Two nonlinear days in Urbino 2017,
Electronic Journal of Differential Equations, Conference 25 (2018), pp. 167–178.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

LINEAR AND SEMILINEAR PROBLEMS INVOLVING Δ_λ-LAPLACIANS

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Dedicated to Anna Aloe

Abstract. In recent years a growing attention has been devoted to Δ_λ-Laplacians, linear second-order degenerate elliptic PDO's contained in the general class introduced by Franchi and Lanconelli in some papers dated 1983–84 [12, 13, 14]. Here we present a survey on several results appeared in literature in the previous decades, mainly regarding: (i) Geometric and functional analysis frameworks for the Δ_λ's; (ii) regularity and pointwise estimates for the solutions to $\Delta_\lambda u = 0$; (iii) Liouville theorems for entire solutions; (iv) Pohozaev identities for semilinear equations involving Δ_λ-Laplacians; (v) Hardy inequalities; (vi) global attractors for the parabolic and damped hyperbolic counterparts of the Δ_λ's.

We also show several typical examples of Δ_λ-Laplacians, stressing that their class contains, as very particular examples, the celebrated Baouendi-Grushin operators as well as the $L_{\alpha,\beta}$ and $P_{\alpha,\beta}$ operators respectively introduced by Thuy and Tri in 2002 [36] and by Thuy and Tri in 2012 [37].

1. Introduction

1.1. **Δ_λ-operators.** In $\mathbb{R}^N$, whose point will be denoted by $x = (x_1, \dots, x_N)$, let us consider a n-tuple $\lambda := (\lambda_1, \dots, \lambda_N)$ of real functions

$$\lambda_j : \mathbb{R}^N \to \mathbb{R}, \quad j = 1, \dots, N,$$

such that $\lambda_1 = 1$ and $\lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1})$ for $j \geq 2$. Define the linear second order partial differential operators Δ_λ as follows:

$$\Delta_\lambda := \sum_{j=1}^{N} \lambda_j \partial_{x_j}(\lambda_j \partial_{x_j}) = \sum_{j=1}^{N} \lambda_j^2 \partial_{x_j}^2. \tag{1.1}$$

1.2. If the λ_j's are non-identically zero polynomial functions then Δ_λ is hypoelliptic, i.e., every distributional solution u to the equation

$$\Delta_\lambda u = f$$

2010 *Mathematics Subject Classification.* 35J70, 35H20, 35K65.
Key words and phrases. Degenerate elliptic PDE; semilinear subelliptic PDE; Δ_λ-Laplacian.
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Published September 15, 2018.

in an open set $\Omega \subseteq \mathbb{R}^N$, is actually of class C^∞ in Ω if f is C^∞ in Ω. This is an easy consequence of the celebrated Hörmander theorem on the hypoellipticity of the "sum of squares of vector fields" [18]. Indeed, let

$$\mathfrak{a} := \mathrm{Lie}\{\lambda_1 \partial_{x_1}, \dots, \lambda_N \partial_{x_N}\}. \tag{1.2}$$

Then, $\lambda_1 \partial_{x_1} = \partial_{x_1} \in \mathfrak{a}$. Moreover, if $j \geq 2$, being λ_j a non zero polynomial function, there exists a multi-index $\beta^{(j)}$ such that $D^{\beta^{(j)}} \lambda_j = c_j$, with c_j non zero real constant. This easily implies that $\partial_{x_2}, \partial_{x_3}, \dots, \partial_{x_N} \in \mathfrak{a}$. Hence

$$\operatorname{rank} \mathfrak{a}(x) = N \quad \forall x \in \mathbb{R}^N,$$

so that, by the Hörmander theorem, Δ_λ is hypoelliptic.

Celebrated typical examples of Δ_λ hypoelliptic PDO's with polynomial coefficients are the Baouendi-Grushin operators [6, 16, 17],

$$\mathcal{L}_{m,p} = \partial_{x_1}^2 + \dots + \partial_{x_p}^2 + (x_1^2 + \dots + x_p^2)^{2m} (\partial_{x_{p+1}}^2 + \dots + \partial_{x_N}^2), \tag{1.3}$$

$m, p \in \mathbb{Z},\ m \geq 0, 1 \leq p < N$, corresponding to the case

$$\lambda_1(x) = \dots = \lambda_p(x) = 1, \quad \lambda_{p+1}(x) = \dots = \lambda_N(x) = (x_1^2 + \dots + x_p^2)^m,$$

and the Baouendi-Goulaouic operator

$$\mathcal{L}_2 = \partial_{x_1}^2 + \partial_{x_2}^2 + x_1^2 \partial_{x_3}^2 \quad (\text{in } \mathbb{R}^3), \tag{1.4}$$

corresponding to the case

$$\lambda_1(x) = \lambda_2(x) = 1, \quad \lambda_3(x) = x_1.$$

The Baouendi-Goulaouic operator was the first example appeared in literature of C^∞-hypoelliptic "sum of squares" operator which is not analytic-hypoelliptic (see [7]).

