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REMARKS AND EXAMPLES ON TRANSIENT PROCESSES AND ATTRACTORS IN BIOLOGICAL EVOLUTION

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Dedicated to Alfonso Casal on his 70th birthday

ABSTRACT. We present a model for the competition of two biological entities into the same species (polyphasic, clonal/sex, cancerous cells), the first one with a birth ratio higher than the second when the resources are abundant, whereas the situation is reversed for scarce resources. The first one rapidly exhausts the resources, improving growth of the second, leading to a auto-sustained cyclic process (ESS = Evolutionary Stable Strategy). We use known models of population dynamics for three agents: two phases asexual and sexual (for instance) of the same species and one of resources. The main feature of the model (for certain values of the parameters) is the very long and entangled transient process, which involves a long period where one of the forms is practically absent, before emerging again to join a stable cycle which implies preservation of both forms. This model should throw some light on the biological problem of the maintenance of sexuality in competition with asexual clones, as well as on the alternated fast growth versus latency in cancer tumors.

1. INTRODUCTION

Biological evolution can be analyzed as the search for new energy resources and new resources in information. Search of energy (food) resources uses pre-existing information, and search of information (variation/selection) resources uses the available energy: a stable trade-off is established between the growth and proliferation of individuals on one side, and the production of genetic diversity on the other. Each species represents its own solution to that double problem of energy-information. For this reason, biodiversity has as a fundamental category the species which can be defined as “an adaptive community”, a concept that goes back to Darwin, and as “the field of recombination” (cf. Carson [2]).

(1) Diversity in the species.

(1-1) Epigenetic or phenotypic diversity: cell diversity (tissues and cancers), direct effects of the environment (living conditions), development (youth, larvae, adult male and female), polyphasy (castes of insects, parthenogenetic forms, locusts).

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(1-2) Genetic diversity: mutation, polymorphism, races, subspecies.

(2) Diversity among species.

(2-1) Spacial or ecological diversity: multiple species of a system at a given time.

(2-2) Evolutionary or time diversity: evolutionary sequence of species in a lineage.

Our model explores the first level of interspecific diversity, i. e. level 1-1). We propose to describe the behavior of a species in which coexist two phases: a one-parent phase (= asexual = parthenogenetic) and a two-parent (= sexual). The interest of this example, albeit very in minority in nature, is to allow a simple modeling: being of the same species, these two phases have a very comparable food skills, and they differ in two points: (i) asexual are more uniform so they operate only a fraction of the matter-energy resources, while sexuals have, through genetic combinations a greater evolutionary diversity that allows to better exploit the various resources; (ii) the asexual phase has a very simple reproductive superiority because it reproduces by clonal proliferation twice faster than the sexual, that work by couples, what was later called the “factor two reproductive burden”. Indeed, according to a very simplified darwinian reasoning, prolificacy is an immediate selective advantage that mechanically causes the disappearance of the less prolific lines. Sexual breeding should therefore disappear very quickly when the asexuals invade a population. However, as a matter of fact, 999/1000 species are sexual. The key question is therefore of importance, since it involves the maintenance of sex in eukaryotes.

The classic hypothesis is that the adaptive information provided by sexuality, must be quite advantageous to balance the factor two burden. The literature is very abundant and easy to summarize [3, 7, 9]: almost all classic, always stochastic, models predict random extinction of asexual lineages, less able to survive changes in living conditions, which is a “lazy” solution, impossible to prove, involving a mystery and also evil-accepted.

Our mathematical model is a deterministic one, giving a cyclic regime inspired by the predator-prey alternation of Lotka - Volterra. In our model the cycle is an alternation of single proliferation in the presence of abundant matter-energy resources whereas shortage of food improves proliferation of males providing genetic diversity and better utilization of resources. This situation is well known as soon as resources are exhausted by seasonal effect (aphids of summer / of winter). But in our model the asexual themselves drain resources due to their proliferation, which then promotes sexuals. Therefore, the cycle of alternating one parent / two parents is self-sustained by its internal structure and not only subject to the vagaries of external resources.

To summarize, the main difference with the traditional model is the existence of two predators of the same species, an asexual that reproduces faster, the other sexual which feeds better. Computations are transparent, everyone may reproduce them using the Mathematica software; it is noting but an exercise in dynamical systems.

The plan of the paper is as follows. In Section 2 we present a little predator/resources material for ulterior utilization. Section 3 contains the description of a numerical computation taken as main example. Close variants are presented in Section 4, whereas Section 5 situates the model in a larger context. Final comments and conclusions are in Section 6.

Definition of variables and parameters: Precise definitions are given later in the text, but it is worthwhile keeping in mind that we mainly handle the resources (= preys) u and two phases of the same species (=predators) x and y . The demographical behavior of the two phases involves three parameters b, δ, α for x and correspondingly c, η, β for y . In fact, the products $\delta.b$ and $\eta.c$ are most significant than δ and η themselves. Specifically:

b (resp c) is the maximal quantity of food u consumed per capita and unit of time by the x (resp y). It describes the behavior of the x when the resources are abundant.

