabla \mathcal{E}(u^*)\| \geqslant \delta$. By continuity of φ at $u^* \in \mathcal{A}$, we obtain $\varphi(u^*) = u^*$ and, so, $u^* \in \operatorname{Ran}\varphi$. By Lemma 2.4, there exists a neighborhood V of u^* and $s^* > 0$ such that $S(u) \subseteq [s^*, +\infty)$ for any $u \in V$. Consequently, there exists n_0 such that, for any $n \geqslant n_0$, $s_n \geqslant s^*$, whence $\sum_{n=0}^{+\infty} s_n$ does not converge, which is a contradiction.

In conclusion, there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ s.t. $\|\nabla \mathcal{E}(u_{n_k})\| \to 0$ when $k \to +\infty$. As $\mathcal{E}$ satisfies the Palais-Smale condition, $(u_{n_k})_{k \in \mathbb{N}}$ possesses a subsequence converging to a critical point of $\mathcal{E}$.

Concerning the second statement of the theorem, the argument is very similar. Let $(u_{n_k})_{k \in \mathbb{N}}$ be a convergent subsequence and assume on the contrary that $u := \lim_{k \to \infty} u_{n_k}$ is not a critical point of $\mathcal{E}$. In that case, on one hand, there exists $\delta > 0$ and $k_1 \in \mathbb{N}$ such that, for any $k \geqslant k_1$, $\|\nabla \mathcal{E}(u_{n_k})\| \geqslant \delta$. By Lemma 2.1, we have

$$\forall k \geqslant k_1, \quad \mathcal{E}(u_{n_{k+1}}) - \mathcal{E}(u_{n_k}) \leqslant -\alpha \delta s_{n_k}.$$

On the other hand, as $u \in \operatorname{Ran}\varphi$, we have by Lemma 2.4 that

$$\exists\, s^* > 0, \ \exists k_2 \in \mathbb{N}, \quad \forall k \geqslant k_2, \ s_n \geqslant s^*.$$

So, for large k, $\mathcal{E}(u_{n_{k+1}}) - \mathcal{E}(u_{n_k}) \leqslant -\frac{\alpha}{2}\delta s^*$, which is a contradiction because $\big(\mathcal{E}(u_n)\big)_{n \in \mathbb{N}}$ is a convergent sequence. $\square$

Remark 2.11. By previous Remarks 2.2 and 2.5, we conclude that we could get the convergence up to a subsequence using the equation (2.1) only at u_0. Thus, the uniform form of Lemma 2.1 is not required (and we could consider that $\varphi(u)$ is a local maximum of φ on C_u instead of a global maximum). Nevertheless, we have kept the uniform setting along the paper as it will be required in Section 2.3.

The following special case of (1.3) is important for the applications.

$$\left.\begin{array}{l}
\text{There exist a closed subspace } E \subseteq \mathcal{H} \text{ (possibly infinite dimensional)} \\
\text{and linear continuous projectors } P_i : \mathcal{H} \to E^\perp, \ i = 1, \ldots, k, \text{ for some} \\
k \in \mathbb{N}, \text{ such that} \\
\qquad \bullet \ \forall \ u \in \mathcal{H}, \ P_i(u) \perp P_j(u) \text{ whenever } i \neq j; \\
\qquad \bullet \ E \oplus \sum_{i=1}^k \operatorname{Ran} P_i = \mathcal{H}. \\
\text{For all } u \in \mathcal{H}, \text{ set } C_u := E \oplus \left\{ \sum t_i P_i(u) \mid t_i \geqslant 0 \text{ for all } i \right\}.
\end{array}\right\} \quad (2.12)$$

Let us now sketch the proof that (2.12) implies both (1.1) and (1.3). Consider $\mathcal{A} := \{u \in \mathcal{H} \mid P_i(u) \neq 0 \text{ for all } i\}$ and $\xi_i(u) := \frac{P_i(u)}{\|P_i(u)\|}$. Clearly $\xi_1, \ldots, \xi_k$ are $\mathcal{C}^1$ functions on $\mathcal{A}$. Moreover $e + \sum t_i P_i(u) \in \operatorname{int} C_u$ if and only if all $t_i > 0$. Since $u = P_E(u) + \sum_{i=1}^k P_i(u)$ where P_E denotes the orthogonal projection onto E, one has $u \in \operatorname{int} C_u$. Given the definitions of $\mathcal{A}$ and ξ_i, points (i), (iii) and (iv) of (1.3) are straightforward. A simple computation shows that $\xi_i'(u)\left[\sum t_j P_j(u)\right]$ is a multiple of $P_i(u)$ whence (ii) follows. Finally, as $\|\xi_i'(u)\| = O(1/\|P_i(u)\|)$, (v) will hold provided $\|P_i(u)\|$ is bounded away from 0 when $u \in \operatorname{Ran} \varphi$. Note that this latter condition also ensures that $\overline{\operatorname{Ran} \varphi} \subseteq \mathcal{A}$. We remark that these cones satisfy property (2.9a) with $\tau_1 = \cdots = \tau_k = 1$ because $\operatorname{dist}(\sum \xi_i(u), \partial \mathcal{A}) = \min_j \|P_j(\sum \xi_i(u))\| = 1$.