1.3. If the λ_j's are (merely) smooth functions, a condition making Δ_λ hypoelliptic in $\mathbb{R}^N$ is the following one:

> For every $x \in \mathbb{R}^N$ and for every $j \in \{1, \dots, N\}$, there exists a multi-index β depending on x and j such that
> $$D^\beta \lambda_j(x) \neq 0. \tag{1.5}$$

If we let, for every fixed $x \in \mathbb{R}$,

$$\mathfrak{a}(x) = \{X(x) \mid X \in \mathfrak{a}\},$$

where $\mathfrak{a}$ is the Lie algebra in (1.2), then, trivially, $\partial_{x_1} \in \mathfrak{a}(x)$. Moreover, using condition (1.5), we can prove that for every $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ there exist smooth functions

$$a_2(x) = a_2(x_1),\ a_3(x) = a_3(x_1, x_2), \dots, a_N(x) = a_N(x_1, \dots, x_{N-1}),$$

such that $a_j(x) \neq 0$ and $a_j \partial_{x_j} \in \mathfrak{a}$, $j = 2, \dots, N$. Then

$$N \geq \operatorname{rank} \mathfrak{a}(x) \geq \dim \operatorname{span}\{\partial_{x_1}, a_1(x)\partial_{x_2}, \dots, a_n(x)\partial_{x_N}\} = N,$$

hence $\operatorname{rank} \mathfrak{a}(x) = N$. Since x is an arbitrary point of $\mathbb{R}^N$, this proves that Δ_λ satisfies the Hörmander rank condition, so that is hypoelliptic. If the λ_j's are real analytic, condition (1.5) is equivalent to say that

$$\lambda_j \not\equiv 0 \quad \forall j = 1, \dots, N.$$

Thus, the hypoellipticity result of subsection 1.2 can be improved as follows:

If the $\lambda_1, \dots, \lambda_N$ are real analytic functions then Δ_λ is hypoelliptic if (and only if)

$$\lambda_j \not\equiv 0 \quad \forall j = 1, \dots, N.$$

1.4. In some papers dated 1982–1984 ([12, 13, 14]), Franchi and Lanconelli studied Δ_λ-operators only assuming the λ_j's locally Lipschitz continuous and of class C^1 out of the coordinate axes. Obviously, in such weak regularity assumptions, Hörmander condition is meaningless. In [12], suitable condition only involving the first derivatives of the λ_j's, are introduced, allowing to get a kind of De Giorgi-Moser theorem for Δ_λ, i.e., the Hölder continuity and the Harnack inequality for the weak solutions.

2. De Giorgi-Moser-type theorem for Δ_λ. Liouville-type theorems

Let us assume the λ_j's satisfy the following hypotheses.

- (H1) $\lambda_1, \dots, \lambda_N$ are continuous and of class C^1 and strictly positive outside the coordinate hyperplanes;
- (H2) $\lambda_1(x) \equiv 1$, $\lambda_j(x) = \lambda_j(x_1, \dots, x_{j-1})$, $j = 2, \dots, N$;
- (H3) $\lambda_j(x) = \lambda_j(x^*)$, where $x^* = (|x_1|, \dots, |x_N|)$;
- (H4) there exists $\rho \geq 0$ such that
$$0 \leq x_k \partial_{x_k} \lambda_j(x) \leq \rho \lambda_j(x), \quad k = 1, \dots, j-1,$$
for every $x \in \mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_i \geq 0 \ \forall i = 1, \dots, N\}$.

Under these hypotheses in [12] a metric d was constructed in $\mathbb{R}^N$ that plays for Δ_λ the same rôle as the Euclidean distance plays for the classical Laplacian. This metric, which actually is the Carnot-Carathéodory distance related to the vector fields

$$X_1 = \lambda_1 \partial_{x_1}, \dots, X_N = \lambda_1 \partial_{x_N},$$

is defined as follows.