$\delta.b$ (resp $\eta.c$) describes the quantity of food u consumed by the x (resp y) when the resources u are scarce; this quantity, per capita and unit of time is $\delta.b.u$ (resp $\eta.c.u$).

α (resp β) is the natural death ratio (or decreasing of the population x (resp y) in the absence of supply of food).

2. THE TWO BASIC (STABLE EQUILIBRIUM AND CYCLE) PREDATOR - RESOURCES PATTERNS

In this section, we are constructing models of evolution of a species along with the resource, which constitute elaborate modifications of the classical Lotka - Volterra model.

The classical Lotka - Volterra model for the evolution of the populations of two species u (preys, or more generically resources) and y (predators, or a species living of the resources)) is

$$\begin{aligned} \dot{u} &= u - Cxu \\ \dot{x} &= -\alpha x + Cxu. \end{aligned} \tag{2.1}$$

Classically, this system has the equilibrium point ($u = \alpha/C, x = 1/C$) (in addition to the origin) and the orbits of the solutions in the plane (u, x) are closed curves around the equilibrium, accounting for oscillations periodic in time.

There are very many modifications of this basic model to render it more realistic and a very abundant bibliography (see for instance [4, 5, 6, 10]). We shall make two modifications:

First, in absence of predation, the equation for the preys (= resource) obviously gives an unrealistic exponential grow of u with time t ; the simplest way to prescribe a bound to this growing (a stable equilibrium) is to put a "logistic term" in the equation for \dot{u} . Specifically, we replaced $u(1 - 0.25u)$ instead of u in the right hand side, which gives the equilibrium $u = 4$ for the resource. This is consistent with our ulterior utilization, as all interesting phenomena will take place for $u < 4$, i. e. under the equilibrium of resources. It is easily seen that this modification in the Lotka - Volterra model is a disadvantage for the preys, which transforms the orbits from closed curves into spirals contracting towards the equilibrium point (which also changes with respect to that of the system (2.1)).

Second, we modify the expression of the quantity of resource consumed per unit of time. In the Lotka - Volterra model (2.1) it is Cxu , which amounts to Cu per capita of the predator, consistent with scarce resources (small u); but it is clear that, when the resources are abundant (large u) this quantity should have an upper limit. This amounts to some sort of "saturation" of the predation. In order to modelling this, we take advantage of the fact the the function $\tanh(u)$ (hyperbolic tangent of u) behaves nearly as u for small u whereas it tends to the limit 1 for

$u \rightarrow \infty$. More precisely, we put $b \tanh(\delta u)$ as the quantity of resource consumed by unit of time and of predators (i. e. per capita), involving the two parameters b and δ ; b is the upper limit of food eaten per capita per unit of time, whereas for small u the expression behaves as $\delta u b$, i. e. it corresponds to the Lotka - Volterra model (2.1) with $C = \delta b$. Obviously, this second modification is an advantage for the preys (which are not eaten beyond the limit). This gives the model

$$\begin{aligned}\dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) \\ \dot{x} &= -\alpha x + bx \tanh(\delta u)\end{aligned}\tag{2.2}$$

Obviously, the model is only viable provided that $b > \alpha$, otherwise the population x is always decreasing. Moreover, the second equation may be written

$$\dot{x}/x = -\alpha + b \tanh(\delta u)\tag{2.3}$$

so that the right hand side is the ratio of growth of the x as a function of u . It behaves as $-\alpha + \delta b u$ for small u and as $-\alpha + b$ for large u . Obviously, the negative term $-\alpha$ is the natural death ratio, whereas the positive one $b \tanh(\delta u)$ is the natural birth ratio, which depends on the food supply u ; for large supply u its value is nearby b , and for small supply it behaves as $\delta b u$.

We then see that both modifications have opposite effects. The first (resp. the second) one is a disadvantage (resp. advantage) for the preys, which leads (when acting alone) to a modification of the Lotka - Volterra cycles, which become spirals contracting (resp. dilating) towards the central equilibrium, which becomes an attractor (resp. to infinity, the equilibrium becoming unstable, i. e. a repulsor). But these advantage and disadvantage are represented in the system by two different nonlinear terms, so that one or the other prevail in different regions, depending on the populations.

As a matter of fact, for certain values of the parameters, the spirals are dilating nearby the equilibrium and contracting far of it, so producing a limit stable cycle (= periodic solution). For instance, that is the case for the system

$$\begin{aligned}\dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) \\ \dot{x} &= -\alpha x + bx \tanh(\delta u) \\ \alpha &= 0.55, \quad b = 1.0, \quad \delta b = 1.0\end{aligned}\tag{2.4}$$

Figure 1 is a plot of the orbits starting from $(u = 1, x = 0.1)$ (out of the cycle) and $(u = 1, x = 1)$ (interior to the cycle). They both converge to the cycle. Moreover, there is an unstable equilibrium inside the cycle, namely $(u = 0.6183, x = 0.9505)$.