Thus, as a corollary of Theorem 2.10, we get the following proposition. It can be thought as an abstract version of the convergence results in [3, 10, 11].

Proposition 2.12. *Let us consider $\varphi : \mathcal{A} \to \mathcal{A}$ a continuous peak selection with the cones $(C_u)_{u \in \mathcal{A}}$ being given by condition (2.12) and $\mathcal{A} := \{u \in \mathcal{H} \mid P_i(u) \neq 0 \text{ for all } i = 1, \ldots, k\}$. Assume that $\inf_{u \in \operatorname{Ran} \varphi} \|P_i(u)\| > 0$ for all $i = 1, \ldots, k$, that $\mathcal{E} \in \mathcal{C}^1(\mathcal{H}; \mathbb{R})$ satisfies the Palais-Smale condition in $\operatorname{Ran} \varphi$ and that $\inf_{u \in \operatorname{Ran} \varphi} \mathcal{E}(u) > -\infty$. Then the sequence $(u_n)_{n \in \mathbb{N}}$ given by the Mountain Pass Algorithm 1.1 possesses a subsequence converging to a critical point of $\mathcal{E}$ in $\operatorname{Ran} \varphi$. In addition, all limit points of $(u_n)_{n \in \mathbb{N}}$ are critical points of $\mathcal{E}$.*

In Theorem 2.10, the Palais-Smale condition is required. For the particular case of $\mathcal{H} = H^1(\mathbb{R}^N)$, this condition does not generally hold, as $\mathbb{R}^N$ is not bounded, some mass of Palais-Smale sequences may be lost at infinity i.e.,

$$\lim_{R \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^N \backslash B(0,R)} u_n^2 > 0.$$

Fortunately, $H^1(\mathbb{R}^N)$ enjoys the following compactness condition (see for example the paper [12] for a proof): for any bounded sequence $(u_n)_{n \in \mathbb{N}} \subseteq H^1(\mathbb{R}^N)$ staying away from zero, there exists $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}^N$ such that the sequence $\big(u_n(\cdot + x_n)\big)$, where $u_n(\cdot + x_n)$ denotes the function $x \mapsto u_n(x + x_n)$, converges weakly, up to a subsequence, to a non-zero function. This is enough to get the convergence up to a subsequence.

Proposition 2.13. *Assume the hypotheses of Theorem 2.10 hold, except for the Palais-Smale condition. Let $\mathcal{H} := H^1(\mathbb{R}^N)$ and $(u_n)_{n \in \mathbb{N}}$ be the sequence given by the Mountain Pass Algorithm 1.1. If, for any $u \in \mathcal{H}$ and $x \in \mathbb{Z}^N$, $\mathcal{E}\big(u(\cdot + x)\big) = \mathcal{E}(u)$ and if $\nabla \mathcal{E} : \mathcal{H} \to \mathcal{H}$ is continuous for the weak topology on $\mathcal{H}$, then there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}^N$ such that $\big(u_n(\cdot + x_n)\big)_{n \in \mathbb{N}}$ converges weakly up to a subsequence to a nontrivial critical point of $\mathcal{E}$.*

Proof. We will only briefly sketch the proof. As $(u_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and stays away from 0, the compactness condition recalled above implies that there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z}^N$ such that $u(\cdot + x_n)$ converges weakly, up to a subsequence, to $u^* \neq 0$. Intuitively, the translations "bring back" some mass that u_n may loose at infinity.

Using the translation invariance of $\mathcal{E}$, the corresponding equivariance of $\nabla \mathcal{E}$ and the weak continuity of $\nabla \mathcal{E}$, we conclude that u^* is a critical point of $\mathcal{E}$.   □

2.3. Convergence of the whole sequence.
In this section, we refine the stepsize used previously to get the convergence of the whole sequence generated by Algorithm 1.1. We require that the stepsize $s_n \in \tilde{S}(u_0) := \tilde{S}^*(u_0) \cap \left(\frac{1}{2} \sup \tilde{S}^*(u_0), +\infty \right)$ where

$$\tilde{S}^*(u_0) := \Big\{ s_0 > 0 : \forall \, s \in (0, s_0], \; u_s := u_0 - s \frac{\nabla \mathcal{E}(u_0)}{\|\nabla \mathcal{E}(u_0)\|} \in \mathcal{A} \text{ and}$$

$$\mathcal{E}\big(\varphi(u_s)\big) - \mathcal{E}(u_0) < -\alpha s \|\nabla \mathcal{E}(u_0)\| \Big\}.$$