An absolutely continuous path $\gamma : [0, T] \to \mathbb{R}^N$, $T > 0$, is λ-subunit if, letting $e_j = (0, \dots, \underset{j}{1}, \dots, 0)$ for every $j = 1, \dots, N$, we have

$$\gamma'(t) = \sum_{j=1}^{N} c_j(t) \lambda_j(\gamma(t)) e_j \quad \text{a.e. in } [0, T], \text{ with } \sum_{j=1}^{m} c_j^2(t) \leq 1.$$

In this case we put $l(\gamma) := T$ and for every $x, y \in \mathbb{R}^N$ we define

$$\mathcal{C}(x, y) := \{\gamma \ \lambda\text{-subunit path } : \gamma : [0, T] \to X, \ \gamma(0) = x, \ \gamma(T) = y\}.$$

Note that Hypotheses (H1) and (H2) imply $\mathcal{C}(x, y) \neq \emptyset$ for all $x, y \in \mathbb{R}^N$. Then, letting

$$d(x, y) := \inf\{l(\gamma) : \gamma \in \mathcal{C}(x, y)\},$$

we have $d(x, y) < \infty$ for every $x, y \in \mathbb{R}^N$.

It is easy to see that $(x, y) \mapsto d(x, y)$ is a *distance* in $\mathbb{R}^N$, which we call the λ-*distance*. In [12] and [14] it is proved that $(\mathbb{R}^N, d)$ is a *doubling metric space*, i.e., that there exists a positive constant c_d such that

$$|B_d(x, 2r)| \leq c_d |B_d(x, r)| \quad \forall x \in \mathbb{R}^N, \ \forall r > 0, \tag{2.1}$$

where $|\cdot|$ stands for the Lebesgue measure and $B_d(x, r)$ denotes the d-ball of center x and radius r,

$$B_d(x, r) = \{y \in \mathbb{R}^N \mid d(x, y) < r\}.$$

It is a standard computation to show that the doubling inequality (2.1) implies

$$|B_d(x,2r)| \le c_d\Big(\frac{R}{r}\Big)^Q |B_d(x,r)|, \tag{2.2}$$

for every $x \in \mathbb{R}^N$ and $0 < r < R < \infty$. Here Q is the constant $Q := \log_2 c_d$, which is called a homogeneous dimension of $(\mathbb{R}^N, d)$.

The natural functional setting for studying Δ_λ-operators is the Sobolev-type space $W^{1,p}_\lambda(\Omega)$, $1 < p < \infty$. More precisely, if Ω is a bounded open subset of $\mathbb{R}^N$ and $1 < p < \infty$, we denote by

$$\mathring{W}^{1,p}_\lambda(\Omega)$$

the closure of $C^1_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{1,p}_\lambda(\Omega)} := \Big(\int_\Omega |\nabla_\lambda u|^p \, dx\Big)^{1/p},$$

where

$$\nabla_\lambda u = (\lambda_1 \partial_{x_1}, \dots, \lambda_N \partial_{x_N}).$$

From [13, Theorem 2.6] (see also [20, Proposition 3.2]), one gets the following result: the embedding

$$\mathring{W}^{1,2}_\lambda(\Omega) \hookrightarrow L^p(\Omega) \tag{2.3}$$

is continuous for every $p \in [1, 2^*]$ and compact for every $p \in [1, 2^*[$, where

$$2^* = \frac{2Q}{Q-2}.$$

Another crucial functional inequality in Δ_λ-setting is the following Poincaré-type inequality: for every $x \in \mathbb{R}^N$ and $r > 0$,

$$\int_{B_d(x,r)} |u - u_r|^2 \, dy \le C r^2 \int_{B_d(x,\theta r)} |\nabla_\lambda u|^2 \, dy \quad \forall u \in C^1(\overline{B_d(x,\theta r)})$$

where $C > 0$ and $\theta > 1$ are suitable constants independent of u, x and r, and u_r denotes the average of u on $B_d(x,r)$:

$$u_r =: \frac{1}{B_d(x,r)} \int_{B_d(x,r)} u(y) \, dy$$

(see [14, 25]).

To complete the list of the key results needed to show a De Giorgi-type theorem for Δ_λ, we recall the existence of global cut-off functions modelled on the geometry of the d-balls. More precisely, the following proposition holds:

Let $B_d(x,r_1)$ and $B_d(x,r_2)$ be concentric d-balls with $0 < r_1 < r_2 < \infty$. Then there exists $\eta \in \mathring{W}^{1,2}_\lambda(B_d(x,r_2))$ such that $\eta \equiv 1$ a.e. in $B_d(x,r_1)$ and

$$|\nabla_\lambda \eta| \le \frac{2}{r_2 - r_1} \quad \text{a.e. in } B_d(0,r_2)$$

(see [19, Theorem 10]).

The doubling condition (2.1), the Sobolev embedding (2.3), and the cut-off function η allow to adapt the Moser's iteration procedure to get the following theorem.