For other values of the parameters, there is a stable equilibrium point. As an example, we consider (note the change of notations, for further utilization)

$$\begin{aligned}\dot{u} &= u(1 - 0.25u) - cy \tanh(\eta u) \\ \dot{y} &= -\beta y + cy \tanh(\eta u) \\ \beta &= 0.7, \quad c = 2.0, \quad \eta c = 1\end{aligned}\tag{2.5}$$

The equilibrium point, obviously stable, is $(u = 0.7308, x = 0.8533)$. Figure 2 is a plot of the orbit of the solution of system (2.5) for the initial values $(u = 2, y = 1)$

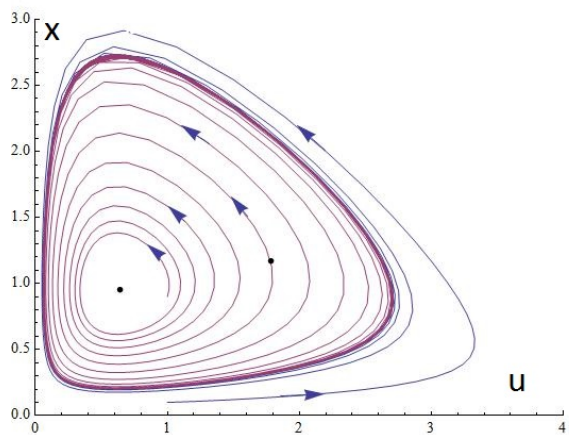


FIGURE 1. Two solutions of 2.4 converging to the stable cycle

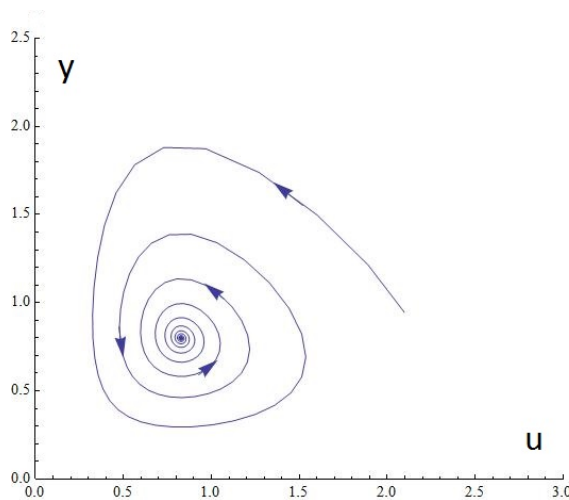


FIGURE 2. A solution of (2.5) converging to the stable equilibrium

3. TWO PREDATORS. FIRST EXAMPLE OF THE EQUILIBRIUM - CYCLE PATTERN AND ITS TRANSIENT PROCESS

We now consider the two predators of the previous section in competition with the same resources. We note that the values of the birth ratios for abundant resources of x and y i.e. $b = 1$ and $c = 2$ respectively, have been chosen to mimic the “burden of order two” of certain of the examples evoked in the introduction: for abundant food, the birth ratio of the y is twice that of the x . To have the possibility of a coexistence, the x should have a demographic advantage; we here took a lower natural death ratio $\alpha = 0.55$ (and $\beta = 0.7$ for the y)(it should be incidentally noted that the ratio is less than two). Other choices of this advantage will be consider in next section, leading to practically the same results. We take

for the parameters precisely the same numerical values of the previous examples, so that the dynamical systems in the planes $x = 0$ and $y = 0$ are those of Section 2. This point is important: in the absence of one of the predators, the system is perfectly viable, becoming one of the previous ones; moreover, if the initial value of x (resp. y) vanishes, x (resp. y) is always null and the problem degenerates in the previous one.

$$\begin{aligned} \dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) - cy \tanh(\eta u) \\ \dot{x} &= -\alpha x + bx \tanh(\delta u) \\ \dot{y} &= -\beta y + cy \tanh(\eta u) \end{aligned} \quad (3.1)$$

$$\begin{aligned} \alpha &= 0.55, & \beta &= 0.7, & b &= 1, \\ c &= 2, & \delta b &= 1.0, & \eta c &= 1.0 \end{aligned}$$

Figures 3 and 4 are two views (in order to exhibit the three-dimensional structure of the curve) of a three-dimensional plot of the orbit corresponding to the initial values $(u(0) = 0.5, x(0) = 0.003, y(0) = 1.5)$. Note the small initial value of x , so the starting point is very near the plane $x = 0$.