Using the Deformation Lemma 2.1, we get that $\tilde{S}(u_n) \neq \varnothing$ as long as u_n is not a critical point. Moreover, working as previously, we get results 2.4, 2.8 and 2.10 for this new choice of stepsizes. Let us remark that, this time, we really need that inequality (2.1) is valid in a neighborhood of u_0 to get Lemma 2.4. This new stepsize will allow us to control the energy for any $0 < s \leqslant s_0$. Under a "localization" assumption, we now prove that the whole sequence $(u_n)_{n \in \mathbb{N}}$ given by the Mountain Pass Algorithm 1.1 converges to a nontrivial critical point of $\mathcal{E}$.

Theorem 2.14. *Assume that u is the unique critical point of $\mathcal{E}$ in the ball $B(u, \delta)$ for some $\delta > 0$. Under the same assumptions as those of Theorem 2.10, if there exists $n^* \in \mathbb{N}$ such that $\mathcal{E}(u_{n^*}) < a := \inf_{v \in \partial B(u,\delta) \cap \mathrm{Ran}\,\varphi} \mathcal{E}(v)$ and $u_{n^*} \in B(u, \delta)$ then the sequence $(u_n)_{n \in \mathbb{N}}$ produced by Algorithm 1.1 with stepsizes $s_n \in \tilde{S}(u_n)$ converges to u.*

Proof. For any $m > n^*$, we claim that $u_m \in B(u, \delta)$. If not, as $u_{n^*} \in B(u, \delta)$, there exists $m \geqslant n^*$ such that $u_m \in B(u, \delta)$ and $u_{m+1} = \varphi\big(u_m - s_m \frac{\nabla \mathcal{E}(u_m)}{\|\nabla \mathcal{E}(u_m)\|}\big) \notin B(u, \delta)$, with $s_m \in \tilde{S}(u_m)$. By continuity, there exists $0 < s \leqslant s_m$ such that $\varphi\big(u_m - s \frac{\nabla \mathcal{E}(u_m)}{\|\nabla \mathcal{E}(u_m)\|}\big) \in \partial B(u, \delta) \cap \mathrm{Ran}\,\varphi$. This is a contradiction because, by the definition of s_m and as $\mathcal{E}$ is decreasing along $(u_n)_{n \in \mathbb{N}}$, we have $a \leqslant \mathcal{E}\big(\varphi(u_m - s \frac{\nabla \mathcal{E}(u_m)}{\|\nabla \mathcal{E}(u_m)\|})\big) \leqslant \mathcal{E}(u_m) \leqslant \mathcal{E}(u_{n^*}) < a$.

As u is the unique critical point in $B(u, \delta)$, by Theorem 2.10, u is the unique accumulation point of $(u_n)_{n \in \mathbb{N}}$. So, u_n converges to u.   □

3. Applications

3.1. Application to indefinite problems.
For problem (1.4), the energy functional $\mathcal{E}$ given by (1.5) is defined on $\mathcal{H} := H_0^1(\Omega)$. Let us denote the decomposition $\mathcal{H} = \mathcal{H}^{(-)} \oplus \mathcal{H}^{(+)}$ corresponding to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum. For any u, we let $u^{(-)} \in \mathcal{H}^{(-)}$ and $u^{(+)} \in \mathcal{H}^{(+)}$ be the unique elements such that $u = u^{(-)} + u^{(+)}$. Let us remark that the case $\mathcal{H}^{(-)} = \{0\}$ corresponds the traditional Mountain Pass Algorithm with a positive definite linear operator.

We choose the following peak selection. Let $\mathcal{A} := \mathcal{H} \setminus \mathcal{H}^-$ and, for any $u \in \mathcal{A}$, let C_u be the cone $C_u := \mathcal{H}^{(-)} \oplus \mathbb{R}^+ u = \mathcal{H}^{(-)} \oplus \mathbb{R}^+ u^{(+)}$. The peak selection φ for $(C_u)_{u \in \mathcal{A}}$ is the map

$$\varphi : \mathcal{A} \to \mathcal{A} : u \mapsto \varphi(u)$$

such that, for all $u \in \mathcal{A}$, $\varphi(u)$ maximizes $\mathcal{E}$ on C_u. To prove that φ is continuous, we refer to the original paper [21]. Is is easy to check that these cones verify (2.12). Indeed it suffices to consider $E = \mathcal{H}^{(-)}$, $k = 1$ and $P_1 : \mathcal{H} \to E^\perp$, the orthogonal projection onto $E^\perp$.

