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EIGENVALUES AND BIFURCATION FOR NEUMANN PROBLEMS WITH INDEFINITE WEIGHTS

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In memory of Alan Lazer with admiration

Abstract. We consider eigenvalue problems and bifurcation of positive solutions for elliptic equations with indefinite weights and with Neumann boundary conditions. We give complete results concerning the existence and non-existence of positive solutions for the superlinear coercive and non-coercive problems, showing a surprising complementarity of the respective results.

1. Introduction

This article concerns the eigenvalues for elliptic equations with an indefinite weight function

$$\begin{gathered} -\Delta u = \lambda a(x) u \quad \text{in } \Omega \subset \mathbb{R}^N \\ \mathcal{B}u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a : \Omega \to \mathbb{R}$ is a continuous and sign-changing function, and $\mathcal{B}$ denotes a homogeneous boundary condition, say Dirichlet or Neumann.

Eigenvalue problems with indefinite weights have numerous applications in engineering, physics, biology, etc.; see the recent work by Sovrano [20] concerning selection-migration models in population genetics.

The second order ODE corresponding to (1.1) has been widely studied, beginning with the work of Bôcher [2], Hilbert [12], and Richardson [18]. The first work for the Dirichlet boundary value problem in higher dimensions goes back to Holmgren (1904) [13] who considered the equation in bounded domains in two dimensions, proving the existence of a sequence of positive and a sequence of negative eigenvalues.

For recent works on such problems we cite the work by de Figueiredo [7], Hess-Kato [11] and Manes-Micheletti [16] in which the indefinite Dirichlet eigenvalue problem in $\Omega \subset \mathbb{R}^N$ was studied. They proved the existence of two sequences of eigenvalues $0 < \lambda_1^+ < \lambda_2^+ \le \dots \to +\infty$ and $0 > \lambda_1^- > \lambda_2^- \ge \dots \to -\infty$, and gave a variational min-max characterization for these eigenvalues. The aim of Manes

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and Micheletti was to generalize the assumptions for the so-called Ambrosetti-Prodi problem: considering a nonlinearity which crosses asymptotically the first eigenvalue of the Laplacian, Ambrosetti and Prodi (1972) [1] gave in their pioneering result a global description of the solutions structure of the associated Dirichlet problem, characterizing it as a *global fold mapping* between Banach spaces. To achieve this, the linearization of the nonlinear mapping needs to be controlled in every point of the domain space, which leads to the indefinite eigenvalue problems studied by Manes-Micheletti. For interesting generalizations of these methods, see [4, 5, 6, 19].

Recently, López-Gómez and Rabinowitz [15] studied bifurcation problems associated to indefinite eigenvalue problems. In particular, for the model problem

$$\begin{gathered} -\Delta u = \lambda a(x)u - |u|^{p-1}u \quad \text{in } \Omega \subset \mathbb{R}^N \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

with $1 < p$ and $a(x)$ continuous and sign-changing, they showed the existence at least k pairs of solutions for $\lambda > \lambda_k^+$, as well as for $\lambda < \lambda_k^-$, implying that all eigenvalues of equation (1.1) are also bifurcation points.

In this article we study the eigenvalue problem with Neumann boundary conditions which has been less studied.

$$\begin{gathered} -\Delta u = \lambda a(x)u \quad \text{in } \Omega \subset \mathbb{R}^N \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

There is again a positive and a negative sequence of eigenvalues, with the peculiarity that $\lambda_1^+ = 0$ if $\int_\Omega a(x)dx > 0$, and $\lambda_1^- = 0$ if $\int_\Omega a(x)dx < 0$; this implies in particular that $\lambda_1^+ = \lambda_1^- = 0$ if $\int_\Omega a(x)dx = 0$. The variational characterization and properties of these eigenvalues are given in section 2.

In section 3 we consider the bifurcation of positive solutions from the first eigenvalues $\lambda_1^\pm$ for the problems

$$\begin{gathered} -\Delta u = \lambda a(x)u \pm u^p \quad \text{in } \Omega \subset \mathbb{R}^N \\ u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

We will prove the following results, which show an interesting complementarity between problems (1.3) with $(-)$ and with and $(+)$.