Theorem 2.1 (De-Giorgi-Moser-type theorem for Δ_λ)**.** *Let Ω be an open subset of $\mathbb{R}^N$ and let $u \in W^{1,2}_{\lambda,\mathrm{loc}}(\Omega)$ be a weak solution to*

$$\Delta_\lambda u = 0 \text{ in } \Omega.$$

Then,

(i) *(Scale invariant Harnack inequality) If $B_d(z,2r) \subseteq \Omega$ and $u \geq 0$, then*
$$\sup_{B_d(z,r)} u \leq C \inf_{B_d(z,r)} u, \tag{2.4}$$
where $C > 0$ is independent of u, z and r.

(ii) *(Local Hölder continuity) If $B_d(z,2r) \subseteq \Omega$, then*
$$|u(x) - u(y)| \leq C\Big(\frac{d(x,y)}{r}\Big)^\alpha \sup_{B(z,2r)} |u| \quad \forall x, y \in B_d\Big(z, \frac{r}{2}\Big), \tag{2.5}$$
where $C > 0$ and $\alpha \in]0,1[$ are independent of u, z and r.

Actually, the conclusions of this theorem hold true for the weak solutions of the λ-*elliptic operators*. A linear second order PDO of the kind
$$\mathcal{L} = \sum_{i,j=1}^N \partial_{x_i}(a_{ij}(x)\partial_{x_j}) = \operatorname{div}(A(x)D)$$
will be called λ-elliptic in $\mathbb{R}^N$ if the quadratic form related to the symmetric matrix $A(x) = a_{ij}(x))_{i,j=1,\dots,n}$ with measurable entries, satisfies
$$\frac{1}{c}\sum_{j=1}^n (\lambda_j(x)\xi_j)^2 \leq \langle A(x)\xi, \xi\rangle \leq c\sum_{j=1}^n (\lambda_j(x)\xi_j)^2 \quad \forall x, \xi \in \mathbb{R}^N.$$

If Ω is an open subset of $\mathbb{R}^N$ we say that $u \in W^{1,2}_{\lambda,\mathrm{loc}}(\Omega)$ if, for every $\varphi \in C_0^\infty(\Omega,\mathbb{R})$ one has $u\varphi \in \mathring{W}^{1,2}_\lambda(\Omega)$. To define the notion of weak solution to the equation $\mathcal{L}u = 0$, we need to introduce the bilinear form
$$\mathcal{L}(u,v) = \int_\Omega \langle A(x)Du(x), Dv(x)\rangle\, dx$$
for $u \in C^1(\Omega,\mathbb{R})$ and $v \in C_0^1(\Omega,\mathbb{R})$. D is the Euclidean gradient $D = (\partial_{x_1},\dots,\partial_{x_N})$. Since $A \geq 0$, we have (because $\mathcal{L}$ is λ-elliptic)
$$\begin{aligned} |\mathcal{L}(u,v)| &\leq \int_\Omega \langle A(x)Du(x), Du(x)\rangle^{\frac12} \langle A(x)Dv(x), Dv(x)\rangle^{\frac12}\, dx \\ &\leq c\int_\Omega |\nabla_\lambda u(x)|\ |\nabla_\lambda v(x)|\, dx\,. \end{aligned}$$
Then the bilinear form a is well defined and, if Ω is bounded, it can be continuously extended to $W^{1,2}_{\lambda,\mathrm{loc}}(\Omega) \times \mathring{W}^{1,2}_\lambda(\Omega)$. A function $u \in W^{1,2}_{\lambda,\mathrm{loc}}(\Omega)$ is a weak solution to $\mathcal{L}u = 0$ in Ω if
$$a(u,v) = 0 \quad \forall v \in C_0^1(\Omega,\mathbb{R}).$$
The Moser iteration procedure works for λ-elliptic operators as for Δ_λ-operators. Then, De Giorgi-Moser Theorem 2.1 extends to the weak solutions to $\mathcal{L}u = 0$ for every λ-elliptic operator $\mathcal{L}$ (for the Δ_λ-case, see [12, 13, 14], for the λ-elliptic case see [24]).

The invariant Harnack inequality (2.4) immediately leads to the following Liouville-type theorem. Here $\mathcal{L}$ stands for any λ-elliptic operator.

Theorem 2.2. *Let $u \in W^{1,2}_{\lambda,\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^N$. If $u \geq 0$, then u is identically constant in $\mathbb{R}^N$.*

From the Hölder estimates (2.5), one obtains another Liouville-type theorem.