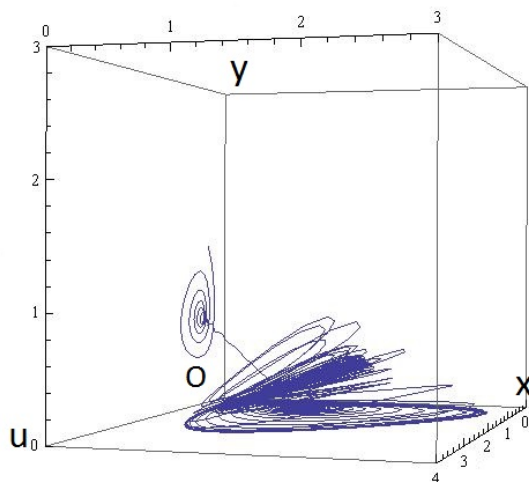


FIGURE 3. The orbit in the space (u, x, y) of a solution of (3.1) starting near the plane $x = 0$. It spirals towards the equilibrium in that plane, then goes towards the unstable equilibrium in the plane $y = 0$, it dilates around it, it converges towards the cycle in that plane and finally takes off to converge to the stable three-dimensional cycle. See Figure 4 for another view of the same curve

The corresponding functions $x(t)$ and $y(t)$ are represented in Figure 5 (which does not exhibit the resources u for the sake of clarity).

The general structure of the curve is easily understood. Clearly, the final state is a stable cycle with preservation of the two predators (i. e. out of the coordinate planes). When starting nearby the plane $x = 0$, the solution is close to that of this plane, so that it spirals contracting towards the equilibrium in the plane. This equilibrium point is obviously stable in the plane $x = 0$, but it is transversally

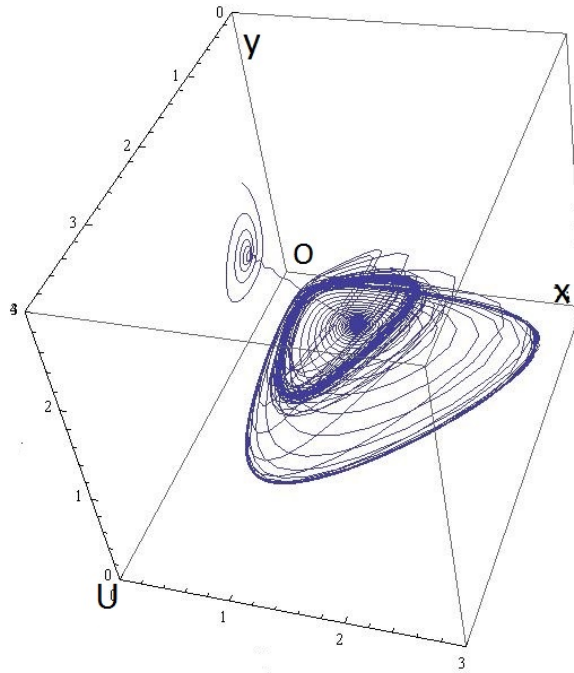


FIGURE 4. The functions $x(t)$ and $y(t)$ for the same solution of system (4.2). The population y practically vanishes for t between 80 and 1000

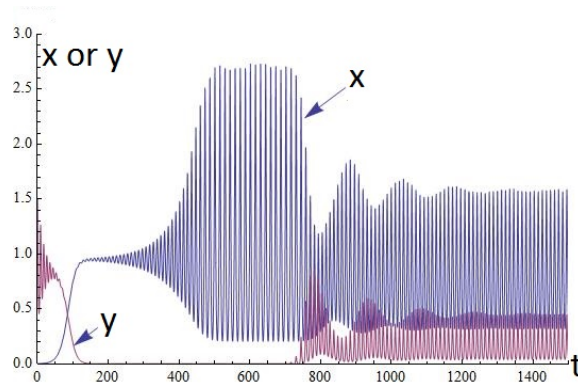


FIGURE 5. The functions $x(t)$ and $y(t)$ for the same solution of system 3.1. The population $y(t)$ practically vanishes for t between 150 and 700.

unstable (this property is numerically evident, but it may be easily checked by computing by (3.1) the quotient \dot{x}/x with the corresponding values of the variables), so that, when arriving near the equilibrium, the point takes off with increasing values of x , passes through the hole of the limit cycle and converges to the equilibrium point in the plane $y = 0$. Obviously, this equilibrium is unstable in its plane but

transversally stable (this may be checked as previously) so that the solution spirals dilating towards the cycle in the plane $y = 0$, which is stable in its plane but transversally unstable (this instability is numerically apparent, but it cannot be explicitly proved as it involves integrals on the trajectory, which is only numerically known). According to this instability, the solution takes off and approaches the final limit cycle. The final structure of the periodic attractor is shown in Figure 6. The periodic motions of the two forms x and y are practically in phase (or with a very small difference of phase) whereas they both exhibit a phase delay of nearly a quarter of a period with respect to the resource u . This seems natural, as the very dynamic process involves each predator with the prey (which implies the classical cycles of Lotka - Volterra or the like, involving inertia effects responsible for the phase delay), whereas the relation linking the predators is merely a quantitative adjustment between them.