Theorem 1.1. *Assume that $a \in C(\overline{\Omega})$ and sign-changing, and $p > 1$. Then equation* (1.3) *with* $(-)$ *has*

- *for $\lambda < \lambda_1^-$ and for $\lambda > \lambda_1^+$ a positive solution,*
- *for $\lambda_1^- \le \lambda \le \lambda_1^+$* no *positive solution.*

Theorem 1.2. *Assume that $a \in C(\overline{\Omega})$ and sign-changing, and $1 < p < \frac{N+2}{N-2}$. Then equation* (1.3) *with* $(+)$ *has*

- *for $\lambda \le \lambda_1^-$ and for $\lambda \ge \lambda_1^+$ no positive solution,*
- *for $\lambda_1^- < \lambda < \lambda_1^+$ a positive solution.*

The complementarity is most striking in the degenerate case $\int_\Omega a(x)dx = 0$. Then we have $\lambda_1^- = \lambda_1^+ = 0$, and hence

Corollary 1.3. *If $\int_\Omega a(x)dx = 0$:*

- *problem (1.3) with $(-)$ has a positive solution for every $\lambda \neq 0$ ($p > 1$);*
- *problem (1.3) with $(+)$ has no positive solution for every $\lambda \in \mathbb{R}$ ($1 < p < (N+2)/(N-2)$).*

2. Eigenvalue problem with indefinite weights

In this section we give a short description of the spectrum, eigenfunctions, and some of their properties for the eigenvalue problem

$$\begin{gathered} -\Delta\phi = \lambda a(x)\phi \quad \text{in } \Omega \\ \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, with $\partial\Omega$ of class C^1, and $a = a(x) \in L^\infty(\Omega)$ is a non trivial function. As mentioned in the introduction (see also Manes-Micheletti [16]), if $a = a(x)$ changes sign then there exist two sequences

(i) $\{\lambda_j^+\}$ of positive eigenvalues, with associated eigenfunctions $\{\phi_j^+\}$,
(ii) $\{\lambda_j^-\}$ of negative eigenvalues, with associated eigenfunctions $\{\phi_j^-\}$.

Manes and Micheletti [16] discussed the Dirichlet case (for more general elliptic operators). Here we focus on the Neumann case. We define the bilinear form

$$S(u,v) := \int_\Omega a(x)uv\,dx$$

Let $\mathcal{B}_+ = \{u : \ S(u,u) = 1\}$, $\mathcal{B}_- = \{u : \ S(u,u) = -1\}$.

Remark 2.1. Since $a = a(x)$ changes sign, both $\mathcal{B}_+$ and $\mathcal{B}_-$ are nonempty.

In what follows, we outline some properties of the eigen-pairs $(\lambda_j^\pm, \phi_j^\pm)$, and rephrase the variational characterization for the eigenvalues given by Manes and Micheletti [16], see also [3].

(a) (Quasi-orthogonality) If λ_* and λ^* are two different eigenvalues of (2.1), and resp. ϕ_*, ϕ^* two associated eigenvectors, then ϕ_*, ϕ^* are orthogonal

$$\int_\Omega \nabla\phi_*\nabla\phi^*\,dx = 0, \quad \int_\Omega a(x)\phi_*\phi^*\,dx = 0.$$

(b) (First eigenvalues)

$$\lambda_1^+ = \inf_{u\in\mathcal{B}_+}\int_\Omega |\nabla u|^2 dx \geq 0, \quad \lambda_1^- = -\inf_{u\in\mathcal{B}_-}\int_\Omega |\nabla u|^2 dx \leq 0$$

are simple, with associated positive eigenfunctions ϕ_1^+ and ϕ_1^-.

