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AN ELLIPTIC EQUATION INVOLVING THE SQUARE ROOT OF THE LAPLACIAN WITHOUT ASYMPTOTIC LIMITS

YUTONG CHEN, JIABAO SU, MINGZHENG SUN, RUSHUN TIAN

In memory of Professor Alan C. Lazer

Abstract. In this article we show the existence of nontrivial solutions for nonlocal elliptic equations involving the square root of the Laplacian with the nonlinearity failing to have asymptotic limits at zero and at infinity. We use a combination of homotopy invariance of critical groups and the topological version of linking theorems.

1. Introduction

This article concerns the nonlocal elliptic equation

$$\begin{aligned} A_{1/2}u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ ($N \geqslant 2$) with Lipschitz boundary $\partial\Omega$, and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the subcritical growth condition

(A1) There exist $C > 0$ and $1 \leqslant p < 2^\star := \frac{2N}{N-1}$ such that

$$|f(x,t)| \leqslant C(1 + |t|^{p-1}) \quad \text{uniformly for a.e. } x \in \Omega \text{ and } t \in \mathbb{R}. \tag{1.2}$$

The nonlocal elliptic operator $A_{1/2}$ in (1.1) is defined as the square root of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary data on $\partial\Omega$. Let $\{\lambda_j, \varphi_j\}_{j=1}^\infty$ satisfy

$$\begin{aligned} -\Delta\varphi_j = \lambda_j\varphi_j \quad x \in \Omega, \\ \varphi_j = 0 \quad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

and $\int_\Omega \varphi_j\varphi_k dx = \delta_{j,k}$. For $u \in H_0^1(\Omega)$,write $u(x) = \sum_{j=1}^\infty \alpha_j\varphi_j(x), x \in \Omega$, the nonlocal operator $A_{1/2}$ appearing in (1.1) is defined (see [12]) as $A_{1/2}u := \sum_{j=1}^\infty \alpha_j\sqrt{\lambda_j}\varphi_j$. It has been proved in [12] that $\{\mu_j := \sqrt{\lambda_j}\}_{j=1}^\infty$ are the eigenvalues of $A_{1/2}$ on Ω with the corresponding eigenfunctions $\{\varphi_j\}_{j=1}^\infty$. The precise mathematical description and basic properties of the operator $A_{1/2}$ will be stated in Section 2.

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In this article, we assume that $f(x,0) \equiv 0$ so that (1.1) has a trivial solution $u = 0$. We will find via Morse theory nontrivial solutions to (1.1) in the situations the nonlinear term f has a linear growth and that there may not be the asymptotic limits of $f(\cdot,t)/t$ near both zero and infinity.

We impose the following assumption on the nonlinearity f:

(A1') There exist $p \in (2, 2^\star)$ and $C > 0$ such that for all $s, t \in \mathbb{R}$,

$$|f(x,s) - f(x,t)| \leqslant C(|s|^{p-2} + |t|^{p-2} + 1)|s-t|, \quad \text{uniformly for a.e. } x \in \Omega. \tag{1.4}$$

It is easy to see that (1.4) implies (1.2). Denote by $0 < \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots \to \infty$ the eigenvalues of $A_{1/2}$. We impose on f the following conditions near zero and near infinity.

(A2) There exist $\delta > 0$ and $k \geqslant 1$ such that for two adjacent eigenvalues $\mu_k < \mu_{k+1}$ of $A_{1/2}$, it holds that

$$\mu_k t^2 \leqslant 2F(x,t) \leqslant \mu_{k+1} t^2, \quad \text{for } |t| \leqslant \delta, \text{ uniformly for a.e. } x \in \Omega. \tag{1.5}$$

(A3) There exist $\delta > 0$ and $k \geqslant 1$ such that for two adjacent eigenvalues $\mu_k < \mu_{k+1}$ of $A_{1/2}$, it holds that

$$\mu_k \leqslant \frac{f(x,t)}{t} \leqslant \mu_{k+1}, \quad \text{for } 0 < |t| \leqslant \delta, \text{ uniformly for a.e. } x \in \Omega. \tag{1.6}$$

(A4) There are $\epsilon > 0$ and $M > 0$ such that for two adjacent eigenvalues $\mu_m < \mu_{m+1}$ of $A_{1/2}$, it holds that

$$\mu_m + \epsilon \leqslant \frac{f(x,t)}{t} \leqslant \mu_{m+1} - \epsilon, \quad \text{for } |t| \geqslant M, \text{ uniformly for a.e. } x \in \Omega. \tag{1.7}$$

(A5) There are $\epsilon > 0$ and $M > 0$ such that for two adjacent eigenvalues $\mu_m < \mu_{m+1}$ of $A_{1/2}$, it holds that

$$\mu_m + \epsilon \leqslant \frac{f(x,t)}{t} \leqslant \mu_{m+1}, \quad 2F(x,t) \leqslant (\mu_{m+1} - \epsilon)t^2, \tag{1.8}$$

for $|t| \geqslant M$ uniformly for a.e. $x \in \Omega$.

(A6) There are $\epsilon > 0$ and $M > 0$ such that for two adjacent eigenvalues $\mu_m < \mu_{m+1}$ of $A_{1/2}$, it holds that

$$\mu_m \leqslant \frac{f(x,t)}{t} \leqslant \mu_{m+1} - \epsilon, \quad 2F(x,t) \geqslant (\mu_m + \epsilon)t^2, \tag{1.9}$$

for $|t| \geqslant M$, uniformly for a.e. $x \in \Omega$.

Our main results are the following two theorems.

Theorem 1.1. *Assume* (A1'). *Then* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

(i) (A2), (A4) *and* $\mu_k \neq \mu_m$ *hold;*
(ii) (A2), (A5) *and* $\mu_k \neq \mu_m$ *hold.*

Theorem 1.2. *Assume* (A1'). *Then* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

(i) (A3), (A4) *and* $\mu_k \neq \mu_m$ *hold;*
(ii) (A3), (A5) *and* $\mu_k \neq \mu_m$ *hold;*
(iii) (A3), (A6) *and* $\mu_k \neq \mu_m$ *hold.*

Now we give some remarks on the conditions and conclusions. Conditions (A2) and (A3) were first introduced in [31], and (A4)–(A6) were first introduced in [29]. Conditions (A2) and (A3) mean that (1.1) may be resonant near zero between any two consecutive eigenvalues of $A_{1/2}$, and there may not be any asymptotic limits of $f(x,t)/t$ as t goes to zero. Obviously (A2) is weaker than (A3) and both of them are very general conditions when the trivial solution of (1.1) acts as a local saddle point. Condition (A4) means that (1.1) is non-resonant at infinity which contains $\lim_{|t|\to\infty} f(x,t)/t \in (\mu_m, \mu_{m+1})$ (see [18, 48]) as a special case. Condition (A5) characterizes (1.1) as resonance near infinity at μ_{m+1} from the left side, and (A6) characterizes (1.1) as resonance near infinity at μ_m from the right side.

Semilinear variational problems with resonance have attracted much attention since the appearance on the great work [26] by Landesman and Lazer in 1970. In the setting of the semilinear elliptic equation

$$\begin{gathered} -\Delta u = f(x,u) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.10}$$

one version of the Landesman-Lazer type resonance condition can be formulated as follows (see [1]),

(LL) $f(x,t) - \lambda_m t$ is bounded, and $\lim_{|t|\to\infty} \int_0^t (f(x,\tau) - \lambda_m \tau) d\tau = \pm\infty$.

The crucial feature of the (LL) condition is the boundedness of $f(x,t) - \lambda_m t$ which implies the asymptotic limit $\lim_{|t|\to\infty} f(x,t)/t = \lambda_m$. In [34] the Saddle Point Theorem was applied to (1.10) with (LL) and infinite dimensional Morse theory was applied to (1.10) with (LL) (see [18, 19, 32, 48]). The study of the Landesman-Lazer type resonance problems motivated a large number of works involving resonance under different situations. Landesman, Robinson and Rumbos [27] considered (1.10) under a generalized Landesman-Lazer resonance condition, and Robinson [35] extended the results in [27]. They treated the double resonance case in the sense that

$$\lambda_m \leqslant \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \lambda_{m+1}$$

and multiple solutions were obtained via Leray-Schauder degree when the trivial solution was nondegenerate. Costa and Magalhães [24] treated (1.10) via the linking theorem under the so-called non-quadratic condition. Su and Tang [43] used via Morse theory and critical groups at infinity to study (1.10) with resonance and the nonlinearity $g(x,t) := f(x,t) - \lambda_m t$ being unbounded and satisfying

$$\begin{gathered} \text{there exist } c_1 > 0,\ c_2 > 0,\ \theta \in (0,1), \text{ and } R > 0 \text{ such that } g(x,t)t \geqslant 0 \text{ or} \\ g(x,t)t \leqslant 0,\ c_1|t|^\theta \leqslant |g(x,t)| \leqslant c_2|t|^\theta, \text{ for all } x \in \overline{\Omega},\ |t| \geqslant R \end{gathered}. \tag{1.11}$$

Su [44] studied (1.10) with the double resonance between two consecutive eigenvalues λ_m and λ_{m+1} and obtained multiple solutions via Morse theory and critical groups when 0 is a degenerate solution of (1.10). For other works involving (1.10) with various resonance we mention the works [1, 7, 8, 17, 27, 31, 35, 37, 38, 39, 40, 42] and their references. The existence of the asymptotic limits of $f(x,t)/t$ near zero and near infinity had been required in most of the works mentioned above. Li, Perera and Su[29] first treated the existence of nontrivial solutions of (1.10) without assuming asymptotic limits of $f(x,t)/t$ near zero and infinity under the conditions in this article. However, the abstract homotopy theorem used in [29] should be modified. Therefore the analogue results for (1.10) are also new.