Theorem 2.3. *Let $u \in W^{1,2}_{\lambda,\mathrm{loc}}(\mathbb{R}^N)$ be a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^N$. Assume that, for a suitable $x_0 \in \mathbb{R}^N$,*

$$\lim_{r\to\infty}\Big(\frac{1}{r^\alpha}\sup_{B(x_0,r)}|u|\Big) = 0,$$

where $\alpha \in]0,1[$ is the Hölder exponent in (2.5). Then, u is identically constant in $\mathbb{R}^N$.

As a last theorem we would like to recall is a Colding-Minicozzi-type Liouville theorem for the λ-elliptic operators $\mathcal{L}$, which is proved in [19].

Theorem 2.4. *Let x_0 be a fixed point of $\mathbb{R}^N$ and denote by $d(x)$ the λ-distance $d(x_0, x)$. Then, for every $m > 0$, the linear space*

$$\{u \in W^{1,2}_{\lambda,\mathrm{loc}}(\mathbb{R}^N) : \mathcal{L}u = 0 \text{ in } \mathbb{R}^N,\ u(x) = O(d(x))^m \text{ as } (d(x)) \to \infty\}$$

has finite dimension.

We would like to close this subsection by quoting the recent paper [4] by Anh and My where a Liouville-type theorem for system of semilinear inequalities involving Δ_λ-operators is proved.

3. Δ_λ-Laplacians

If the functions λ_j's, together with hypotheses (H1), (H2), (H3) and (H4), are supposed to be homogeneous with respect to a fixed group of dilations in $\mathbb{R}^N$, the corresponding Δ_λ-operators have been called in [20] Δ_λ-*Laplacians*, since they share some important homogeneity properties with the classical Laplacian. The corresponding geometry of the λ-distance achieves some crucial analogies with the Euclidean ones.

Let $(\delta_r)_{r>0}$ be a group of dilations in $\mathbb{R}^N$ of the kind

$$\delta_r : \mathbb{R}^N \to \mathbb{R}^N, \quad \delta_r(x) = \delta_r(x_1,\dots,x_N) = (r^{\varepsilon_1}x_1,\dots,r^{\varepsilon_N}x_N), \tag{3.1}$$

where $1 \le \varepsilon_1 \le \varepsilon_2 \le \dots \le \varepsilon_N$. Assume λ_j is δ_r-*homogeneous of degree* $\varepsilon_j - 1$, i.e.,

$$\lambda_j(\delta_r(x)) = r^{\varepsilon_j-1}\lambda_j(x), \quad \forall x \in \mathbb{R}^N,\ r > 0,\ j = 1,\dots,N. \tag{3.2}$$

Under this new assumption, Δ_λ becomes δ_r-homogeneous of degree two, i.e.,

$$\Delta_\lambda(u(\delta_r(x))) = r^2(\Delta_\lambda u)(\delta_r(x)) \quad \forall x \in \mathbb{R}^N, \quad \forall r > 0,$$

and for every $u \in C^\infty(\mathbb{R}^N)$. The positive real number

$$Q := \varepsilon_1 + \dots + \varepsilon_N$$

is the *homogeneous dimension* of $\mathbb{R}^N$ with respect to the group of dilations $(\delta_r)_{r>0}$. With respect to the Lebesque measure of the λ-balls and the Sobolev-type embedding Theorems, it plays the rôle of the dimension N in the classical Laplacian case. Indeed, it works as the optimal exponent Q in the inequality (2.2) and in the embedding (2.3).

In the present homogeneous assumption, precise estimates of both the λ-distance d and the Lebesque measure of the d-balls are showed by Kogoj and Lanconelli in [20]. A deep study of the λ-geometry for particular form of the λ_j's have been recently performed by Wu in [42].

By crucially exploiting the homogeneity (3.2), in [20] the following Pohozaev-type identities are proved. We stress that the constant Q which will appear in

(3.3), (3.5) and (3.6), is exactly the homogeneous dimension of $\mathbb{R}^N$ with respect to $(\delta_r)_{r>0}$.
Let T be linear first order PDO

$$T:\mathbb{R}^N\to\mathbb{R}^N,\quad T(x)=T(x_1,\dots,x_N)=\sum_{j=1}^N \varepsilon_j x_j\partial_{x_j},$$

i.e., the generator of dilation group $(\delta_r)_{r>0}$. Then, if Ω is a C^1 bounded open subset of $\mathbb{R}^N$, we have

$$\begin{aligned}&\int_\Omega T(u)\Delta_\lambda u\,dx\\&=\int_{\partial\Omega}\langle\nabla_\lambda u,\nu_\lambda\rangle T(u)\,ds-\frac12\int_{\partial\Omega}|\nabla_\lambda u|^2\langle T,\nu\rangle\,ds+\Big(\frac{Q}{2}-1\Big)\int_\Omega|\nabla_\lambda u|^2\,dx\end{aligned}\tag{3.3}$$

for every $u\in C^1(\overline{\Omega}),\mathbb{R})$. Here $\langle\cdot,\cdot\rangle$ stands for the Euclidean inner product, ν is the outward normal to Ω and $\nu_\lambda=(\lambda_1\nu_1,\dots,\lambda_N\nu_N)$.