(c) (Higher eigenvalues) For $k \geq 2$,

$$\lambda_k^+ = \inf_{\dim F = k}\sup_{u\in\mathcal{B}_+\cap F}\int_\Omega |\nabla u|^2 dx > 0, \quad \lambda_k^- = -\inf_{\dim F = k}\sup_{u\in\mathcal{B}_-\cap F}\int_\Omega |\nabla u|^2 dx < 0,$$

or equivalently, using MM characterization,

$$\frac{1}{\lambda_k^+} = \sup_{\dim F = k}\min_{u\in F, u\neq 0}\frac{\int_\Omega a(x)u^2}{\int_\Omega |\nabla u|^2}, \quad \frac{1}{\lambda_k^-} = -\sup_{\dim F = k}\min_{u\in F, u\neq 0} -\frac{\int_\Omega a(x)u^2}{\int_\Omega |\nabla u|^2}\,. \tag{2.2}$$

(d) $\lambda_k^+ \to +\infty$ and $\lambda_k^- \to -\infty$ as $k \to +\infty$.

(e) (Positivity of first eigenfunctions) The eigenfunctions corresponding to the first eigenvalues have constant sign. Moreover, the eigenvalues $\lambda \neq \lambda_1^{\pm}$ do not posses a positive eigenfunction.

The same results occur in both the Dirichlet and the Neumann case, but a distinction is a must: while for the Dirichlet case the inequality for the first eigenvalues $\lambda_1^{\pm}$ is strict, i.e. $\lambda_1^- < 0 < \lambda_1^+$, we have that $\lambda = 0$ is always an eigenvalue in the Neumann case. Indeed, for Neumann boundary conditions, when $\int_\Omega a(x)dx = 0$, then both first eigenvalues λ_1^+ and λ_1^- coincide with zero. On the other hand, when the mass of the sign-changing weight $a(x)$ is unbalanced (say, $\int_\Omega a(x)dx < 0$), we still have that $\lambda_1^- = 0$ fits in the characterization described above. Roughly speaking, if the negative part is dominant, the first "negative" eigenvalue is the trivial one, as stated in the following proposition.

Proposition 2.2 (Neumann case)**.** *Let $a : \overline{\Omega} \to \mathbb{R}$ be a continuous, sign changing function.*

(1) *If $\int_\Omega a(x)\,dx < 0$ (resp. > 0), then*
$$\lambda_1^- = 0 \text{ and } \lambda_1^+ > 0 \quad \big(\text{resp. } \lambda_1^- < 0 \text{ and } \lambda_1^+ = 0\big).$$

(2) *If $\int_\Omega a(x)\,dx = 0$, then*
$$\lambda_1^- = 0 = \lambda_1^+.$$

Proof. (1) The first statement is trivial. Indeed, $\lambda_1^- := -\inf_{u\in\mathcal{B}_-}\int_\Omega |\nabla u|^2 dx = 0$, since the infimum is attained by the constant function $u(x) = \alpha$, where α satisfies
$$\alpha^2 = -\frac{1}{\int_\Omega a(x)\,dx}.$$
For λ_1^+ we argue by contradiction. If $\lambda_1^+ := \inf_{u\in\mathcal{B}_+}\int_\Omega |\nabla u|^2 dx = 0$, there is a sequence $u_n = w_n + s_n$, with $\int_\Omega w_n = 0$ and $s_n \in \mathbb{R}$ such that
$$\int_\Omega a(x)u_n^2 = 1, \quad \int_\Omega |\nabla w_n|^2 dx \to 0, \quad \text{as } n \to +\infty.$$
Therefore $w_n \to 0$ strongly in $H^1(\Omega)$ and s_n is bounded: otherwise we would have (up to subsequences)
$$1 = \int_\Omega a(x)u_n^2 = \int_\Omega a(x)(s_n^2 + 2w_n s_n + w_n^2)dx = s_n^2\Big(\int_\Omega a(x)\,dx + o(1)\Big) \to -\infty.$$
Since s_n is bounded, up to subsequences, $s_n \to s$ and $u_n \to s$ strongly, from which we obtain
$$1 = \int_\Omega a(x)u_n^2 \to s^2 \int_\Omega a(x) \le 0,$$
which is a contradiction.