The fractional powers of the Laplacian, such as the square root $A_{1/2}$ of the Laplacian considered in this paper, appear in flame propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, anomalous diffusions in plasmas, and American options in finances (see [3, 25, 47]). In their well-known work [12], Cabré and Tan explored an essential characteristic of the nonlocal operator $A_{1/2}$ in the sense that it could be realized in a local manner through the notion of harmonic extension and the Dirichlet-Neumann map due to Stein([41]) on Ω. More precisely, by introducing an harmonic extension problem in a cylinder $\mathcal{C} = \Omega \times (0, \infty)$ in one more dimension, the nonlocal problem (1.1) is transformed to a local problem in the half cylinder $\mathcal{C} = \Omega \times (0, \infty)$ with mixed boundary data which has a variational structure(see Section 2). Based on the variationl framework from [12], many efforts have been made in the applications of the variational and topological methods to (1.1) with various nonlinearities in getting the existence and multiplicity and many known results for (1.10) in literature have been extended to (1.1), see [2, 4, 6, 9, 10, 11, 12, 13, 14, 16, 20, 21, 36, 45, 46, 49] and some references therein. For examples, the existence of a positive solution of (1.1) for $f(u) = |t|^{q-1}t$ with $1 < q < \frac{N+1}{N-1}$ was obtained in [12] by constrained minimization method, Tan studied in [45] the existence of a positive solution of (1.1) with critical nonlinearity case of $f(t) = \mu t + |t|^{\frac{2}{N-1}}t$ by the mountain pass theorem. In [23], nontrivial solutions and multiple solutions for (1.1) were obtained by comparing the critical groups at zero and infinity. To the authors' knowledge, there are few works in literature for (1.1) with resonance near zero or near infinity at higher eigenvalues. Thus the results in this paper are quite new in the setting of the nonlocal problem considered here.

This article is organized as follows. In Section 2 we present the functional framework related to (1.1) together with basic properties of the operator $A_{1/2}$. Then we recall some preliminary results about Morse theory and critical groups. In Section 3 we compute the critical groups at zero, and in Section 4 we compute the critical groups at infinity. Finally in Section 5 we give the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section we will give the preliminaries for the variational settings related to the nonlocal problem (1.1) and some abstract results in Morse theory.

2.1. **Variational framework.** We first recall briefly the functional framework of (1.1) given by Cabré and Tan [12]. Denote the half cylinder standing on Ω by

$$\mathcal{C} = \{(x, y) : x \in \Omega, y > 0\} = \Omega \times (0, +\infty) \subset \mathbb{R}^{N+1}_{+}$$

and its lateral boundary by

$$\partial_L \mathcal{C} = \partial\Omega \times (0, +\infty).$$

Consider the Sobolev space of functions with trace vanishing on $\partial_L \mathcal{C}$:

$$H^1_{0,L}(\mathcal{C}) = \big\{v \in L^2(\mathcal{C}) : v = 0 \text{ on } \partial_L \mathcal{C},\ \int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy < \infty\big\}.$$

Then $H^1_{0,L}(\mathcal{C})$ is a Hilbert space with the scalar product

$$\langle v, w\rangle = \int_{\mathcal{C}} \nabla v \nabla w \,dx\,dy$$

and the norm

$$\|v\| = \Big(\int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy\Big)^{1/2}.$$

From [12, Lemmas 2.4 and 2.5] we have the following embedding results.

Proposition 2.1. *The embedding from $H^1_{0,L}(\mathcal{C})$ into $L^q(\Omega)$ is continuous for all $q \in [1, 2^\star]$ and is compact for all $q \in [1, 2^\star)$. Moreover, there is $c_q > 0$ such that*

$$\Big(\int_\Omega |v(x,0)|^q dx\Big)^{1/q} \leqslant c_q \Big(\int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy\Big)^{1/2} \quad \text{for all } v \in H^1_{0,L}(\mathcal{C}). \tag{2.1}$$

We denote by tr_Ω the trace operator on $\Omega \times \{0\}$ for functions in $H^1_{0,L}(\mathcal{C})$:

$$\operatorname{tr}_\Omega v := v(\cdot, 0), \text{ for } v \in H^1_{0,L}(\mathcal{C}).$$

Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega \times \{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$, that is,

$$\mathcal{V}_0(\Omega) := \big\{u = \operatorname{tr}_\Omega v : v \in H^1_{0,L}(\mathcal{C})\big\}.$$

Then by [12, Lemma 2.10], $\mathcal{V}_0(\Omega)$ can be characterized as

$$\mathcal{V}_0(\Omega) = \big\{u \in L^2(\Omega) : u = \sum_{j=1}^\infty \alpha_j \varphi_j \text{ satisfies } \sum_{j=1}^\infty \alpha_j^2 \sqrt{\lambda_j} < +\infty\big\} \tag{2.2}$$

and the space $H^1_{0,L}(\mathcal{C})$ can be characterized as (see the proof of [12, Lemma 2.10])

$$H^1_{0,L}(\mathcal{C}) = \big\{v \in L^2(\mathcal{C}) : v(x,y) = \sum_{j=1}^\infty \alpha_j \varphi_j \exp(-\sqrt{\lambda_j} y), \sum_{j=1}^\infty \alpha_j^2 \sqrt{\lambda_j} < +\infty\big\}.$$

Where the pair $\{\lambda_j, \varphi_j\}_{j\in\mathbb{N}}$ are the eigenvalue and the corresponding eigenfunction of $-\Delta$ on Ω with zero boundary value on $\partial\Omega$, as stated in (1.3).

For a given function $u \in \mathcal{V}_0(\Omega)$, its harmonic extension v to the cylinder $\mathcal{C}$ is the weak solution of the problem

$$\begin{gathered} -\Delta v = 0 \quad \text{in } \mathcal{C}, \\ v = 0 \quad \text{on } \partial_L \mathcal{C}, \\ v = u \quad \text{on } \Omega \times \{0\}. \end{gathered} \tag{2.3}$$

The idea of the harmonic extension was introduced in the pioneering work of Caffarelli and Silvestre [15] where the fractional Laplacian in the whole space was considered.

The definition and properties of the operator $A_{1/2}$ are stated as follows.

Proposition 2.2 ([12])**.** *For $u = \sum_{j=1}^\infty \alpha_j \varphi_j \in \mathcal{V}_0(\Omega)$, there exists a unique harmonic extension v in $\mathcal{C}$ of u such that $v \in H^1_{0,L}(\mathcal{C})$, and it is given by the expansion*

$$v(x,y) = \sum_{j=1}^\infty \alpha_j \varphi_j(x) \exp(-\sqrt{\lambda_j} y), \quad \text{for all } (x,y) \in \mathcal{C}. \tag{2.4}$$

The operator $A_{1/2} : \mathcal{V}_0(\Omega) \to \mathcal{V}_0^*(\Omega)$ is given by the Dirichlet-Neumann map

$$A_{1/2} u := \frac{\partial v}{\partial \nu}\Big|_{\Omega \times \{0\}}, \tag{2.5}$$

where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$ and where ν is the unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$. We have

$$A_{1/2}u = \sum_{j=1}^{\infty} \alpha_j \sqrt{\lambda_j} \varphi_j, \tag{2.6}$$

and that $A_{1/2} \circ A_{1/2}$ is equal to $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. The inverse $A_{1/2}^{-1}$ is the unique positive square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary values on $\partial\Omega$.

Now we consider the linear eigenvalue problem

$$\begin{gathered} A_{1/2}u = \mu u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.7}$$

By the definition of $A_{1/2}$, we see that a nontrivial function $u \in \mathcal{V}_0(\Omega)$ is an eigenfunction associated to the eigenvalue μ if and only if the harmonic extension v of u to the cylinder $\mathcal{C}$ satisfies

$$\begin{gathered} -\Delta v = 0 \quad \text{in } \mathcal{C}, \\ v = 0 \quad \text{on } \partial_L \mathcal{C}, \\ \frac{\partial v}{\partial \nu} = \mu u \quad \text{on } \Omega \times \{0\}. \end{gathered} \tag{2.8}$$

We have that $\{\sqrt{\lambda_j}, \varphi_j\}_{j\in\mathbb{N}}$ are the eigenvalues and the corresponding eigenfunctions of (2.7) (see [12, Lemma 2.13]). Setting

$$\mu_j = \sqrt{\lambda_j} \text{ and } e_j(x,y) = \varphi_j(x)\exp(-\mu_j y) \quad \text{for all } j \in \mathbb{N}. \tag{2.9}$$

Then all the pairs $\{\mu_j, e_j\}_{j\in\mathbb{N}}$ satisfy (2.8), the eigenfunction sequence $\{e_j\}_{j\in\mathbb{N}}$ forms an orthogonal basis of $H^1_{0,L}(\mathcal{C})$. The eigenvalue sequence $\{\mu_j\}_{j\in\mathbb{N}}$ has the following variational characterizations:

$$\mu_1 = \min_{v \in H^1_{0,L}(\mathcal{C})\setminus\{0\}} \frac{\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy}{\int_\Omega |v(x,0)|^2 dx} = \int_{\mathcal{C}} |\nabla e_1|^2 \,dx\,dy, \tag{2.10}$$

and

$$\mu_j = \min_{v \in \mathbb{P}_j\setminus\{0\}} \frac{\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy}{\int_\Omega |v(x,0)|^2 dx} = \int_{\mathcal{C}} |\nabla e_j|^2 \,dx\,dy, \tag{2.11}$$

where

$$\mathbb{P}_j = \{v \in H^1_{0,L}(\mathcal{C}) : \langle v, e_i\rangle = 0 \text{ for } i = 1, 2, \dots, j-1\}.$$