From this identity, we easily obtain an integral identity for the solutions to

$$\Delta_\lambda u+f(u)=0,\tag{3.4}$$

$f:\mathbb{R}\to\mathbb{R}$ is a continuous function. We let

$$F(t):=\int_0^t f(s)ds,\quad t\in\mathbb{R}.$$

Then, if $u\in C^2(\overline{\Omega})$ is a solution to (3.4) the following identity holds

$$\begin{aligned}&\int_\Omega\Big(F(u)+\Big(\frac1Q-\frac12\Big)uf(u)\Big)\,dx\\&=\frac1Q\int_{\partial\Omega}\Big(\langle T,\nu\rangle\big(F(u)-\frac12|\nabla_\lambda u|^2\big)+\langle\nabla_\lambda u,\nu\rangle\Big(T(u)+\Big(\frac{Q}{2}-1\Big)u\Big)\Big)ds\,.\end{aligned}\tag{3.5}$$

Moreover, if $u=0$ on $\partial\Omega$,

$$\int_\Omega\Big(F(u)+\Big(\frac1Q-\frac12\Big)u\ f(u)\Big)\,dx=\frac{1}{2Q}\int_{\partial\Omega}\Big(\frac{\partial u}{\partial\nu}\Big)^2|\nu_\lambda|^2\langle T,\nu\rangle\,ds\,.$$

Pohozaev-type identities for particular Δ_λ-Laplacians were previously proved in [39, 40, 36, 10, 37].

If the domain $\Omega\subseteq\mathbb{R}^N$ is $(\delta_r)_{r>0}$ *starlike*, i.e.,

$$\langle T,\nu\rangle\ge 0 \text{ at every point of } \partial\Omega,$$

and C^1 bounded open set, then the following non-existence result, extending to the Δ_λ a celebrated theorem by Pohozaev, holds. The problem

$$\Delta_\lambda u+f(u)=0\quad\text{in }\Omega,\quad u|_{\partial\Omega}=0,\tag{3.6}$$

has non trivial non-negative solution in $C^2(\overline{\Omega})$ if

$$F(t)+\Big(\frac1Q-\frac12\Big)tf(t)<0\quad\forall t\neq0.$$

Thanks to the properties of the Δ_λ's previously recalled, the techniques of the variational theory of the critical points work equally well for the Δ_λ-Laplacian as for the classical Laplacian. Many existence and non-existence results are today present in literature for semilinear Δ_λ boundary value problem, both in subcritical and critical behaviour assumption on the semilinear term $f(u)$ (see, e.g.,[3, 5, 9, 20, 28, 29, 32, 36, 37, 41]).

The homogeneity properties of the Δ_λ-Laplacians have been also exploited in [23] to prove Hardy-type inequalities, which extend previous results by Garofalo and D'Ambrosio for the Baouendi-Grushin case [11, 15].

Before closing this section, we have to mention that initial value problems for evolution equations modelled on Δ_λ-Laplacians have been studied in these last years.

In [2] Anh, Hung, Ke and Phong have proved the existence of the global attractor for semilinear parabolic equations involving Baouendi-Grushin-type operators. Kogoj and Sonner have extended this result for Δ_λ-Laplacians (and showed the finite fractal dimension of the attractor) in [21] and for more general degenerate parabolic equations in [22]. We stress that in this last paper semilinear damped hyperbolic equations involving Δ_λ-Laplacians are also considered.

Extensions to the critical cases of the results in [21] and in [22] have been proved in [26, 27]. We also quote the papers [1, 30, 31, 34, 35, 38] where evolution equations related to classes of Δ_λ operators are studied.