To apply Proposition 2.12, we need to verify the following assumptions on $\mathcal{E}$: on a bounded domain Ω,

- (i) 0 does not belong to $\overline{\mathrm{Ran}(P_1 \circ \varphi)}$: it comes from the fact that 0 is a strict local minimum of $\mathcal{E}$ on $E^\perp = \mathcal{H}^{(+)}$ (see [21]);
- (ii) $\mathcal{E} \in \mathcal{C}^1\big(H_0^1(\Omega); \mathbb{R}\big)$: it follows from standard arguments;
- (iii) $\mathcal{E}$ verifies the Palais-Smale condition on $\mathrm{Ran}\,\varphi$: see [21];
- (iv) $\inf_{u \in \mathrm{Ran}\,\varphi} \mathcal{E}(u) > -\infty$: actually $\mathcal{E}$ is bounded from below by 0 on $\mathrm{Ran}\,\varphi$, see [21].

In conclusion, Proposition 2.12 applies and gives the convergence up to a subsequence of the sequence (u_n) generated by Mountain Pass Algorithm 1.1 for this indefinite problem provided that the domain Ω is bounded.

Let us now sketch what happens about the convergence up to a subsequence when $\Omega = \mathbb{R}^N$. As $(\mathcal{E}(u_n))_{n \in \mathbb{N}}$ is decreasing (see Proposition 2.3) and is bounded away from zero, we have that $(u_n)_{n \in \mathbb{N}}$ is bounded and stays away from zero in $H^1(\mathbb{R}^N)$ (see [21]). On the other hand, V is assumed to be 1-periodic, thus $\mathcal{E}\big(u(\cdot + x)\big) = \mathcal{E}(u)$ for any $u \in \mathcal{H}$ and $x \in \mathbb{Z}^N$. It is not difficult to check that $\nabla \mathcal{E} : \mathcal{H} \to \mathcal{H}$ is continuous for the weak topology on $\mathcal{H}$. Thus, Theorem 2.13 asserts that, if (u_n) is the sequence generated by the MPA, there exists a sequence of translations $(x_n) \subseteq \mathbb{Z}^N$ such that $(u_n(\cdot + x_n))_{n \in \mathbb{N}}$ converges weakly, up to a subsequence, to a nontrivial critical point u^* of $\mathcal{E}$. Moreover, if $\mathcal{E}(u_n) \to \inf_{u \in \mathrm{Ran}\,\varphi} \mathcal{E}(u)$, then it can be proved that the above convergence is strong. The idea is that, if it does not converge strongly, some mass is lost at infinity. At the limit, this mass will take away a quantity of energy greater or equal to $\inf_{u \in \mathrm{Ran}\,\varphi} \mathcal{E}(u) > 0$, a contradiction.

Numerical experiments. Let us start by giving some details on the computation of various objects used in the MPA. Functions in $\mathcal{H}$ will be approximated using P^1-finite elements on a Delaunay triangulation of Ω generated by the software Triangle [20]. The matrix of the quadratic form $(u_1, u_2) \mapsto \int_\Omega \nabla u_1 \nabla u_2$ is readily evaluated on the finite elements basis. For $(u_1, u_2) \mapsto \int_\Omega V(x) u_1 u_2 \, \mathrm{d}x$ and the various integrals involving u to a power, a quadratic integration formula on each triangle is used. The gradient $g := \nabla \mathcal{E}(v)$ is computed in the usual way: the function $g \in \mathcal{H}$ is the solution of the linear system of equations $\forall \varphi \in \mathcal{H}$, $(g|\varphi)_{\mathcal{H}} = \mathcal{E}'(v)[\varphi]$. In practice, the peak selection φ must be evaluated with great accuracy to obtain satisfying results. For this, we use a limited-memory quasi-Newton code for bound-constrained optimization [13]. The program stops when the gradient of the energy functional at the approximation has a norm less than 10^{-4}.

As an illustration, we consider $\Omega = (0,1)^2$, $V \in \mathbb{R}$ constant and $p = 4$. Let us remark that $\mathcal{H}^{(-)}$ is then formed by eigenfunctions of $-\Delta + V$ with negative eigenvalues. In dimension 2, the eigenvalues λ_i of $-\Delta$ on the square $(0,1)^2$ with zero Dirichlet boundary conditions are given by $\pi^2(n^2 + m^2)$ with $n, m = 1, 2, \ldots$ The related eigenfunctions are given by $\sin(n\pi x) \sin(m\pi y)$. We get $\lambda_1 = 2\pi^2 \approx 19.76$,

$\lambda_2 = \lambda_3 = 5\pi^2 \approx 49.48$ (a double eigenvalue), $\lambda_4 = 8\pi^2 \approx 78.95$, $\lambda_5 = \lambda_6 = 10\pi^2 \approx 98.69$,...