Moreover, μ_1 is simple and $0 < \mu_1 < \mu_2 \leqslant \dots \leqslant \mu_j \leqslant \dots \to \infty$ as $j \to \infty$, and that each μ_j has finite multiplicity. For $j \in \mathbb{N}$, let τ_j be the multiplicity of μ_j, that is

$$\mu_{j-1} < \mu_j = \mu_{j+1} = \dots = \mu_{j+\tau_j-1} < \mu_{j+\tau_j}.$$

Set

$$\begin{gathered} H^-(\mu_j) = \operatorname{span}\{e_1, \dots, e_{j-1}\}, \quad H(\mu_j) = \operatorname{span}\{e_j, \dots, e_{j+\tau_j-1}\}, \\ H^+(\mu_j) = \overline{\operatorname{span}\{e_{j+\tau_j}, e_{j+\tau_j+1}, \dots, \}} = \left[H^-(\mu_j) \oplus H(\mu_j)\right]^\perp. \end{gathered}$$

Then

$$H^1_{0,L}(\mathcal{C}) = H^-(\mu_j) \oplus H(\mu_j) \oplus H^+(\mu_j). \tag{2.12}$$

Proposition 2.3. *The following variational inequalities hold:*

$$\int_{\mathcal{C}} |\nabla v|^2 \, dx\, dy \leqslant \mu_{j-1} \int_\Omega |v(x,0)|^2 dx \quad \text{for all } v \in H^-(\mu_j),$$
$$\int_{\mathcal{C}} |\nabla v|^2 \, dx\, dy = \mu_j \int_\Omega |v(x,0)|^2 dx \quad \text{for all } v \in H(\mu_j),$$
$$\int_{\mathcal{C}} |\nabla v|^2 \, dx\, dy \geqslant \mu_{j+\ell_j} \int_\Omega |v(x,0)|^2 dx \quad \text{for all } v \in H^+(\mu_j).$$

We say that a function $u \in \mathcal{V}_0(\Omega)$ is a weak solution of (1.1) if the function $v \in H^1_{0,L}(\mathcal{C})$ with $\operatorname{tr}_\Omega v = v(\cdot,0) = u$ weakly solves the extended problem

$$\begin{gathered} -\Delta v = 0 \quad \text{in } \mathcal{C}, \\ v = 0 \quad \text{on } \partial_L \mathcal{C}, \\ \frac{\partial v}{\partial \nu} = f(x, v(\cdot,0)) \quad \text{on } \Omega \times \{0\}, \end{gathered} \tag{2.13}$$

that is the function v satisfies the variational formula

$$\int_{\mathcal{C}} \nabla v \nabla \phi \, dx\, dy = \int_\Omega f(x, v(x,0))\phi(x,0) dx \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}). \tag{2.14}$$

Since f satisfies (A1), it follows by Proposition 2.1 that the functional

$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2 \, dx\, dy - \int_\Omega F(x, v(x,0)) dx, \quad v \in H^1_{0,L}(\mathcal{C}) \tag{2.15}$$

is well-defined on $H^1_{0,L}(\mathcal{C})$ and is of class C^1 with derivative given by

$$\langle \mathcal{J}'(v), \phi\rangle = \int_{\mathcal{C}} \nabla v \nabla \phi \, dx\, dy - \int_\Omega f(x, v(x,0))\phi(x,0) dx. \tag{2.16}$$

Therefore critical points of $\mathcal{J}$ are exactly weak solutions of (2.13) and then the traces of critical points of $\mathcal{J}$ are exactly weak solutions of (1.1).

We will apply Morse theory and critical groups to find critical points of $\mathcal{J}$ and the following results will be necessary.

Proposition 2.4. *Assume that* (A1') *holds. Then* $\mathcal{J} \in C^{2-0}(H^1_{0,L}(\mathcal{C}), \mathbb{R})$.

Proof. The arguments are similar to that in [5] and we sketch out them for the readers' convenience. We only need to prove $\mathcal{I}(v) = \int_\Omega F(x, v(x,0))dx$ is C^{2-0} on $H^1_{0,L}(\mathcal{C})$. For any $v, w, \phi \in H^1_{0,L}(\mathcal{C})$ with $\|\phi\| = 1$, we have by (A1'), Proposition 2.1 and Hölder inequality that

$$\begin{aligned} &|\langle \mathcal{I}'(v) - \mathcal{I}'(w), \phi\rangle| \\ &\leqslant \Big(\int_\Omega |f(x,v(x,0)) - f(x,w(x,0))|^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_\Omega |\phi(x,0)|^p dx\Big)^{1/p} \\ &\leqslant C\Big(\int_\Omega |v(x,0) - w(x,0)|^{\frac{p}{p-1}} \big(1 + |v(x,0)|^{p-2} + |w(x,0)|^{p-2}\big)^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \\ &\leqslant C\Big(\int_\Omega |v(x,0) - w(x,0)|^p dx\Big)^{1/p} \\ &\quad \times \Big(\int_\Omega \big(1 + |v(x,0)|^{p-2} + |w(x,0)|^{p-2}\big)^{\frac{p}{p-2}} dx\Big)^{\frac{p-2}{p}} \\ &\leqslant C\,(1 + \|v\| + \|w\|)^{p-2}\, \|v - w\|. \end{aligned} \tag{2.17}$$

For $\varsigma > 0$ and $\|v\| \leqslant \varsigma, \|w\| \leqslant \varsigma$, it follows from (2.17) that

$$\|\mathcal{I}'(v) - \mathcal{I}'(w)\| = \sup_{\phi \in H^1_{0,L}(\mathcal{C}), \|\phi\|=1} |\langle \mathcal{I}'(v) - \mathcal{I}'(w), \phi\rangle| \leqslant C(\varsigma)\|v - w\|,$$

where $C(\varsigma)$ is a constant depending on ς. Therefore $\mathcal{I}'$ is locally Lipschitz continuous. □

Proposition 2.5 ([23, Lemma 3.1]). *Assume that* (A1) *holds. Then any a bounded sequence* $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ *satisfying* $\mathcal{J}'(v_n) \to 0$ *as* $n \to \infty$ *has a convergent subsequence.*

2.2. **Preliminaries about Morse theory.** In this subsection we collect some abstract results on Morse theory [18, 32] for a C^1 functional defined on a Banach space X.

Let $\mathcal{J} \in C^1(X, \mathbb{R})$ and $\mathcal{K} = \{v \in X : \mathcal{J}'(v) = 0\}$. For $c \in \mathbb{R}$ we denote $\mathcal{J}^c = \{v \in X : \mathcal{J}(v) \leqslant c\}$ and $\mathcal{K}_c = \mathcal{K} \cap \{v \in X : \mathcal{J}(v) = c\}$. We say that $\mathcal{J}$ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ if any sequence $\{v_n\} \subset X$ satisfying $\mathcal{J}(v_n) \to c$ and $\mathcal{J}'(v_n) \to 0$ as $n \to \infty$ has a convergent subsequence. We say that $\mathcal{J}$ satisfies the Palais-Smale condition if $\mathcal{J}$ satisfies the Palais-Smale condition at each $c \in \mathbb{R}$.

Let v_0 be an isolated critical point of $\mathcal{J}$ with $\mathcal{J}(v_0) = c \in \mathbb{R}$, and U be a neighborhood of v_0 such that $U \cap \mathcal{K}_c = \{v_0\}$. The group $C_q(\mathcal{J}, v_0) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U \setminus \{v_0\}), q \in \mathbb{Z}$ is called the q-th critical group of $\mathcal{J}$ at v_0, where $H_*(A, B)$ denotes a singular relative homology group of the pair (A, B) with integer coefficients (see [18, 32]).

Assume that $\mathcal{J}(\mathcal{K})$ is bounded from below by $a \in \mathbb{R}$ and $\mathcal{J}$ satisfies the Palais-Smale condition at all $c \leqslant a$. The group $C_q(\mathcal{J}, \infty) := H_q(X, \mathcal{J}^a), q \in \mathbb{Z}$, is called the q-th critical group of $\mathcal{J}$ at infinity([8]).

Assume that $\mathcal{J}$ satisfies the Palais-Smale condition and $\mathcal{K}$ is a finite set containing 0. Then the critical groups of $\mathcal{J}$ at infinity and at 0 are well-defined. The basic idea of Morse theory tells us that if $\mathcal{K} = \{0\}$ then $C_q(\mathcal{J}, \infty) \cong C_q(\mathcal{J}, 0)$ for all $q \in \mathbb{Z}$. It follows that if $C_q(\mathcal{J}, \infty) \ncong C_q(\mathcal{J}, 0)$ for some $q \in \mathbb{Z}$ then $\mathcal{J}$ must have a nontrivial critical point. Therefore the basic method in applying Morse theory to find nontrivial critical points of $\mathcal{J}$ is to compute critical groups $C_q(\mathcal{J}, 0)$ and $C_q(\mathcal{J}, \infty)$.

The groups $C_q(\mathcal{J}, 0)$ can be computed partially when $\mathcal{J}$ has a local linking structure at zero.

Proposition 2.6 ([30]). *Let* $\mathcal{J} \in C^1(X, \mathbb{R})$ *satisfy the Palais-Smale condition and* $0 \in \mathcal{K}$. *Assume that* $\mathcal{J}$ *has a local linking structure at* 0 *with respect to* $X = X_0^- \oplus X_0^+$, *i.e. there exists* $\rho > 0$ *such that*

$$\mathcal{J}(v) > 0 \text{ for } v \in X_0^+,\ 0 < \|v\| \leqslant \rho, \quad \mathcal{J}(v) \leqslant 0 \text{ for } v \in X_0^-,\ \|v\| \leqslant \rho. \tag{2.18}$$

Then $C_{\ell_0}(\mathcal{J}, 0) \ncong 0$ *if* $\ell_0 = \dim X_0^- < \infty$.

The concept of local linking in Proposition 2.6 was introduced by Li and Liu [28] for the existence of nontrivial critical points. If X is a Hilbert space and $\mathcal{J}$ is of C^2 then $C_q(\mathcal{J}, 0)$ can be computed clearly provided ℓ_0 is the Morse index or augmented Morse index of $\mathcal{J}$ at 0. See [44, Proposition 2.3].