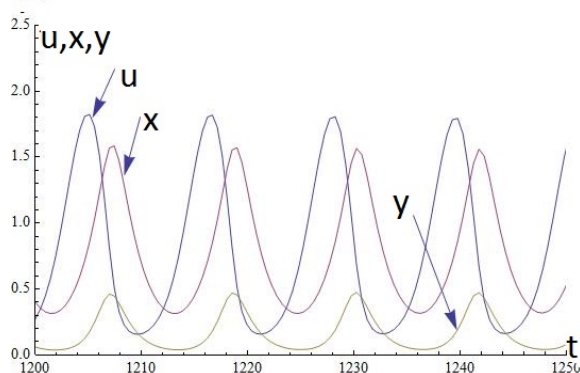


FIGURE 6. The three functions $u(t)$ (the upper one), $x(t)$ and $y(t)$ for the same solution of system (3.1) in the final periodic state.

The timing of the various phases of the motion is easily seen in the figures. The initial convergence nearby the plane $x = 0$ corresponds to t between 0 and 60. The drastic transition from the vicinity of $x = 0$ to that of $y = 0$ is for t between 60 and 150. The dilatation nearby the plane $y = 0$ then goes until $t = 500$ and is practically accomplished for t between 500 and 700, when y begins to take off to join the final attractor. The transient process is practically finish at $t = 1200$. The computation was stopped at $t = 1400$, which amounts to the practical capabilities of the Mathematica software.

Clearly, the final pattern (the attractor in mathematical terminology) is a periodic cycle involving non-vanishing u, x, y , or an ESS (= Evolutionary stable Strategy in biology, see [7]).

The most striking feature of this solution is that it leads to a final periodic state implying preservation of the diversity, whereas y practically vanishes between $t = 150$ and $t = 700$. Moreover, it seems to match the (tree-dimensionally unstable!) cycle in the plane $y = 0$ for t between 150 and 700. In other words, if the computation was stopped at t between 500 and 700, it should clearly exhibit a convergence to a cycle involving only u and x , with vanishing y . A crucial point in this context is the explicit description of the small values of y for t between 150

and 700. This is shown in Figure 7. In fact the values are extremely small, but obviously not zero and the curve is well defined and oscillating.

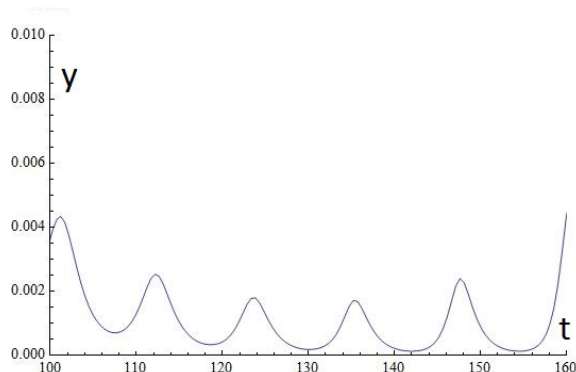


FIGURE 7. The function $y(t)$ in the region of its smallest values. Compare with Figure 5

The previous feature deserves a comment. The order of magnitude of y in that region is practically one thousandth of that of x . One may wonder if this is significant in the framework of the necessarily very coarse mathematical modeling of biological phenomena. In other words, one may wonder if in practice, such a result shows that the final issue may be either the preservation of the diversity or or the disappearance of the y .

4. COMPLEMENTS AND VARIANTS

The properties of the solutions of system (3.1) with the specific values of the parameters and of the initial values in the previous section are common to a large class of analogous systems.

An interesting property of system (3.1) is that it has no equilibrium positions with the three unknowns (u,x,y) different from 0 (unless for very special values of the parameters). In other words, if the attractor implies preservation of the various forms, this is not in an equilibrium, this necessarily implies a non-rest motion, (periodic oscillations, for instance, but not necessarily). This is easily seen by inspection of the system for the equilibria (u,x,y) obtained equating to 0 the equations in (3.1):

$$\begin{aligned}
 0 &= u(1 - 0.25u) - bx \tanh(\delta u) - cy \tanh(\eta u) \\
 0 &= -\alpha x + bx \tanh(\delta u) \\
 0 &= -\beta y + cy \tanh(\eta u) \\
 \alpha &= 0.55, \quad \beta = 0.7, \quad b = 1, \\
 c &= 2, \quad \delta b = 1.0, \quad \eta c = 1.0
 \end{aligned}
 \tag{4.1}$$

If x and y are different from 0, we may divide the second equation by x and the third by y and we obtain two equations for the only unknown u , so that generically there is no solution.