Figure 1 depicts four non-zero solutions approximated by the Algorithm 1.1 for four different values of V. The algorithm was always started from $u_0(x,y) := xy(x-1)(y-1)$. The graphs on the left-hand side are given for the values $V = 0$ ($\dim \mathcal{H}^{(-)} = 0$) and $-\lambda_2 < V = -21 < -\lambda_1$ ($\dim \mathcal{H}^{(-)} = 1$). The graphs on the right-hand side are given for $-\lambda_4 < V = -50 < -\lambda_3$ ($\dim \mathcal{H}^{(-)} = 3$) and $-\lambda_5 < V = -80 < -\lambda_4$ ($\dim \mathcal{H}^{(-)} = 4$). In Table 1, we present some characteristics of the solutions.

$V = 0$ $V = -50$

$V = -21$ $V = -80$
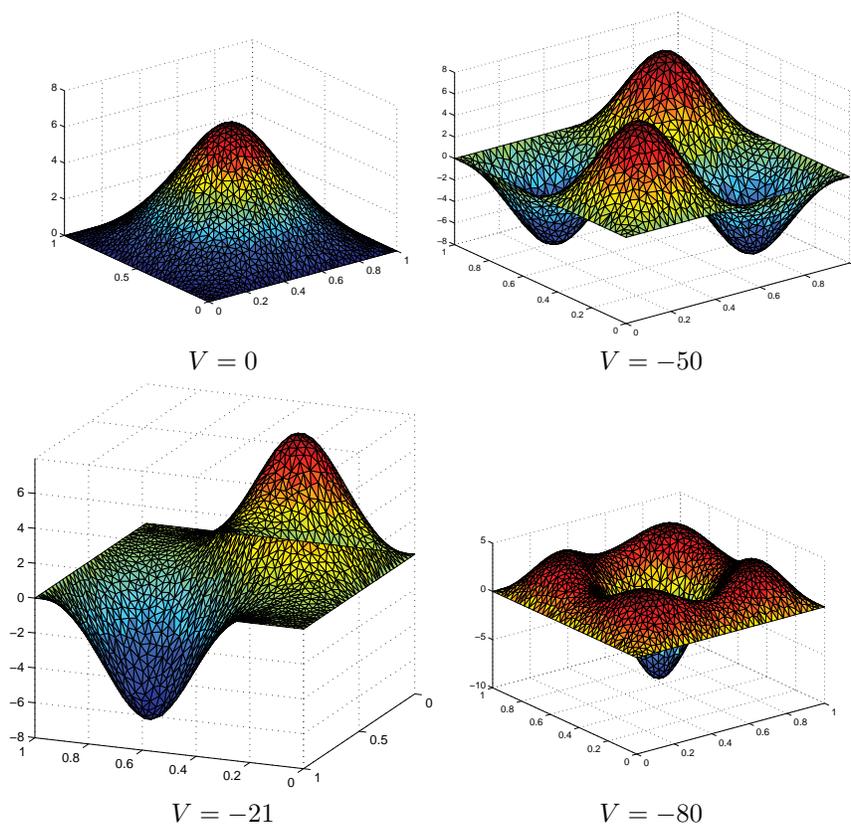
FIGURE 1. MPA solutions for an indefinite problem on a square

V	$\|\nabla \mathcal{E}\|$	# of steps	$\mathcal{E}(u)$
0	6.0×10^{-5}	7	37.89
-21	6.4×10^{-5}	48	70.43
-50	5.3×10^{-5}	113	91.42
-80	6.5×10^{-5}	44	35.06

TABLE 1. Characteristics of approximate solutions to an indefinite problem.

For $V = 0$, we remark that the approximation is even w.r.t. any symmetry of the square and is positive. It was expected and it is actually already known in this case (i.e. for the problem $-\Delta u = |u|^{p-2}u$) that ground state solutions have the same symmetries as the first eigenfunctions of $-\Delta$ (see [9, 1]).

For $V = -21$, the approximation has two nodal domains and a diagonal as nodal line. It seems to respect the symmetries of a second eigenfunction of $-\Delta$, which can be explained as follows. When $V = 0$, it is proved [1] that, for p close to 2, least energy nodal solutions have the same symmetries as their projections onto the second eigenspace of $-\Delta$. On the square, it is even conjectured that the projection must be a function odd w.r.t. a diagonal. In view of the bifurcation diagrams computed by J. M. Neuberger [15, 16], the least energy nodal solution for $V \in (-\lambda_1, 0]$ becomes the solution with lowest energy when $V \in (-\lambda_2, -\lambda_1]$ and no bifurcation happens along the way. So it is reasonable (and this is supported by the bifurcation diagrams) that they keep the same symmetries along the whole branch.