Proposition 2.7 ([8]). *Assume* $X = X_1 \oplus X_2$ *and* $\mathcal{J} \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition. If* $\mathcal{J}$ *is bounded from below on* X_2 *and is anti-coercive* *on* X_1*, i.e.* $\mathcal{J}(v) \to -\infty$ *as* $v \in X_1$ *with* $\|v\| \to \infty$*, then* $C_\ell(\mathcal{J}, \infty) \ncong 0$ *if* $\ell = \dim X_1 < \infty$*.*

The above proposition is a version of the famous Rabinowitz's Saddle Point Theorem [34, Theorem 4.6]. We regard Propositions 2.6 and 2.7 as the topological versions of corresponding linking theorems since there are no minimax values involved. Next we give two theorems about the homotopy invariance of critical groups that can be used to compute directly the critical groups at isolated critical point and infinity respectively.

Theorem 2.8 ([18, 32])**.** *Let* X *be a Hilbert space and* $\{\mathcal{J}_t \in C^{2-0}(X, \mathbb{R}) : t \in [0, 1]\}$ *be a family of functional satisfying the Palais-Smale condition. Assume that there exists an open set* U *such that* $\mathcal{J}_t$ *has a unique critical point* $v_t \in U$ *for each* $t \in [0, 1]$ *and* $t \mapsto \mathcal{J}_t$ *is continuous in* $C^1(\bar{U})$ *topology. Then* $C_q(\mathcal{J}_t, v_t)$ *is independent of* $t \in [0, 1]$*.*

Theorem 2.9 ([22])**.** *Let* X *be a Hilbert space and let* $\mathcal{J}_t \in C^1(X, \mathbb{R})$ *be a family of functionals,* $t \in [0, 1]$*. Assume that each* $\mathcal{J}_t$ *satisfies the Palais-Smale condition,* $\mathcal{J}_t'$ *and* $\partial_t \mathcal{J}_t$ *are locally Lipschitz continuous in* u*. If there exists* $a \in \mathbb{R}$ *and* $\delta > 0$ *such that for some* $C > 0$

$$\mathcal{J}_t(u) \leqslant a \;\Rightarrow\; \|\partial_t \mathcal{J}_t(u)\| \leqslant C\|u\|^2, \quad \text{for all } \; t \in [0, 1], \tag{2.19}$$

$$\mathcal{J}_t(u) \leqslant a \;\Rightarrow\; \|\mathcal{J}_t'(u)\| \geqslant \delta\|u\|, \quad \text{for all } t \in [0, 1], \tag{2.20}$$

then

$$C_q(\mathcal{J}_0, \infty) \cong C_q(\mathcal{J}_1, \infty). \tag{2.21}$$

We point out that Theorem 2.9 is a new modification of [29, Theorem 3.1] (see [33]), where the given conditions were not sufficient to ensure the existence of flow. This new version of homotopy theorem has its own meanings and can be applied to many variational problems. The proof of Theorem 2.9 has been given in [22] where another type of nonlocal variational problem was studied.

3. Critical groups at zero

In this section we compute $C_*(\mathcal{J}, 0)$ under the assumptions (A2) and (A3). We make a conventional assumption that the trivial solution 0 of (1.1) is isolated. By Proposition 2.5 we see that $\mathcal{J}$ satisfies the Palais-Smale condition over any a closed ball centered at 0. We use the following orthogonal decomposition:

$$\begin{gathered} H^1_{0,L}(\mathcal{C}) = H^-(\mu_k) \oplus H(\mu_k) \oplus H^+(\mu_k), \\ H_k = \oplus_{\mu_j \leqslant \mu_k} H(\mu_j), \quad \ell_k = \dim H_k. \end{gathered} \tag{3.1}$$

Proposition 3.1. *Assume that* (A1) *and* (A2) *hold. Then* $C_{\ell_k}(\mathcal{J}, 0) \neq 0$*.*

Proof. We will show that $\mathcal{J}$ has a local linking structure at 0 with respect to $H^1_{0,L}(\mathcal{C}) = H_k \oplus H_k^\perp$.

(i) Since H_k is finite dimensional, by (A2), Propositions 2.1 and 2.3 we can find $\rho > 0$ small such that for all $v \in H_k$ with $\|v\| \leqslant \rho$,

$$\mathcal{J}(v) \leqslant -\int_\Omega \Big(F(x, v(x, 0)) - \frac{1}{2}\mu_k |v(x, 0)|^2\Big) dx \leqslant 0. \tag{3.2}$$

(ii) For $v \in H_k^\perp$, we write $v = z + w$, where $z \in H(\mu_{k+1})$, $w \in H_{k+1}^\perp$. By Proposition 2.3 we have

$$\mathcal{J}(v) \geqslant \frac{1}{2}\Big(1 - \frac{\mu_{k+1}}{\mu_{k+2}}\Big)\|w\|^2 - \int_\Omega \Big(F(x, v(x,0)) - \frac{1}{2}\mu_{k+1}|v(x,0)|^2\Big)dx. \tag{3.3}$$

For $|v(x,0)| \leqslant \delta$, by (A2) we have

$$\int_{\{|v(x,0)|\leqslant\delta\}} \Big(F(x, v(x,0)) - \frac{1}{2}\mu_{k+1}|v(x,0)|^2\Big)dx \leqslant 0. \tag{3.4}$$

It follows from (A1) and Proposition 2.1 that for some $q \in (2, 2^\star]$,

$$\int_{\{|v(x,0)|>\delta\}} \Big(F(x, v(x,0)) - \frac{1}{2}\mu_{k+1}|v(x,0)|^2\Big)dx \leqslant C\|w\|^q. \tag{3.5}$$

Hence by (3.3)–(3.5) we have

$$\mathcal{J}(v) \geqslant \frac{1}{2}\Big(1 - \frac{\mu_{k+1}}{\mu_{k+2}}\Big)\|w\|^2 - C\|w\|^q. \tag{3.6}$$

Since $q > 2$, it follows from (3.6) that there is $\rho > 0$ small such that

$$\mathcal{J}(v) > 0, \quad \forall\, 0 < \|v\| \leqslant \rho \text{ with } w \neq 0. \tag{3.7}$$

We can choose $\rho > 0$ so small that $\|v\| \leqslant \rho \Rightarrow \|z\| \leqslant \rho \Rightarrow |z(x,0)| \leqslant \delta$ for all $x \in \Omega$. Then by (A2) we have

$$F(x, z(x,0)) - \frac{1}{2}\mu_{k+1}z^2(x,0) \leqslant 0 \quad \text{uniformly in } x \in \Omega.$$

Thus

$$\mathcal{J}(z) = -\int_\Omega \Big(F(x, z(x,0)) - \frac{1}{2}\mu_{k+1}z^2(x,0)\Big)dx \geqslant 0.$$

Let $z_* \in H(\mu_{k+1})$ be such that $0 < \|z_*\| \leqslant \rho$ and $\mathcal{J}(z_*) = 0$. Then

$$F(x, z_*(x,0)) - \frac{1}{2}\mu_{k+1}z_*^2(x,0) = 0 \quad \text{uniformly in } x \in \Omega,$$

and so

$$f(x, z_*(x,0)) = \mu_{k+1}z_*(x,0) \quad \text{uniformly in } x \in \Omega.$$

As $z_* \in H(\mu_{k+1})$, going back to (2.13), we see that z_* is a nontrivial solution for (2.13) and $z_*(\cdot,0)$ is a solution of (1.1). We conclude that there is $\rho > 0$ small such that for all $\|v\| \leqslant \rho$ with $w = 0$ and $z \neq 0$,

$$\mathcal{J}(v) = \mathcal{J}(z) = -\int_\Omega \Big(F(x, z(x,0)) - \frac{1}{2}\mu_{k+1}z^2(x,0)\Big)dx > 0. \tag{3.8}$$

Otherwise, for any $\epsilon > 0$ there exists $0 \neq z_\epsilon \in H(\mu_{k+1})$ such that $\|z_\epsilon\| < \epsilon$ and $\mathcal{J}(z_\epsilon) = 0$. Then $z_\epsilon(\cdot,0)$ is a nontrivial solution of (1.1). It contradicts the isolation of the trivial solution. It follows from (3.7) and (3.8) that

$$\mathcal{J}(v) > 0, \quad \text{for all } v \in H_k^\perp \text{ with } 0 < \|v\| \leqslant \rho.$$

Therefore $\mathcal{J}$ has a local linking structure at 0 with respect to $H_{0,L}^1(\mathcal{C}) = H_k \oplus H_k^\perp$. Since $\ell_k = \dim H_k < \infty$, it follows from Proposition 2.6 that $C_{\ell_k}(\mathcal{J}, 0) \neq 0$. □

Proposition 3.2. *Assume that* (A1') *and* (A3) *hold. Then* $C_q(\mathcal{J}, 0) = \delta_{q,\ell_k}\mathbb{F}$.