4. Examples of Δ_λ-Laplacians

The following examples are taken from [21]. We split $\mathbb{R}^N$ as $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k}$, and write

$$x = \big(x^{(1)}, \dots, x^{(k)}\big), \quad x^{(i)} = \big(x_1^{(i)}, \dots, x_{N_i}^{(i)}\big) \in \mathbb{R}^{N_i}, \quad i = 1, \dots, k.$$

We denote the classical Laplace operator in $\mathbb{R}^{N_i}$ by

$$\Delta_{x^{(i)}} = \sum_{j=1}^{N_i} \partial^2_{x_j^{(i)}},$$

and we write Δ_λ operators in the form

$$\Delta_\lambda = (\lambda^{(1)})^2 \Delta_{x^{(1)}} + \cdots + (\lambda^{(k)})^2 \Delta_{x^{(k)}} \quad \text{in } \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_k},$$

where

$$\lambda = \big(\lambda^{(1)}, \dots, \lambda^{(k)}\big), \quad \lambda^{(i)} = \big(\lambda_1^{(i)}, \dots, \lambda_{N_i}^{(i)}\big),$$

and the functions $\lambda^{(i)}$ are continuous in $\mathbb{R}^{N_i}$, $i = 1, \dots, k$.

Example 4.1. Let α be a real positive constant and $k = 2$. We consider at first the Baouendi-Grushin-type operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}},$$

where $\lambda = (\lambda^{(1)}, \lambda^{(2)})$, with $\lambda_j^{(1)}(x) = 1, j = 1, \dots, N_1$ and $\lambda_j^{(2)}(x) = |x^{(1)}|^\alpha, j = 1, \dots, N_2$. A group of dilations making Δ_λ homogeneous of degree two is $(\delta_r)_{r>0}$ with

$$\delta_r\Big(x^{(1)}, x^{(2)}\Big) = \Big(r x^{(1)}, r^{\alpha+1} x^{(2)}\Big).$$

In this case the homogenous dimension of $\mathbb{R}^N$ with respect to $(\delta_r)_{r>0}$ is

$$Q = N_1 + (\alpha + 1) N_2.$$

More generally, for a given multi-index $\alpha = (\alpha_1, \dots, \alpha_{k-1})$ with real constants $\alpha_j \geq 0$, $j = 1, \dots, k-1$, we define

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha_1} \Delta_{x^{(2)}} + \cdots + |x^{(k-1)}|^{2\alpha_{k-1}} \Delta_{x^{(k)}}.$$

Then, in our notation $\lambda = \big(\lambda^{(1)}, \dots, \lambda^{(k)}\big)$ with

$$\begin{gathered}\lambda_j^{(1)}(x) \equiv 1, \quad j = 1, \dots, N_1 \\ \lambda_j^{(i)}(x) = |x^{(i-1)}|^{\alpha_{i-1}} \quad i = 2, \dots, k, \quad j = 1, \dots, N_i,\end{gathered}$$

and the group of dilations such that λ satisfies (3.2) is given by

$$\delta_r\big(x^{(1)}, \dots, x^{(k)}\big) = \big(r^{\varepsilon_1} x^{(1)}, \dots, r^{\varepsilon_k} x^{(k)}\big),$$

with $\varepsilon_1 = 1$ and $\varepsilon_i = \alpha_{i-1}\varepsilon_{i-1} + 1$ for $i = 2, \dots, k$. In particular, if $\alpha_1 = \dots = \alpha_{k-1} = \alpha$, the dilations become

$$\delta_r\Big(x^{(1)}, \dots, x^{(k)}\Big) = \Big(r x^{(1)}, r^{1+\alpha} x^{(2)}, \dots, r^{1+\alpha+\alpha^2+\dots+\alpha^{k-1}} x^{(k)}\Big).$$

Remark 4.2. A trivial change of variable makes the operator

$$\Delta_{x^{(1)}} + \frac{1}{4}|x^{(1)}|^2 \Delta_{x^{(2)}}$$

a Δ_λ-Laplacian in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ of the previous type.

Moreover, if the dimensions N_1 and N_2 satisfy the inequality $N_2 < \rho(N_1)$, where ρ is the so called Hurwitz-Radon function, then there exists a composition law $\circ$ in $\mathbb{R}^N$ such that $\mathbb{H}_N := (\mathbb{R}^N, \circ, \delta_\lambda)$ is a group of Heinsenberg type (see [8, Remark 3.6.7],) and, denoting by $\Delta_{\mathbb{H}_N}$ the canonical sub-Laplacian on $\mathbb{H}_N$, we have

$$\Big(\Delta_{x^{(1)}} + \frac{1}{4}|x^{(1)}|^2 \Delta_{x^{(2)}}\Big) u = \Delta_{\mathbb{H}_N} u$$

for every smooth function $u : \mathbb{R}^N \to \mathbb{R}$ which is radially symmetric in the variable $x^{(1)}$ (see [8, p. 251]).