We also observe that, for $V = -50$ (resp. -80), the approximation seems to respect the symmetries of (and has the "same form" as) a fourth (resp. fifth) eigenfunction of $-\Delta$. Their number of bumps corresponds to their Morse index $(\dim \mathcal{H}^{(-)} + 1)$.

All those considerations support the conjecture that if $-\lambda_n < V < -\lambda_{n-1}$ then, at least for p small enough, ground state solutions respect the symmetries of a nth eigenfunction of $-\Delta$.

3.2. **Application to systems.** In this section we will perform numerical experiments for the system (1.6). The corresponding energy functional (1.7) is defined on $\mathcal{H} = H_0^1(\Omega, \mathbb{R}^k)$ endowed with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 = \sum_i \int_\Omega |\nabla u_i|^2 \, dx$. In [18], B. Noris and G. Verzini prove that the minimization of $\mathcal{E}$ on

$$\mathcal{N} := \Big\{ u \in \mathcal{A} : \forall i = 1, \ldots, k, \ \int_\Omega |\nabla u_i|^2 \, dx = \int_\Omega \partial_i F(u) u_i \, dx \Big\},$$

where $\mathcal{A} := \{u \in \mathcal{H} \mid u_i \neq 0 \text{ for every } i\}$, yields a solution $u = (u_1, \ldots, u_k) = \sum u_i e_i$ with $u_i \neq 0$ for all $i = 1, \ldots, k$ provided that the following assumptions are satisfied: there exist $p \in (2, 2^*)$, $C_F > 0$ and $\delta > 0$ such that, for any $u, \lambda \in \mathbb{R}^k$, one has

  (i) $\sum_{i,j} |\partial_{i,j}^2 F(u)| \leqslant C_F |u|^{p-2}$, $\sum_i |\partial_i F(u)| \leqslant C_F |u|^{p-1}$ and $|F(u)| \leqslant C_F |u|^p$;
  (ii) $\sum_{i,j} \partial_{i,j}^2 F(u)\lambda_i u_i \lambda_j u_j - (1+\delta) \sum_i \partial_i F(u)\lambda_i^2 u_i \geqslant 0$;
  (iii) for every i there exists $\bar{u}_i > 0$ such that $\partial_i F(\bar{u}_i e_i) > 0$;
  (iv) $\partial_i F(u)u_i \leqslant \partial_i F(u_i e_i)u_i$ for every i.

The first three assumptions are traditional in the framework of variational methods. The last one insures $u_i \neq 0$ for all i. In this section, we will use the Mountain Pass Algorithm 1.1 with the following peak selection. For any $u = (u_1, \ldots, u_k) \in \mathcal{A}$, we consider the cone $C_u := \{(t_1 u_1, \ldots, t_k u_k) \mid t_i \geqslant 0 \text{ for all } i = 1, \ldots, k\}$. The peak selection φ for $(C_u)_{u \in \mathcal{A}}$ is the map

$$\varphi : \mathcal{A} \to \mathcal{A} : u \mapsto \varphi(u)$$

such that $\varphi(u)$ maximizes $\mathcal{E}$ on C_u. Under the additional hypothesis that $\sum_i \partial_i F(u)u_i \geqslant 0$, the second assumption plays the role of the traditional super-quadraticity and implies that φ is well-defined as a peak selection. In fact, if $u \in \mathcal{A}$ verifies

$\mathcal{E}'(u)[(\lambda_1 u_1, \ldots, \lambda_k u_k)] = 0$ for any $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ then u is a strict local maximum of $\mathcal{E}$ on C_u. It implies the uniqueness of the global maximum of $\mathcal{E}$ on C_u. Moreover, φ is continuous.

To see that assumption (2.12) is satisfied, it suffices to take $E = \{0\}$ and, for $i = 1, \ldots, k$, $P_i(u) = P_i((u_1, \ldots, u_k)) := u_i e_i$, i.e., P_i is the projection onto the ith component of u. Finally, let us quickly run through the assumptions of Proposition 2.12:

(i) $\text{dist}(\text{Ran}\,\varphi, \partial\mathcal{A}) > 0$: see [18];
(ii) $\mathcal{E} \in \mathcal{C}^1(\mathcal{H}; \mathbb{R})$: it follows from standard arguments;
(iii) $\mathcal{E}$ verifies the Palais-Smale condition on $\text{Ran}\,\varphi$: see [18];
(iv) $\inf_{u \in \text{Ran}\,\varphi} \mathcal{E}(u) > -\infty$: actually $\mathcal{E}$ is bounded from below by 0 on $\text{Ran}\,\varphi$ (see [18]).

In conclusion, Proposition 2.12 applies and gives the convergence, up to a subsequence, of the sequence (u_n) generated by the Mountain Pass Algorithm 1.1.