Proof. For $v \in H^1_{0,L}(\mathcal{C})$, we write $v = z + \phi + w$ and set $\hat{v} = -z + \phi + w$, where $z \in H_k$, $\phi \in H(\mu_{k+1})$ and $w \in H^{\perp}_{k+1}$. We define a family of functionals

$$\mathcal{J}_t(v) = (1-t)\mathcal{J}(v) + \frac{t}{2}\left(-\|z\|^2 + \|\phi\|^2 + \|w\|^2\right), \quad t \in [0,1]. \tag{3.9}$$

By (A1') and Proposition 2.4 we have that $\mathcal{J}_t \in C^{2-0}(H^1_{0,L}(\mathcal{C}), \mathbb{R})$ and

$$\langle \mathcal{J}_t'(v), \varphi\rangle = (1-t)\langle \mathcal{J}'(v), \varphi\rangle + t\langle \hat{v}, \varphi\rangle. \tag{3.10}$$

Now we show that there is $\rho > 0$ such that $v = 0$ is a unique critical point of $\mathcal{J}_t$ in the ball $\overline{B}_\rho(0)$ for all $t \in [0,1]$. Denote $g(x,t) = f(x,t) - \mu_{k+1}t$. Then by (A3) we have that

$$0 < -\frac{g(x,t)}{t} \leqslant \mu_{k+1} - \mu_k, \quad 0 < |t| \leqslant \delta.$$

Then for $|v(x,0)| \leqslant \delta$,

$$g(x, v(x,0))\hat{v}(x,0) \leqslant \begin{cases} 0, & \text{if } v(x,0)\tilde{v}(x,0) \geqslant 0, \\ (\mu_{k+1} - \mu_k)z^2(x,0), & \text{if } v(x,0)\tilde{v}(x,0) < 0. \end{cases} \tag{3.11}$$

Hence

$$\int_{\{|v(x,0)|\leqslant\delta\}} g(x, v(x,0))\hat{v}(x,0)dx \leqslant (\mu_{k+1} - \mu_k)\int_\Omega z^2(x,0)dx. \tag{3.12}$$

Since $\operatorname{tr}_\Omega H_k$ and $\operatorname{tr}_\Omega H(\mu_{k+1})$ are finite dimensional, there is a $\rho > 0$ such that

$$\|z\| \leqslant \rho \Rightarrow |z(x,0)| \leqslant \frac{\delta}{3}, \quad \|\phi\| \leqslant \rho \Rightarrow |\phi(x,0)| \leqslant \frac{\delta}{3}.$$

For $\|v\| \leqslant \rho$ and $|v(x,0)| > \delta$,

$$|v(x,0)| \leqslant |w(x,0)| + |\phi(x,0)| + |z(x,0)| \leqslant |w(x,0)| + \frac{2}{3}\delta,$$

and so

$$|v(x,0)| < 3|w(x,0)|, \quad |\hat{v}(x,0)| < 3|w(x,0)|.$$

Thus by (A1') we have

$$\begin{aligned} &\int_{\{|v(x,0)|>\delta\}} |g(x, v(x,0))\hat{v}(x,0)|dx \\ &\leqslant C\int_{\{|v(x,0)|>\delta\}} |v(x,0)|^{p-1}|\hat{v}(x,0)|dx \\ &\leqslant C\int_{\{|v(x,0)|>\delta\}} |w(x,0)|^p dx \leqslant C\|w\|^p. \end{aligned} \tag{3.13}$$

Now for $\|v\| \leqslant \rho$, it follows from (3.12) and (3.13) that

$$\begin{aligned} &\langle \mathcal{J}'(v), \hat{v}\rangle \\ &= \langle v, \hat{v}\rangle - \mu_{k+1}\int_\Omega v\hat{v}dx - \int_\Omega g(x, v(x,0))\hat{v}(x,0)dx \\ &\geqslant \Big(1 - \frac{\mu_{k+1}}{\mu_{k+2}}\Big)\|w\|^2 - \Big(\|z\|^2 - \mu_{k+1}\int_\Omega |z(x,0)|^2dx\Big) \\ &\quad - \int_{\{|v(x,0)|\leqslant\delta\}} g(x, v(x,0))\hat{v}(x,0)dx - \int_{\{|v(x,0)|>\delta\}} g(x, v(x,0))\hat{v}(x,0)dx \\ &\geqslant \Big(1 - \frac{\mu_{k+1}}{\mu_{k+2}}\Big)\|w\|^2 - \Big(\|z\|^2 - \mu_k\int_\Omega |z(x,0)|^2dx\Big) - C\|w\|^p. \end{aligned} \tag{3.14}$$

Therefore for $\|v\| \leqslant \rho$, take $\varphi = \hat{v}$ in (3.10), we obtain from (3.14) that

$$\begin{aligned}\langle \mathcal{J}_t'(v), \hat{v}\rangle &\geqslant (1-t)\Big[\big(1-\frac{\mu_{k+1}}{\mu_{k+2}}\big)\|w\|^2 - \Big(\|z\|^2 - \mu_k\int_\Omega |z(x,0)|^2dx\Big)\Big] \\ &\quad - (1-t)C\|w\|^p + t\|v\|^2.\end{aligned} \tag{3.15}$$

Since $p > 2$, it follows that 0 is the only critical point of $\mathcal{J}_t$ in $\overline{B}_\rho(0)$ for all $t \in [0,1]$ if $\rho > 0$ is sufficiently small. Since

$$\mathcal{J}_1(v) = \frac{1}{2}\left(-\|z\|^2 + \|\phi\|^2 + \|w\|^2\right) \tag{3.16}$$

is a C^2 functional and has 0 as a non-degenerate critical point with Morse index $\ell_k = \dim H_k$, it follows that

$$C_q(\mathcal{J}_1, 0) = \delta_{q,\ell_k}\mathbb{F}, \quad \forall q \in \mathbb{Z}. \tag{3.17}$$

By Theorem 2.8 and (3.17) we have

$$C_q(\mathcal{J}, 0) = C_q(\mathcal{J}_0, 0) \cong C_q(\mathcal{J}_1, 0) = \delta_{q,\ell_k}\mathbb{F}. \tag{3.18}$$

The proof is complete. □

4. Critical groups at infinity

In this section we compute $C_*(\mathcal{J}, \infty)$ under the corresponding assumptions (A4)–(A6). We use the following orthogonal decomposition:

$$\begin{gathered} H^1_{0,L}(\mathcal{C}) = H^-(\mu_m) \oplus H(\mu_m) \oplus H^+(\mu_m), \\ H_m = \oplus_{\mu_j \leqslant \mu_m} H(\mu_j), \quad \ell_m = \dim H_m. \end{gathered} \tag{4.1}$$

We will use C to denote various positive constants in the sequel.

Proposition 4.1. *Assume that* (A1') *and* (A4) *hold. Then* $\mathcal{J}$ *satisfies the Palais-Smale condition and* $C_q(\mathcal{J}, \infty) \cong \delta_{q,\ell_m}\mathbb{F}$.

Proof. We will apply Theorem 2.9 to prove this proposition. Set

$$\tilde{f}(x,t) = f(x,t) - (\mu_{m+1} - \epsilon)t, \quad \tilde{F}(x,t) = \int_0^t \tilde{f}(x,\zeta)d\zeta.$$

Then $\mathcal{J}$ can be rewritten as

$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \frac{1}{2}(\mu_{m+1} - \epsilon)\int_\Omega |v(x,0)|^2dx - \int_\Omega \tilde{F}(x, v(x,0))dx.$$

For $v \in H^1_{0,L}(\mathcal{C})$, we write $v = z + w$ and set $\tilde{v} = -z + w$ where $z \in H_m$ and $w \in H_m^\perp$. We define a family of functionals

$$\mathcal{J}_t(v) = (1-t)\mathcal{J}(v) + \frac{t}{2}\left(-\|z\|^2 + \|w\|^2\right), \quad t \in [0,1]. \tag{4.2}$$

By (A1') we have that $\mathcal{J}_t \in C^{2-0}(H^1_{0,L}(\mathcal{C}), \mathbb{R})$ and

$$\langle \mathcal{J}_t'(v), \varphi\rangle = (1-t)\langle \mathcal{J}'(v), \varphi\rangle + t\langle \tilde{v}, \varphi\rangle, \quad \forall v, \varphi \in H^1_{0,L}(\mathcal{C}). \tag{4.3}$$

By (A4) we have

$$0 \leqslant -\frac{\tilde{f}(x,t)}{t} \leqslant \mu_{m+1} - \mu_m - 2\epsilon, \quad \forall |t| \geqslant M,\ x \in \Omega.$$

For $|v(x,0)| \geqslant M$ we have that

$$\tilde{f}(x, v(x,0))\tilde{v}(x,0) \leqslant \begin{cases} 0, & v(x,0)\tilde{v}(x,0) \geqslant 0, \\ (\mu_{m+1} - \mu_m - 2\epsilon) z^2(x,0), & v(x,0)\tilde{v}(x,0) < 0. \end{cases}$$

Hence

$$\int_{\{|v(x,0)| \geqslant M\}} \tilde{f}(x, v(x,0))\tilde{v}(x,0)dx \leqslant (\mu_{m+1} - \mu_m - 2\epsilon) \int_\Omega |z(x,0)|^2 dx. \tag{4.4}$$

By (A1) there is $C > 0$ such that

$$\int_{\{|v(x,0)| < M\}} |\tilde{f}(x, v(x,0))\tilde{v}(x,0)| dx \leqslant C\|\tilde{v}\|. \tag{4.5}$$

Now it follows from (4.4) and (4.5) that

$$\begin{aligned} &\langle \mathcal{J}'(v), \tilde{v}\rangle \\ &= \langle v, \tilde{v}\rangle - (\mu_{m+1} - \epsilon) \int_\Omega v(x,0)\tilde{v}(x,0)dx - \int_\Omega \tilde{f}(x, v(x,0))\tilde{v}(x,0)dx \\ &\geqslant \frac{\epsilon}{\mu_{m+1}} \|w\|^2 - \Big[\|z\|^2 - (\mu_{m+1} - \epsilon) \int_\Omega |z(x,0)|^2 dx\Big] \\ &\quad - \Big(\int_{\{|v(x,0)| < M\}} + \int_{\{|v(x,0)| \geqslant M\}}\Big) \tilde{f}(x, v(x,0))\tilde{v}(x,0)dx \\ &\geqslant \frac{\epsilon}{\mu_{m+1}} \|w\|^2 - \Big[\|z\|^2 - (\mu_m + \epsilon) \int_\Omega |z(x,0)|^2 dx\Big] - C\|\tilde{v}\| \\ &\geqslant \frac{\epsilon}{\mu_{m+1}} \|w\|^2 + \frac{\epsilon}{\mu_m}\|z\|^2 - C\|\tilde{v}\| \\ &\geqslant \frac{\epsilon}{\mu_{m+1}} \|v\|^2 - C\|v\|. \end{aligned} \tag{4.6}$$