Example 4.3. Let α, β and γ be nonnegative real constants. We consider the operator

$$\Delta_\lambda = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha} \Delta_{x^{(2)}} + |x^{(1)}|^{2\beta} |x^{(2)}|^{2\gamma} \Delta_{x^{(3)}},$$

where $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$ with

$$\begin{gathered}\lambda_j^{(1)}(x) \equiv 1, \quad j = 1, \dots, N_1 \\ \lambda_j^{(2)}(x) = |x^{(1)}|^{\alpha}, \quad j = 1, \dots, N_2, \\ \lambda_j^{(3)}(x) = |x^{(1)}|^{\beta} |x^{(2)}|^{\gamma}, \quad j = 1, \dots, N_3.\end{gathered}$$

The dilations become

$$\delta_r\Big(x^{(1)}, x^{(2)}, x^{(3)}\Big) = \Big(r x^{(1)}, r^{\alpha+1} x^{(2)}, r^{\beta+(\alpha+1)\gamma+1} x^{(3)}\Big).$$

Similarly, for operators of the form

$$\begin{aligned}\Delta_\lambda = \Delta_{x^{(1)}} &+ |x^{(1)}|^{2\alpha_{1,1}} \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha_{2,1}} |x^{(2)}|^{2\alpha_{2,2}} \Delta_{x^{(3)}} + \dots \\ &+ \Big(\prod_{i=1}^{k-1} |x^{(i)}|^{2\alpha_{k-1,i}}\Big) \Delta_{x^{(k)}},\end{aligned}$$

where $\alpha_{i,j} \geq 0$, $i = 1, \dots, k-1, j = 1, \dots, i$, are real constants, the group of dilations is given by

$$\delta_r\Big(x^{(1)}, \dots, x^{(k)}\Big) = \Big(r^{\varepsilon_1} x^{(1)}, \dots, r^{\varepsilon_k} x^{(k)}\Big)$$

with $\varepsilon_1 = 1$ and $\varepsilon_j = 1 + \sum_{i=1}^{j-1} \alpha_{j-1,i}\varepsilon_i$, for $i = 2, \dots, k$. In particular, if $\alpha_{1,1} = \dots = \alpha_{k-1,k-1} = \alpha$,

$$\delta_r\Big(x^{(1)}, \dots, x^{(k)}\Big) = \Big(rx^{(1)}, r^{\alpha+1}x^{(2)}, \dots, r^{(\alpha+1)^{k-1}}x^{(k)}\Big).$$

Remark 4.4. We would like to remark that this class of operators contains the operators

$$L_{\alpha,\beta} = \Delta_{x^{(1)}} + |x^{(1)}|^{2\alpha}\Delta_{x^{(2)}} + |x^{(1)}|^{2\beta}\Delta_{x^{(3)}},$$

introduced by Thuy and Tri in [36], and the operators

$$P_{\alpha,\beta} = \Delta_{x^{(1)}} + \Delta_{x^{(2)}} + |x^{(1)}|^{2\alpha}|x^{(2)}|^{2\beta}\Delta_{x^{(3)}},$$

introduced by Thuy and Tri in [37]. We also want to mention that the class of the Grushin-like operators very recently introduced by Maldonado in [33, Subsection 4.1] extends the one described above.

Example 4.5. The Δ_λ-operators of the following type

$$\Delta_\lambda = \Delta_{x^{(1)}} + \Big(\mu_1(x^{(1)})\Big)^2\Delta_{x^{(2)}} + \Big(\mu_2(x^{(1)})\Big)^2\Big(\mu_3(x^{(2)})\Big)^2\Delta_{x^{(3)}},$$

where $\mu_1, \mu_2 : \mathbb{R}^{N_1} \to \mathbb{R}$ and $\mu_3 : \mathbb{R}^{N_2} \to \mathbb{R}$ are continuous functions satisfying (H1)–(H4) and

$$\mu_1(sx^{(1)}) = s^\alpha\mu_1(x^{(1)}), \quad \mu_2(sx^{(1)}) = s^\beta\mu_2(x^{(1)}), \quad \mu_3(sx^{(2)}) = s^\gamma\mu_3(x^{(2)}), \quad \forall s > 0,$$

with α, β and γ nonnegative real constants, are Δ_λ-Laplacians with the group of dilations $(\delta_r)_{r>0}$,

$$\delta_r\Big(x^{(1)}, x^{(2)}, x^{(3)}\Big) = \Big(rx^{(1)}, r^{\alpha+1}x^{(2)}, r^{\beta+(\alpha+1)\gamma+1}x^{(3)}\Big).$$

Acknowledgements. A. E. Kogoj was partially supported by the Gruppo Nazionale per l' Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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