Another interesting feature of the pattern in the previous section is its structural stability. This amounts to saying that it is preserved under small perturbation of

the functions in the right hand side of the system. This obviously involves small changes of the parameters, but also modification by very small functions (provided they preserve the property that the planes $x = 0$ and $y = 0$ are invariant, i. e. \dot{x} (resp. \dot{y}) vanishes for $x = 0$ (resp. $y = 0$). Indeed, the peculiar structure of the solutions nearby the plane $x = 0$ is due to the presence of an equilibrium which is stable in that plane and transversally unstable, i. e. two of the eigenvalues of the linearized matrix (those associated with the invariant plane $x = 0$) have negative real parts, whereas the third one is positive. All these properties are preserved by small perturbation. The same applies to the equilibrium in $y = 0$. As for the cycle in the plane $y = 0$, the properties of being stable in that plane and transversally unstable are also structurally stable for reasons analogous to the previous ones, applied to the dynamics induced on a Poincaré section. The same is true for the attractor (stable cycle). All these properties are classical in the theory of hyperbolic dynamical systems.

As a variant, we now consider the system

$$\begin{aligned}\dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) - cy \tanh(\eta u) \\ \dot{x} &= -\alpha x + bx \tanh(\delta u) \\ \dot{y} &= -\beta y + cy \tanh(\eta u)\end{aligned}\tag{4.2}$$

$$\begin{aligned}\alpha &= 0.45, & \beta &= 0.45, & b &= 1, \\ c &= 2, & \delta b &= 1.1, & \eta c &= 0.7\end{aligned}$$

which is the same of section 3 with different numerical values. We keep the “burden of order two” $b = 1$, $c = 2$, but the necessary demographic advantage of x with respect to y is now concerned with more efficient utilization of the scarce resources (instead of lower death ratio). Specifically, the natural birth ratio of the x is $bx \tanh(\delta u)$, which behaves as $\delta b u$ for small u . We took $\delta b = 1.1$, larger than the analogous for the y , $\eta c = 0.7$ (it should be incidentally noted that the ratio is less than two).

The solution corresponding to the initial values ($u = 0.5$, $x = 0.003$, $y = 1.5$) is represented in Figs 8 and 9: it exhibits a very small population of y for t between 80 and 950.

Obviously, most qualitative features of the model are exactly the same as for the previous on (3.1). After a very entangled transient process, the final pattern is a periodic cycle involving non-vanishing u, x, y , or an ESS (= Evolutionary Stable Strategy in biology, see [7]).

Obviously, the main feature of these models is the very long and entangled transient process in the example of the previous section. It is clear that it takes place when the initial position is nearby the plane $x = 0$ (i. e. the initial population $x(0)$ is small). But the above description of the general pattern allows to foresee without new computations what happens when starting nearby $y = 0$: merely the initial portion of the transient disappears, starting nearby $t = 100$ (if the initial position is near the unstable equilibrium) or $t = 500$ (if it is near the cycle). Moreover, the sensitivity of the orbit to the small value of $x(0)$ is apparent by comparing the previous solution with that corresponding to the initial values ($u(0) = 0.5$, $x(0) = 0.3$, $y(0) = 1.5$) (where $x(0)$ is 100 times larger than in the previous example), which is plotted in Figures 10 and 11.

In a very general and coarse sense, it may be said that a passage near to an (unstable) equilibrium induces a delay proportional to $\log(1/\varepsilon)$ where ε denotes

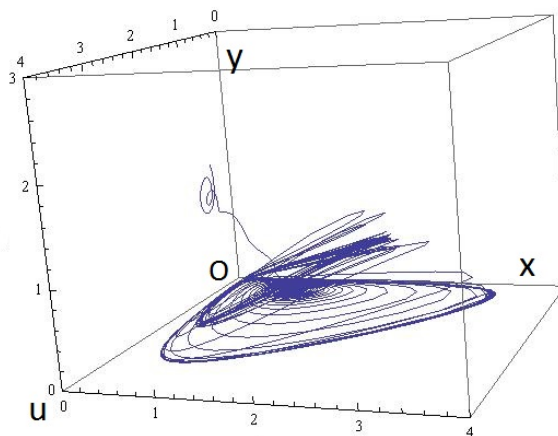


FIGURE 8. The orbit of a solution of system (4.2) starting very near the plane $x = 0$.