Numerical experiments. For the numerical experiments, we will consider the following particular case of equation (1.6):

$$-\Delta u_i(x) = \mu_i u_i^3 + u_i \sum_{j \neq i} \beta_{i,j} u_j^2, \quad x \in \Omega,$$
$$u_i(x) = 0, \quad x \in \partial\Omega, \qquad \qquad i = 1, \ldots, k, \qquad (3.1)$$

where $\beta_{i,j} = \beta_{j,i}$ and Ω is a bounded domain of $\mathbb{R}^2$. This system is modeling a competition between k populations. We will focus on the case $\Omega = (0,1)^2$ and $k = 2$. In this setting, the assumptions (3.2)–(3.2) stated above boil down to

$$\mu_1 > 0, \quad \mu_2 > 0, \quad \text{and} \quad -\sqrt{\mu_1 \mu_2} \leqslant \beta_{1,2} \leqslant 0. \qquad (3.2)$$

Let us remark that the condition $\sum_i \partial_i F(u) u_i \geqslant 0$ discussed in the previous section is also verified in this range.

Let us now give the outcome of the algorithm for various choices of $(\mu_1, \mu_2, \beta_{1,2})$. The MPA will always start with the function $u_0 = (u_{0,1}, u_{0,2}) \in \mathcal{A}$ where $u_{0,1}(x,y) = u_{0,2}(x,y) = xy(1-x)(1-y)$ and stops when the norm of the gradient is less than 10^{-4}.

First we choose $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, -1)$. The numerical solution (u_1, u_2) is depicted on Figure 2 and some characteristics are given in Table 2. In this case, the assumptions (3.2) are satisfied so the algorithm converges to a solution (u_1, u_2) with $u_1 > 0$ and $u_2 > 0$ as expected. Notice also that the solutions u_1 and u_2 are even w.r.t. axes of symmetry of the square.

As a second choice, we consider $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, 0.5)$. The MPA solution (u_1, u_2) is depicted on Figure 3 and some characteristics are given in the second row of Table 2. Despite the fact that the assumptions (3.2) are not satisfied anymore, the solution is similar to that found in the first case. If we enlarge $\beta_{1,2}$ further and choose $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, 1.2)$, the algorithm still converges (see the third row of Table 2), but this time, the second component vanishes (see Figure 4). What happens is that, at the very first step, $u_2 = 0$ and then the MPA essentially proceeds as if the system was only consisting in the first equation.
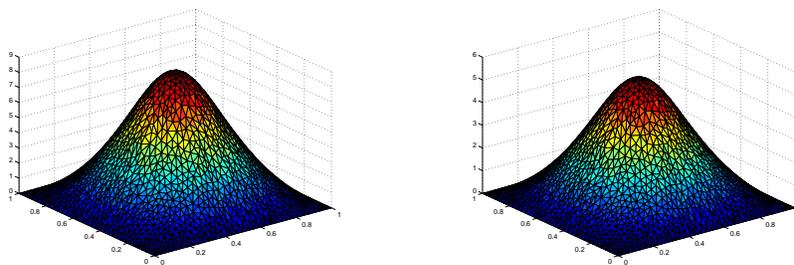
FIGURE 2. MPA solution for the system with $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, -1)$.
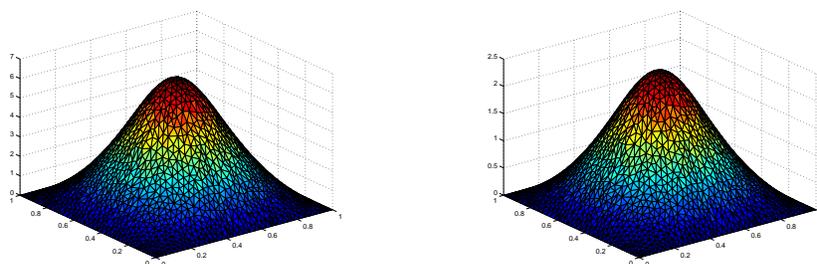
FIGURE 3. MPA solution for the system with $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, 0.5)$
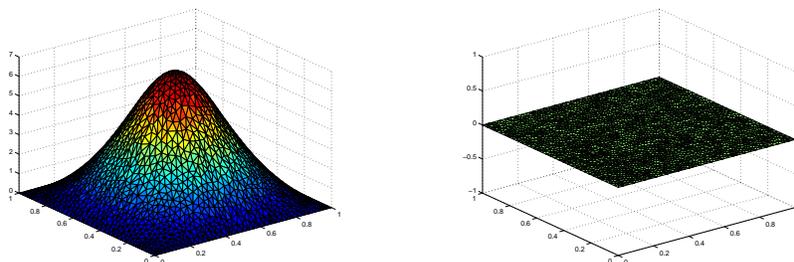
FIGURE 4. MPA solution for the system with $(\mu_1, \mu_2, \beta_{1,2}) = (1, 4, 1.2)$.