Taking $\varphi = \tilde{v}$ in (4.3), we obtain from (4.6) that

$$\langle \mathcal{J}_t'(v), \tilde{v}\rangle \geqslant (1-t) \left[\frac{\epsilon}{\mu_{m+1}} \|v\|^2 - C\|v\|\right] + t\|\tilde{v}\|^2 \geqslant C_\epsilon \|v\|^2 - C\|v\|. \tag{4.7}$$

where $C_\epsilon = \min\{1, \epsilon/\mu_{m+1}\}$. By (4.7) we see that any a Palais-Smale sequence of $\mathcal{J}_t$ must be bounded. By Proposition 2.5([23, Lemma 3.1]) we have that $\mathcal{J}_t$ satisfies the Palais-Smale condition for all $t \in [0,1]$. Moreover, it follows from (4.7) that there are $a \ll -1$ and $\delta > 0$ such that

$$\mathcal{J}_t(v) \leqslant a \Rightarrow \|\mathcal{J}_t'(v)\| \geqslant \delta\|v\|. \tag{4.8}$$

For any $a < -1$ being fixed, it always holds that

$$\mathcal{J}_t(v) \leqslant a \Rightarrow |\partial_t \mathcal{J}_t(v)| \leqslant C\|v\|^2. \tag{4.9}$$

From the definition we see that $\mathcal{J}_0(v) = \mathcal{J}(v)$ and

$$\mathcal{J}_1(v) = \frac{1}{2}(-\|z\|^2 + \|w\|^2). \tag{4.10}$$

Then $\mathcal{J}_1$ is a C^2 functional and has 0 as a unique non-degenerate critical point with Morse index $\ell_m = \dim H_m$. It follows that

$$C_q(\mathcal{J}_1, \infty) = C_q(\mathcal{J}_1, 0) \cong \delta_{q,\ell_m}\mathbb{F}, \quad \forall q \in \mathbb{Z}. \tag{4.11}$$

By Theorem 2.9 and (4.11) we have

$$C_q(\mathcal{J},\infty)=C_q(\mathcal{J}_0,\infty)\cong C_q(\mathcal{J}_1,\infty)=\delta_{q,\ell_m}\mathbb{F}. \tag{4.12}$$

The proof is complete. □

We note here that the conclusion of Proposition 4.1 is valid in the case that (A1) holds and $\lim_{|t|\to\infty}\frac{f(x,t)}{t}=\xi\in(\mu_m,\mu_{m+1})$. This is the completely non-resonant case at infinity. See [18] for a proof in abstract version. However, the arguments in [18] would not be applied to the case of Proposition 4.1.

The next two results involve with (1.1) being slight resonant near infinity from one side of an eigenvalue of $A_{1/2}$.

Proposition 4.2. *Assume that* (A1') *and* (A5) *hold. Then* $\mathcal{J}$ *satisfies the Palais-Smale condition and* $C_q(\mathcal{J},\infty)=\delta_{q,\ell_m}\mathbb{F}$.

Proof. We set $\tilde f(x,t)=f(x,t)-\mu_{m+1}t$, $\tilde F(x,t)=\int_0^t\tilde f(x,\zeta)d\zeta$ and rewrite $\mathcal{J}$ as

$$\mathcal{J}(v)=\frac12\|v\|^2-\frac12\mu_{m+1}\int_\Omega|v(x,0)|^2dx-\int_\Omega\tilde F(x,v(x,0))dx.$$

For $v\in H^1_{0,L}(\mathcal{C})$, we write $v=z+\phi+w$, where $z\in H_m$, $\phi\in H(\mu_{m+1})$, $w\in H^\perp_{m+1}$ and set $\tilde v=-z+\phi+w$. Define a family of functionals

$$\mathcal{J}_t(v)=(1-t)\mathcal{J}(v)+\frac t2(-\|z\|^2+\|\phi\|^2+\|w\|^2),\quad t\in[0,1]. \tag{4.13}$$

By (A1') we have that $\mathcal{J}_t\in C^{2-0}(H^1_{0,L}(\mathcal{C}),\mathbb{R})$ and

$$\langle\mathcal{J}_t'(v),\varphi\rangle=(1-t)\langle\mathcal{J}'(v),\varphi\rangle+t\langle\tilde v,\varphi\rangle. \tag{4.14}$$

By (A5) we have that

$$0\leqslant-\frac{\tilde f(x,t)}{t}\leqslant\mu_{m+1}-\mu_m-\epsilon,\quad\forall|t|\geqslant M,\ x\in\Omega.$$

Thus for $|v(x,0)|\geqslant M$ we have

$$\tilde f(x,v(x,0))\tilde v(x,0)\leqslant\begin{cases}0, & v(x,0)\tilde v(x,0)\geqslant0,\\(\mu_{m+1}-\mu_m-\epsilon)z^2(x,0), & v(x,0)\tilde v(x,0)<0.\end{cases} \tag{4.15}$$

Hence

$$\int_{\{|v(x,0)|\geqslant M\}}\tilde f(x,v(x,0))\tilde v(x,0)dx\leqslant(\mu_{m+1}-\mu_m-\epsilon)\int_\Omega|z(x,0)|^2dx, \tag{4.16}$$

and there is $C>0$ such that

$$\int_{\{|v(x,0)|<M\}}|\tilde f(x,v(x,0))\tilde v(x,0)|dx\leqslant C\|\tilde v\|. \tag{4.17}$$

Now it follows from (4.16), (4.17) and Proposition 2.3 that

$$\begin{aligned}\langle \mathcal{J}'(v),\tilde{v}\rangle &\geqslant \Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\|w\|^2-\Big[\|z\|^2-\mu_{m+1}\int_\Omega |z(x,0)|^2dx\Big]\\&\quad -\Big(\int_{\{|v(x,0)|<M\}}+\int_{\{|v(x,0)|\geqslant M\}}\Big)\tilde{f}(x,v(x,0))\tilde{v}(x,0)dx\\&\geqslant \Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\|w\|^2-\Big[\|z\|^2-\mu_{m+1}\int_\Omega |z(x,0)|^2dx\Big]\\&\quad -\int_\Omega(\mu_{m+1}-\mu_m-\epsilon)|z(x,0)|^2dx-C\|\tilde{v}\|\\&\geqslant \Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\|w\|^2+\frac{\epsilon}{\mu_m}\|z\|^2-C\|\tilde{v}\|.\end{aligned}\tag{4.18}$$

Taking $\varphi=\tilde{v}$ in (4.14), then we obtain from (4.18) that

$$\langle \mathcal{J}_t'(v),\tilde{v}\rangle \geqslant (1-t)\Big[\Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\|w\|^2+\frac{\epsilon}{\mu_m}\|z\|^2-C\|\tilde{v}\|\Big]+t\|\tilde{v}\|^2.\tag{4.19}$$

We prove that there exists $\delta>0$ such that for any $a\in\mathbb{R}$ fixed

$$\mathcal{J}_t(v)\leqslant a\Rightarrow \|\mathcal{J}_t'(v)\|\geqslant \delta\|v\|.\tag{4.20}$$

Arguing by contradiction, we assume that there exists $t_n\in[0,1]$, $v_n\in H^1_{0,L}(\mathcal{C})$ such that

$$\mathcal{J}_{t_n}(v_n)\to-\infty \text{ and } \|\mathcal{J}_{t_n}'(v_n)\|<\frac{1}{n}\|v_n\|,\tag{4.21}$$

this means that

$$\|v_n\|\to\infty,\quad \text{as } n\to\infty.\tag{4.22}$$

We denote

$$\hat{v}_n=\frac{v_n}{\|v_n\|}=\hat{z}_n+\hat{\phi}_n+\hat{w}_n.$$

Then $\|\hat{v}_n\|\equiv 1$. It follows from (4.21) that

$$\frac{\langle \mathcal{J}_{t_n}'(v_n),\tilde{v}_n\rangle}{\|v_n\|^2}\to 0,\quad \text{as } n\to\infty.\tag{4.23}$$

We have by (4.19) that

$$\begin{aligned}&\frac{\langle \mathcal{J}_{t_n}'(v_n),\tilde{v}_n\rangle}{\|v_n\|^2}\\&\geqslant (1-t_n)\Big[\Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\|\hat{w}_n\|^2+\frac{\epsilon}{\mu_m}\|\hat{z}_n\|^2-\frac{C}{\|\tilde{v}_n\|}\Big]+t_n.\end{aligned}\tag{4.24}$$

Since $t_n\in[0,1]$, $\|\hat{w}_n\|^2\leqslant 1$ and $\|\hat{z}_n\|^2\leqslant 1$, we may assume, up to a subsequence, that

$$t_n\to t_*\in[0,1],\quad \|\hat{z}_n\|^2\to\alpha\in[0,1],\quad \|\hat{w}_n\|^2\to\beta\in[0,1],\quad n\to\infty.\tag{4.25}$$

It follows from (4.23)–(4.25) that

$$(1-t_*)\Big[\Big(1-\frac{\mu_{m+1}}{\mu_{m+2}}\Big)\beta+\frac{\epsilon}{\mu_m}\alpha\Big]+t_*\leqslant 0.\tag{4.26}$$