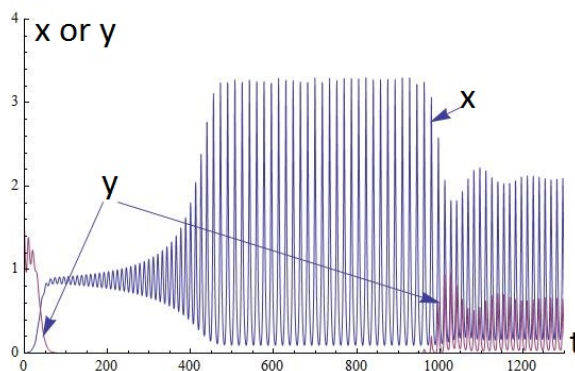


FIGURE 9. The functions $x(t)$ and $y(t)$ for the same solution of Figure 8 of system (4.2). The population y practically vanishes for t between 80 and 1000.

the distance between the orbit and the equilibrium. This easily follows from the fact that, near the equilibrium, in the linearized approximation, the speed at any point is proportional to the distance to the equilibrium. The same applies to the passage nearby an (unstable) cycle, for analogous reasons, applied to the dynamics of the first return to a Poincaré section. In the case of Figures 10 and 11, the first passage was avoided, and the two others (nearby the equilibrium and the cycle in the plane $y = 0$) were approached a little less.

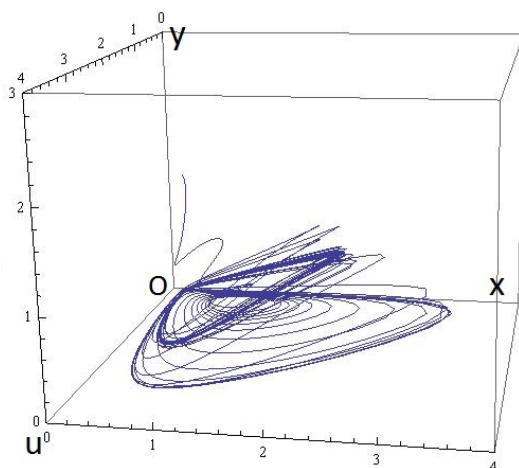


FIGURE 10. The orbit of a solution of a solution of system (4.2) starting not very near the plane $x = 0$. Compare with Figure 8 and the text.

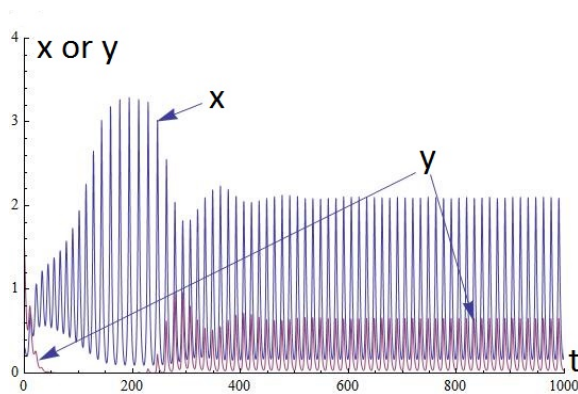


FIGURE 11. The functions $x(t)$ and $y(t)$ for the same solution of Figure 10 of system (4.2)) The population y practically vanishes for t between 50 and 200.

5. THE CYCLE - CYCLE AND THE EQUILIBRIUM - EQUILIBRIUM PATTERNS

As we shall see, the previous scheme is not preserved in cases when the two predators have the same pattern (i. e. they are both cycle-like or stable equilibrium-like). The global behavior in that cases is much more conventional.

Let us consider a system analogous to the previous one, but with values of the parameters giving cycles for both predators:

$$\begin{aligned}
 \dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) - cy \tanh(\eta u) \\
 \dot{x} &= -\alpha x + bx \tanh(\delta u) \\
 \dot{y} &= -\beta y + cy \tanh(\eta u)
 \end{aligned} \tag{5.1}$$

$$\begin{aligned}
 \alpha &= 0.95, & \beta &= 0.95, & b &= 1.38, \\
 c &= 1.5, & \delta &= 2.6, & \eta &= 1.6
 \end{aligned}$$

The solutions for $(u(0) = 2.5, x(0) = 0.002, y(0) = 2.0)$ and $(u(0) = 2.5, x(0) = 2.0, y(0) = 0.002)$ are both plotted in Figure 12.

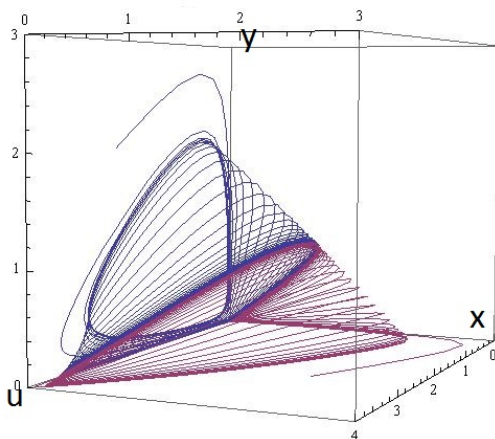


FIGURE 12. Two orbits of solutions of system (5.1) starting very near the planes $x = 0$ and $y = 0$. They both converge to the (three-dimensionally stable) oblique cycle.