	$\|\nabla\mathcal{E}(u)\|$	# steps	$\mathcal{E}(u)$	$\max u_1$	$\max u_2$
$(1, 4, -1)$	7.9×10^{-5}	11	88.4	8.6	5.4
$(1, 4, 0.5)$	5.4×10^{-5}	11	40.4	6.4	2.4
$(1, 4, 1.2)$	5.2×10^{-5}	11	39.9	6.6	0.0

TABLE 2. Characteristics of the solution to system (3.1).

References

[1] Denis Bonheure, Vincent Bouchez, Christopher Grumiau, and Jean Van Schaftingen. Asymptotics and symmetries of least energy nodal solutions of Lane-Emden problems with slow growth. *Commun. Contemp. Math.*, 10(4):609–631, 2008.

[2] Xianjin Chen and Jianxin Zhou. A local min-max-orthogonal method for finding multiple solutions to noncooperative elliptic systems. *Math. Comp.*, 79(272):2213–2236, 2010.

[3] Xianjin Chen, Jianxin Zhou, and Xudong Yao. A numerical method for finding multiple co-existing solutions to nonlinear cooperative systems. *Appl. Numer. Math.*, 58(11):1614–1627, 2008.

[4] Yung Sze Choi and P. Joseph McKenna. A mountain pass method for the numerical solution of semilinear elliptic problems. *Nonlinear Anal.*, 20(4):417–437, 1993.

[5] M. Conti, S. Terracini, and G. Verzini. Nehari's problem and competing species systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 19(6):871–888, 2002.

[6] M. Conti, S. Terracini, and G. Verzini. An optimal partition problem related to nonlinear eigenvalues. *J. Funct. Anal.*, 198(1):160–196, 2003.

[7] David G. Costa, Zhonghai Ding, and John M. Neuberger. A numerical investigation of sign-changing solutions to superlinear elliptic equations on symmetric domains. *J. Comput. Appl. Math.*, 131(1-2):299–319, 2001.

[8] E. N. Dancer, Juncheng Wei, and Tobias Weth. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):953–969, 2010.

[9] Basilis Gidas, Wei Ming Ni, and Louis Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[10] Youngxin Li and Jianxin Zhou. A minimax method for finding multiple critical points and its applications to semilinear elliptic PDE's. *SIAM Sci.Comp.*, 23:840–865, 2001.

[11] Youngxin Li and Jianxin Zhou. Convergence results of a local minimax method for finding multiple critical points. *SIAM Sci. Comp.*, 24:865–885, 2002.

[12] Elliott H. Lieb. On the lowest eigenvalue of the Laplacian for the intersection of two domains. *Invent. Math.*, 74(3):441–448, 1983.

[13] J.L. Morales and J. Nocedal. Remark on algorithm 778: L-bfgs-b, fortran subroutines for large-scale bound constrained optimization. *ACM Transactions on Mathematical Software (TOMS)*, 38(1), November 2011.

[14] John M. Neuberger. A numerical method for finding sign-changing solutions of superlinear dirichlet problems. *Nonlinear World*, 4(1):73–83, 1997.

[15] John M. Neuberger. GNGA: recent progress and open problems for semilinear elliptic pde. *Contemp. Math.*, 357:201–237, 2004. Amer. Math. Soc., Providence, RI.

[16] John M. Neuberger and James W. Swift. Newton's method and morse index for semilinear elliptic pdes. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 11(3):801–820, 2001.

[17] Jorge Nocedal and Stephen J. Wright. *Numerical Optimization*. Springer Series in Operations Research and Financial Engineering. Springer, 2nd edition, 2006.

[18] Benedetta Noris and Gianmaria Verzini. A remark on natural constraints in variational methods and an application to superlinear schrödinger systems. *J. Differential Equations*, 254(2013) No. 3, 1529–1547.

[19] Paul H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[20] Jonathan Richard Shewchuk. Delaunay refinement algorithms for triangular mesh generation. *Comput. Geom.*, 22(1-3):21–74, 2002. 16th ACM Symposium on Computational Geometry (Hong Kong, 2000).

[21] Andrzej Szulkin and Tobias Weth. Ground state solutions for some indefinite variational problems. *J. Funct. Anal.*, 257(12):3802–3822, 2009.

[22] N. Tacheny and C. Troestler. A mountain pass algorithm with projector. *J. Comput. Appl. Math.*, 236(7):2025–2036, 2012.

[23] Hugo Tavares, Susanna Terracini, Gianmaria Verzini, and Tobias Weth. Existence and nonexistence of entire solutions for non-cooperative cubic elliptic systems. *Comm. Partial Differential Equations*, 36(11):1988–2010, 2011.

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