It must be that $t_*=0$ and $\alpha=\beta=0$. This means that

$$\hat{z}_n\to 0,\quad \hat{w}_n\to 0,\quad \text{as } n\to\infty.$$

Since $\|\hat{v}_n\| \equiv 1$, it holds that

$$\hat{\phi}_n \to \hat{\phi} \neq 0, \quad \|\hat{\phi}\| = 1.$$

It follows that

$$\begin{aligned}
&\mathcal{J}_{t_n}(v_n) \\
&= (1-t_n)\Big(\frac{1}{2}\|v_n\|^2 - \int_\Omega F(x, v_n(x,0))dx\Big) + \frac{t_n}{2}(-\|z_n\|^2 + \|\phi_n\|^2 + \|w_n\|^2) \\
&\geqslant (1-t_n)\|v_n\|^2\Big(\frac{1}{2}\frac{\epsilon}{\mu_{m+1}}\|\hat{\phi}_n\|^2 - C(\|\hat{z}_n\|^2 + \|\hat{w}_n\|^2)\Big) - C \\
&\quad + \frac{1}{2}t_n\|v_n\|^2(-\|\hat{z}_n\|^2 + \|\hat{\phi}_n\|^2 + \|\hat{w}_n\|^2) \to \infty, \quad n \to \infty.
\end{aligned} \tag{4.27}$$

This proves (4.20). Using the same arguments above we can show that for each $t \in [0,1]$, $\mathcal{J}_t$ satisfies the Palais-Smale condition. Moreover, it is easy to see that for any $a < -1$ being fixed, it always holds that

$$\mathcal{J}_t(v) \leqslant a \Rightarrow |\partial_t \mathcal{J}_t(v)| \leqslant C\|v\|^2. \tag{4.28}$$

Since

$$\mathcal{J}_1(v) = \frac{1}{2}(-\|z\|^2 + \|\phi\|^2 + \|w\|^2) \tag{4.29}$$

is a C^2 functional and has 0 as a unique non-degenerate critical point with Morse index $\ell_m = \dim H_m$, it follows that

$$C_q(\mathcal{J}_1, \infty) = C_q(\mathcal{J}_1, 0) \cong \delta_{q,\ell_m}\mathbb{F}, \quad \forall q \in \mathbb{Z}. \tag{4.30}$$

By (4.20), (4.28), Theorem 2.9 and (4.30) we have

$$C_q(\mathcal{J}, \infty) = C_q(\mathcal{J}_0, \infty) \cong C_q(\mathcal{J}_1, \infty) = \delta_{q,\ell_m}\mathbb{F}. \tag{4.31}$$

The proof is complete. □

Proposition 4.3. *Assume that* (A1) *and* (A6) *hold. Then $\mathcal{J}$ satisfies the Palais-Smale condition and $C_{\ell_m}(\mathcal{J}, \infty) \ncong 0$.*

Proof. We will apply Proposition 2.7. We first prove that $\mathcal{J}$ satisfies the Palais-Smale condition. Although the argument is somewhat similar to that of the previous proposition, we prefer to give the details. Assume that $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ satisfies

$$|\mathcal{J}(v_n)| \leqslant C, \quad \mathcal{J}'(v_n) \to 0, \quad \text{as } n \to \infty. \tag{4.32}$$

By Proposition 2.5, we only need to prove that $\{v_n\}$ is bounded. Assume that $\|v_n\| \to \infty$ as $n \to \infty$. Set $\hat{v}_n = \frac{v_n}{\|v_n\|} = \hat{z}_n + \hat{\phi}_n + \hat{w}_n$ where $\hat{z}_n \in H_{m-1}$, $\hat{\phi}_n \in H(\mu_m)$ and $\hat{w}_n \in H_m^\perp$. Then $\|\hat{v}_n\| \equiv 1$. Set

$$\tilde{f}(x,t) = f(x,t) - \mu_m t, \quad \tilde{F}(x,t) = \int_0^t \tilde{f}(x,\zeta)d\zeta.$$

By (A6) we have

$$0 \leqslant \frac{\tilde{f}(x,t)}{t} \leqslant \mu_{m+1} - \mu_m - \epsilon, \quad \forall |t| \geqslant M,\ x \in \Omega.$$

For $v \in H^1_{0,L}(\mathcal{C})$, set $\tilde{v} = -(z+\phi) + w$. Then for $|v(x,0)| \geqslant M$, we have

$$\tilde{f}(x, v(x,0))\tilde{v}(x,0) \leqslant \begin{cases} 0, & v(x,0)\tilde{v}(x,0) < 0, \\ (\mu_{m+1} - \mu_m - \epsilon)w^2(x,0), & v(x,0)\tilde{v}(x,0) \geqslant 0. \end{cases} \tag{4.33}$$

Hence

$$\int_{\{|v(x,0)|\geqslant M\}} \tilde{f}(x,v(x,0))\tilde{v}(x,0)dx \leqslant (\mu_{m+1}-\mu_m-\epsilon)\int_\Omega |w(x,0)|^2dx. \tag{4.34}$$

There is $C>0$ such that

$$\int_{\{|v(x,0)|<M\}} |\tilde{f}(x,v(x,0))\tilde{v}(x,0)|dx \leqslant C\|\tilde{v}\|. \tag{4.35}$$

It follows from (4.34) and (4.35) that

$$\begin{aligned}
&\langle \mathcal{J}'(v),\tilde{v}\rangle \\
&= \langle v,\tilde{v}\rangle - \mu_m\int_\Omega v(x,0)\tilde{v}(x,0)dx - \int_\Omega \tilde{f}(x,v(x,0))\tilde{v}(x,0)dx \\
&= \Big(\|w\|^2 - \mu_m\int_\Omega |w(x,0)|^2dx\Big) - \Big(\|z\|^2 - \mu_m\int_\Omega |z(x,0)|^2dx\Big) \\
&\quad - \Big(\int_{\{|v(x,0)|<M\}} + \int_{\{|v(x,0)|\geqslant M\}}\Big)\tilde{f}(x,v(x,0))\tilde{v}(x,0)dx \\
&\geqslant \Big(\|w\|^2 - \mu_m\int_\Omega |w(x,0)|^2dx\Big) - \Big(\|z\|^2 - \mu_m\int_\Omega |z(x,0)|^2dx\Big) \\
&\quad - \int_\Omega (\mu_{m+1}-\mu_m-\epsilon)|w(x,0)|^2dx - C\|\tilde{v}\| \\
&\geqslant \Big(\frac{\mu_m}{\mu_{m-1}}-1\Big)\|z\|^2 + \frac{\epsilon}{\mu_{m+1}}\|w\|^2 - C\|\tilde{v}\|.
\end{aligned} \tag{4.36}$$

By (4.32) and (4.36), we obtain

$$o(\|v_n\|) = \langle \mathcal{J}'(v_n),\tilde{v}_n\rangle \geqslant \Big(\frac{\mu_m}{\mu_{m-1}}-1\Big)\|z_n\|^2 + \frac{\epsilon}{\mu_{m+1}}\|w_n\|^2 - C\|\tilde{v}_n\|. \tag{4.37}$$

Therefore,

$$\frac{o(1)}{\|v_n\|} \geqslant \Big(\frac{\mu_m}{\mu_{m-1}}-1\Big)\|\hat{z}_n\|^2 + \frac{\epsilon}{\mu_{m+1}}\|\hat{w}_n\|^2 - \frac{C}{\|v_n\|}. \tag{4.38}$$

Since $\|\hat{w}_n\|^2 \leqslant 1$ and $\|\hat{z}_n\|^2 \leqslant 1$, we assume, up to a subsequence, that

$$\|\hat{z}_n\|^2 \to \alpha \in [0,1], \quad \|\hat{w}_n\|^2 \to \beta \in [0,1], \quad n\to\infty. \tag{4.39}$$

It follows from (4.38) that

$$\Big(\frac{\mu_m}{\mu_{m-1}}-1\Big)\alpha + \frac{\epsilon}{\mu_{m+1}}\beta \leqslant 0. \tag{4.40}$$

It must be that $\alpha=\beta=0$ and thus $\hat{z}_n\to 0$ and $\hat{w}_n\to 0$ as $n\to\infty$. Since $\|\hat{v}_n\|\equiv 1$, it follows that

$$\hat{\phi}_n \to \hat{\phi} \neq 0, \quad \|\hat{\phi}\| = 1.$$

Now we have

$$\begin{aligned}
\mathcal{J}(v_n) &= \frac{1}{2}\|v_n\|^2 - \int_\Omega F(x,v_n(x,0))dx \\
&\leqslant \|v_n\|^2\Big(-\frac{1}{2}\frac{\epsilon}{\mu_m}\|\hat{\phi}_n\|^2 + C(\|\hat{z}_n\|^2+\|\hat{w}_n\|^2)\Big) + C \to -\infty
\end{aligned} \tag{4.41}$$

as $n\to\infty$. This contradicts (4.32).