It is apparent that there is a stable cycle (the oblique one in Figure 12) which is the attractor of all the orbits, starting nearby anyone of the planes $x = 0$ and $y = 0$. The behavior of these orbits is clear: starting near a plane, in a first phase, it converges nearby the cycle in this plane, then taking off towards the three-dimensional stable cycle. This is somewhat analogous to the last phase of the previous cases. A plot of x and y as functions of t for the initial values $(u(0) = 2.5, x(0) = 0.002, y(0) = 2.0)$ is shown in Figure 13.

The case when both predators (alone with the resources) have stable equilibria seems uninteresting from our present viewpoint. In the examples that we worked out, there is always a (three-dimensionally stable) equilibrium point, which is located on one or the other of the planes. In other words, one of the two predators

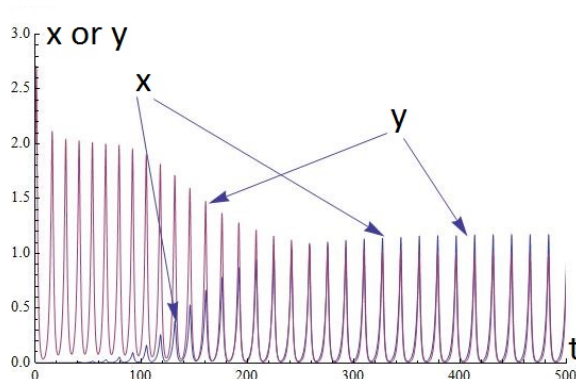


FIGURE 13. The functions $x(t)$ and $y(t)$ for one of the solutions of Figure 12. After a short transient, they evolve analogously

disappears. As an example, we consider the system

$$\begin{aligned}
 \dot{u} &= u(1 - 0.25u) - bx \tanh(\delta u) - cy \tanh(\eta u) \\
 \dot{x} &= -\alpha x + bx \tanh(\delta u) \\
 \dot{y} &= -\beta y + cy \tanh(\eta u)
 \end{aligned} \tag{5.2}$$

$$\begin{aligned}
 \alpha &= 0.95, & \beta &= 0.95, & b &= 1.3, \\
 c &= 2.3, & \delta b &= 0.91, & \eta c &= 0.76
 \end{aligned}$$

which have the two equilibria $(u = 1.3293, x = 0, y = 0.9342)$ and $(u = 1.3291, x = 0.9341, y = 0)$. They are both stable in their planes; moreover, the first one is three-dimensionally stable whereas the second is not, so that the three-dimensional attractor is the first one, leading to the disappearing of x .

It is not difficult to understand the reasons of this very simple behavior. When both predators (alone with the resources) have stable equilibria, most of the limit process takes place nearby the segment joining these two equilibria. But, according to the above values, this segment concerns points with practically the same value of u . The consequence is that the qualitative differences between the advantages of x and y (involving small versus large values of the resources u) disappear; the advantages are merely quantitative, so that one of the phases x or y wins.

6. CONCLUSION

Traditional models always conclude that asexuality is an unbeatable reproductive advantage in the presence of unlimited food or energy resources. Oppositely, the logistic model shows that, when resources diminish, the growth rate of asexuals decreases and in this case sexuality acquires a new advantage by providing diversity and genetic information. In particular, when asexuals have partially or totally exhausted their privileged resource, sexuals prevail because their adaptive diversity gives them access to new and varied resources. The interest of the model is to resolve the issue of the factor two burden showing that the advantage of sexuality can be very large when males appear in times of food crisis, because in this situation the burden of growth is reversed.

On one side the production of males halves population growth but on the other they brings genetic or informational resources. It may be said that there is a trade - off between food energy allowing rapid and immediate growing and adaptive information allowing a slow and late growth; the rapid proliferation of parthenogenetic females implies better utilization of energy but is poor in information, while the appearance of males wastes energy but provides genetic resources, which is their sole reason for being and the sole function of sexuality.

An ESS shows that we succeeded in maintaining both forms at the end of a long unstable phase. This model is founder as it is also concerned with the very important enigma of the maintenance of sex in eukaryotes. Indeed, the sexuality of eukaryotes is the basis of genetic biodiversity (c.f. Otto [8]) and therefore of the evolution of species, since the birth and the existence of each species is based in its integrality on the balance between the survival of individuals and the success of their sexual exchanges.

Another application of our model is that it should also describe certain behavior of cancer cells. Aktipis et al [1] show that there is a trade - off between rapid proliferation of short - lived cells and dormancy of long - lived cells in tumors. “Thus, cancer cell dormancy may represent an example of the life history trade -off between producing offsprings as soon as possible and producing offsprings later”. Obviously, cancer cells are not sexual, however the cells have quite similar genetic structure, live in the same environment (the bearing organism) and follow two alternating strategies, it would therefore be interesting to test the proposed model.

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