(2) Let $\Omega^+ = \{x \in \Omega : \ a(x) > 0\}$ and $B \subset\subset \Omega^+$ a ball. Let $v_\epsilon(x) = 1 + \epsilon\,\eta$, where $\eta \in C_0^\infty(\Omega)$ is a positive smooth function with compact support in B, and $\epsilon > 0$. Then
$$\frac{\int_\Omega |\nabla v_\epsilon|^2}{\int_\Omega a(x)v_\epsilon^2} = \frac{\epsilon^2\int_\Omega |\nabla \eta|^2}{\int_\Omega a(x)(1+\epsilon\eta)^2} = \frac{\epsilon^2\int_\Omega |\nabla \eta|^2}{\epsilon\int_\Omega a(x)(2\eta + \epsilon\eta^2)} \to 0, \quad \text{as } \epsilon \to 0.$$
This proves that $\lambda_1^+ = 0$. With a similar argument, taking a smooth function with support in Ω^- we obtain that $\lambda_1^- = 0$. $\square$

3. Superlinear equations - bifurcation of positive solutions

We now consider superlinear equations of the form

$$
\begin{gathered}
-\Delta u = \lambda a(x) u \pm h(u) \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{3.1}
$$

where $a : \overline{\Omega} \to \mathbb{R}$ is continuous and sign-changing, and $h \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a superlinear function. As a model function we will consider $h(s) = s^p, p > 1$, but the results will remain valid for a large class of superlinear nonlinearities.

We investigate bifurcation results when the parameter λ crosses the eigenvalues $\lambda_1^{\pm}$. We obtain a rather complete picture of existence and non-existence of solutions. We will see (see Figures 1 and 2) that the existence and non-existence results for the equations with $-u^p$, resp. $+u^p$, have a completely complementary behavior: for λ's for which there exists a solution for (1.3) with $(-)$ there exists no solution for (1.3) with $(+)$, and vice versa.

3.1. Existence and non-existence of solutions for problem (1.3) with $(-)$.

We remark that the corresponding Dirichlet problem has been treated by López-Gómez and Rabinowitz [15], emphasizing on the existence of a growing number of (pairs of) solutions for increasing $|\lambda|$. Here we consider the Neumann problem, which presents some peculiarities, and we restrict attention to the existence and non-existence of positive solutions.

Let us now consider the model problem (1.3) with $(-)$:

$$
\begin{gathered}
-\Delta u = \quad \lambda a(x) u - u^p \quad \text{in } \Omega \\
u > 0 \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega
\end{gathered}
\tag{3.2}
$$

First, we prove a non-existence result.

Theorem 3.1. *Let $a = a(x) \in C^0(\overline{\Omega})$, and suppose that $a(x)$ changes sign. By Proposition 2.2, we have that $\lambda_1^- \le \lambda_1^+$. Then, for every $\lambda \in [\lambda_1^-, \lambda_1^+]$, the problem* (3.2) *has no non-trivial solution.*

Proof. If $\int_\Omega a(x)dx = 0$, then $[\lambda_1^-, \lambda_1^+] = \{0\}$ and the assertion is trivial. Let $\int_\Omega a(x)dx < 0$ (the other case is similar); suppose that u is a positive solution of (1.3) with $(-)$. Multiplying by u and integrating we obtain

$$
\int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega a(x) u^2 + \int_\Omega u^{p+1} \, dx = 0.
$$

Now, if $\lambda \int_\Omega a(x) u^2 \le 0$ the assertion is obvious. If not, $\int_\Omega a(x) u^2 > 0$, and using the characterization of λ_1^+ we obtain

$$
0 = \int_\Omega |\nabla u|^2 \, dx - \lambda \int_\Omega a(x) u^2 + \int_\Omega u^{p+1} \, dx \ge \big(1 - \frac{\lambda}{\lambda_1^+}\big) \int_\Omega |\nabla u|^2 + \int_\Omega u^{p+1} \, dx > 0.
$$

□

Next, we show that for λ outside of the interval $[\lambda_1^-, \lambda_1^+]$, problem (3.2) has always a positive solution.

Theorem 3.2. *For every $\lambda > \lambda_1^+$ or $\lambda < \lambda_1^-$ the Neumann problem* (3.2) *has at least one positive solution. In particular, if $\int_\Omega a(x) = 0$, then for every $\lambda \neq 0$ the problem has a positive solution.*

Proof. We proceed by steps. First, we observe that the solutions of (3.2) correspond to critical points of the functional

$$\Phi_\lambda : H \to \mathbb{R}, \quad \Phi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{p+1}\int_\Omega u^{p+1} - \frac{\lambda}{2}\int_\Omega a(x)u^2 \tag{3.3}$$

where $H := H^1(\Omega)$.