Next we prove that $\mathcal{J}$ satisfies the geometrical assumptions of Proposition 2.7 with respect to $H^1_{0,L}(\mathcal{C}) = H_m \oplus H_m^\perp$. From (A6) it follows that

$$(\mu_m + \epsilon)t^2 - C \leqslant 2F(x,t) \leqslant (\mu_{m+1} - \epsilon)t^2 + C \tag{4.42}$$

for some $C > 0$. Then for $w \in H_m^\perp$,

$$\begin{aligned} \mathcal{J}(w) &\geqslant \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_\Omega (\mu_{m+1} - \epsilon)|w(x,0)|^2 dx - C \\ &\geqslant \frac{\epsilon}{2}\int_\Omega |w(x,0)|^2 dx - C \geqslant -C. \end{aligned} \tag{4.43}$$

For $z \in H_m$,

$$\begin{aligned} \mathcal{J}(z) &\leqslant \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega (\mu_m + \epsilon)|z(x,0)|^2 dx + C \\ &\leqslant -\frac{\epsilon}{2\mu_m}\|z\|^2 + C \to -\infty, \quad \|z\| \to \infty. \end{aligned} \tag{4.44}$$

As $\dim H_m = \ell_m < \infty$, we have by Proposition 2.7 that $C_{\ell_m}(\mathcal{J}, \infty) \ncong 0$. The proof is complete. □

5. Proofs of main theorems

Proof of Theorem 1.1. (i) By Proposition 3.1, we have that $C_{\ell_k}(\mathcal{J}, 0) \neq 0$. By Proposition 4.1, $\mathcal{J}$ satisfies the Palais-Smale condition and $C_q(\mathcal{J}, \infty) = \delta_{q,\ell_m}\mathbb{F}$. Since $\mu_k \neq \mu_m$ implies $\ell_k \neq \ell_m$, it follows that $C_{\ell_k}(\mathcal{J}, \infty) \ncong C_{\ell_k}(\mathcal{J}, 0)$. Therefore $\mathcal{J}$ has at least one nontrivial critical point. The case (ii) is proved in a similar way. □

Proof of Theorem 1.2. (iii) By Proposition 3.2, we have that $C_q(\mathcal{J}, 0) = \delta_{q,\ell_k}\mathbb{F}$. By Proposition 4.3, $\mathcal{J}$ satisfies the Palais-Smale condition and $C_{\ell_m}(\mathcal{J}, \infty) \ncong 0$. Since $\mu_k \neq \mu_m$ implies $\ell_k \neq \ell_m$, it follows that $C_{\ell_m}(\mathcal{J}, \infty) \ncong C_{\ell_m}(\mathcal{J}, 0)$. Therefore $\mathcal{J}$ has at least one nontrivial critical point. The other cases are proved in a similar way. □

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References

[1] S. Ahmad, A.C. Lazer, J. L. Paul; Elementary critical point theory and perturbations of elliptic boundary value problems at resonance. *Indiana Univ. Math. J.*, **25** (1976), 933–944.
[2] V. Ambrosio, G. M. Bisci, D. Repovs; Nonlinear equations involving the square root of the Laplacian. *Discrete Contin. Dyn. Syst. Ser. S*, **12** (2019), 151–170.
[3] D. Applebaum; Lévy processes – from probability to finance and quantum groups. *Notices Amer. Math. Soc.*, **51**(2004), 1336–1347.
[4] D. Arcoya, E. Colorado, T. Leonori; Asymptotically linear problems and antimaximum principle for the square root of the Laplacian. *Advanced Nonlinear Studies*, **12** (2012), 683–701.
[5] M. Badiale, E. Serra; *Semilinear Elliptic Equations for Beginners.* Springer, Berlin, 2011.
[6] B. Barrios, E. Colorado, A. De Pablo, U. Sanchez; On some critical problems for the fractional Laplacian operator. *J. Differential Equations*, **252** (2012), 6133–6162.
[7] P. Bartolo, V. Benci, D. Fortunato; Abstract critical point theorems and applications to nonlinear problems with "strong" resonance at infinity. *Nonlinear Anal.,* **7** (1983), 981–1012.
[8] T. Bartsch, S. Li; Critical point theory for asymptotically quadratic functionals and applications to problems with resonance. *Nonlinear Anal.,* **28** (1997), 419–441.
[9] C. Brädle, E. Colorado, A. de Pablo, U. Sáchez; A concave-convex elliptic problem involving the fractional Laplacian. *Proc. Roy. Soc. Edinburgh Sect. A,* **143** (2013), 39–71.

[10] X. Cabré, Y. Sire; Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31** (2014), 23–53.
[11] X. Cabré, Y. Sire; Nonlinear equations for fractional Laplacians, II: Existence, uniqueness, and qualitative properties of solutions. *Trans. Amer. Math. Soc.*, **367** (2015), 911–941.
[12] X. Cabré, J. Tan; Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.*, **224** (2010), 2052–2093.
[13] L. Caffarelli, A. Vasseur; Drift diffusion equation with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math.* **171** (2010), 903–930.
[14] L. Caffarelli, J. M. Roquejoffre, Y. Sire; Variational problems for free boundaries for the fractional Laplacian. *J. Eur. Math. Soc.*, **12** (2010), 1151–1179.
[15] L. Caffarelli, L. Silvestre; An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, **32** (2007), 1245–1260.
[16] A. Capella, J. D'avila, L. Dupaigne, Y. Sire; Regularity of radial extremal solutions for some nonlocal semilinear equations. *Comm. Partial Differential Equations*, **36** (2011), 1353–1384.
[17] A. Castro, A. C. Lazer; Critical point theory and the number of solutions of nonlinear Dirichlet problem. *Ann. Mat. Pura Appl.,* **70** (1979), 113–137.
[18] K. -C. Chang; *Infinite dimensional Morse theory and multiple solutions*. Birkhäuser-Bosten, 1993.
[19] K. -C. Chang, S. Li, J. Liu; Remarks on multiple solutions for asymptotically linear elliptic boundary value problems. *Topo. Meth. Nonl. Anal.,* **3** (1994),179–187.
[20] X. Chang, Z.-Q. Wang; Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. *Nonlinearity,* **26** (2013), 479–494.
[21] X. Chang, Z.-Q. Wang; Nodal and multiple solutions of nonlinear problems in volving the fractional Laplacian. *J. Differential Equations*, **256** (2014), 2965–2992.
[22] Y. Chen and J. Su, Nontrivial solutions for the fractional Laplacian problems without asymptotic limits near both infinity and zero. *Discrete Contin. Dyn. Syst-S.,* to appear.
[23] Y. Chen, J. Su, H. Yan; Existence and multiplicity results for nonlinear equations involving the square root of the Laplacian. *Electronic Journal of Differential Equations.,* **2017** no. 234 (2017), 1–26.
[24] D. G. Costa, C. A. Magalhães; Variational elliptic problems which are nonquadratic at infinity. *Nonlinear Anal.,* **23**, 11(1994), 1401–1412.
[25] A. Garroni, S. Müller; Γ-limit of aphase-field model of dislocations. *Siam J. Math. Anal.,* **36** (2005), 1943–1964.
[26] E. M. Landesman, A. C. Lazer; Nonlinear perturbations of linear elliptic boundary value problems at resonance. *J. Math. Mech.*, **19** (1970), 609–623.
[27] E. M. Landesman, S. Robinson, A. Rumbos; Multiple solutions of semilinear elliptic problems at resonance. *Nonlinear. Anal.*, **24** (1995), 1049–1059.
[28] S. Li, J. Liu; Some existence theorems on multiple critical points and their applications. *Kexue Tongbao*, **17** (1984), 1025–1027.
[29] S. Li, K. Perera, J. Su; Computations of critical groups in elliptic boundary value problems where the asymptotic limits may not exist. *Proc. Roy. Soc. Edinburgh Sect. A*, **131** (2001), 721–732.
[30] J. Liu; A Morse index for a saddle point. *Syst. Sci. Math. Sci.,* **2** (1989), 32–39.
[31] J. Liu, J. Su; Remarks on multiple nontrivial solutions for Quasi-Linear resonant problems. *J. Math. Anal. Appl.,* **258** (2001), 209–222.
[32] J. Mawhin, M. Willem; *Critical point theory and Hamiltonian systems*. Springer, Berlin, 1989.
[33] K. Perera, M. Schechter; Solution of nonlinear equations having asymptotic limits at zero and infinity. *Calc. Var. Partial Differential Equations,* **12** (2001), 359–369.
[34] P. Rabinowitz; *Minimax methods in critical point theory with application to differential equations.* CBMS,Vol. 65 AMS: Providence 1986.
[35] S. Robinson; Multiple solutions for semilinear elliptic boundary value problems at resonance. *Elec. J. Diff. Equa.,* **1995** no. 1(1995), 1–14.
[36] L. Silvestre; Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.,* **60** (2006), 67–112.
[37] M. Schechter; Elliptic resonance problems with inequal limits at infinity. *Math. Ann.,* **300** (1994), 629–642.

[38] M. Schechter; Asympototically linear elliptic boundary value problems. *Monatsh Math.,* **117** (1994), 121–137.
[39] M. Schechter; Bounded resonance problem for semilinear elliptic equations. *Nonlinear Anal.,* **24** (1995), 1471–1482.
[40] S. Solimini; On the solvability of some elliptic partial differential equations with the linear part at resonance. *J. Math. Anal. Appl.*, **117**(1986), 138–152.
[41] E. M. Stein; *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, vol. 30 (Princeton University Press, 1970).
[42] J. Su; Semilinear elliptic resonant problems at higher eigenvalue with unbounded nonlinear terms. *Acta Math. Sinica, N. S.*, **14** (1998), 411–419.
[43] J. Su, C. Tang; Multiplicity results for semilinear elliptic equations with resonance at higher eigenvalues. *Nonlinear Anal.,* **44** (2001), 311–321.
[44] J. Su; Semilinear elliptic boundary value problems with double resonance between two consecutive eigenvalues. *Nonlinear Anal.,* **48** (2002), 881–895.
[45] J. Tan; The Brezis-Nirenberg type problem involving the square root of the Laplacian. *Calc. Var. Partial Differential Equations,* **36** (2011), 21–41.
[46] J. Tan; Positive solutions for nonlocal elliptic problems. *Discrete Contin. Dyn. Syst.,* **33** (2013), 837–859.
[47] E. Valdinoci; From the long jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl. SMA,* **49** (2009)33–44.
[48] Z.-Q. Wang; Multiple solutions for indefinite functionals and applications to asymptotically linear problems. *Acta Math. Sinica, N.S.*, **5** (1989), 101–113.
[49] X. Yu; The Nehari manifold for elliptic equation involving the square root of the Laplacian. *J. Differential Equations,* **252** (2012), 1283–1308.

Yutong Chen
School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
Email address: chenyutong@cnu.edu.cn

Jiabao Su
School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
Email address: sujb@cnu.edu.cn

Mingzheng Sun
College of Sciences, North China University of Technology, Beijing 100144, China
Email address: suncut@163.com

Rushun Tian
School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
Email address: rushun.tian@cnu.edu.cn