Actually, the functional is well defined on H only for $1 < p+1 \le \frac{2N}{N-2}$. This is not an obstacle. Lemma 3.3 below provides an *a priori* estimate, which allows to use the following functional instead of (3.3):

$$\widetilde{\Phi}_\lambda : H \to \mathbb{R} \quad , \quad \tilde{\Phi}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega G(u) - \frac{\lambda}{2}\int_\Omega a(x)u^2.$$

where

$$G(s) = \begin{cases} \frac{s^{p+1}}{p+1}, & \text{if } 0 \le s \le C_\lambda \\ \frac{p}{p+1}C_\lambda^{p-1}s^2 - \frac{p-1}{p+1}C_\lambda^p s & \text{if } s > C_\lambda \\ 0 & \text{if } s < 0, \end{cases}$$

and $C_\lambda := \big(|\lambda|\, \|a\|_\infty\big)^{\frac{1}{p-1}}$, as suggested by the next lemma.

Lemma 3.3. *All solutions of* (3.2) *and of*

$$\begin{gathered} -\Delta u = \lambda a(x)u - g(u) \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.4}$$

where $g(s) := G'(s)$, satisfy the estimate

$$0 \le u(x) \le \big(|\lambda|\, \|a\|_\infty\big)^{\frac{1}{p-1}} := C_\lambda, \quad x \in \Omega$$

Proof. If $u \in H$ is a weak solution of (3.2), it satisfies

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega \big(u^p - \lambda a(x)u\big)v \, dx = 0 \quad \forall v \in H.$$

Take $v = (u - C_\lambda)^+$, where $w^+(x) = \max\{0, w(x)\}$, and $\Omega_\lambda^+ = \{x \in \Omega : u > C_\lambda\}$. We have

$$\int_{\Omega_\lambda^+} \nabla u \nabla (u - C_\lambda)^+ \, dx = -\int_{\Omega_\lambda^+} u\left(u^{p-1} - \lambda a(x)\right)(u - C_\lambda)^+ \, dx.$$

i.e.

$$\int_\Omega |\nabla (u - C_\lambda)^+|^2 \, dx = -\int_{\Omega_\lambda^+} u\left(u^{p-1} - \lambda a(x)\right)(u - C_\lambda)^+ \, dx \le 0$$

since $u^{p-1} > \|a\|_\infty |\lambda|$ and $u > 0$ on Ω_λ^+. This proves that $u \le C_{a\lambda}$.

In a similar way, if u is a weak solution of (3.4), it satisfies

$$\int_\Omega \nabla u \nabla v \, dx + \int_\Omega (G'(u) - \lambda a(x)u)v \, dx = 0 \quad \forall v \in H.$$

For $v = (u - C_\lambda)^+$ it holds

$$
\begin{aligned}
&\int_\Omega |\nabla (u - C_\lambda)^+|^2\,dx \\
&= -\int_{\Omega_\lambda^+} \big[\frac{2p}{p+1} C_\lambda^{p-1} u - \frac{p-1}{p+1} C_\lambda^{p} - \lambda a(x) u\big](u - C_\lambda)^+\,dx \\
&\le -\int_{\Omega_\lambda^+} \big[\frac{2p}{p+1} C_\lambda^{p-1} u - \frac{p-1}{p+1} C_\lambda^{p-1} u - \lambda a(x) u\big](u - C_\lambda)^+\,dx \\
&= -\int_{\Omega_\lambda^+} [C_\lambda^{p-1} - \lambda a(x)] u (u - C_\lambda)^+\,dx \le 0,
\end{aligned}
$$

so that $u \le C_\lambda$. □

Thus, all positive critical points of $\widetilde{\Phi}_\lambda$ will satisfy $0 \le u \le C_\lambda$, and will hence be also critical points of Φ_λ, and thus solutions of (3.2). In the next proposition we show that for $\lambda \notin [\lambda_1^-, \lambda_1^+]$ the functional $\widetilde{\Phi}_\lambda$ has a negative minimum.

Proposition 3.4. *Let $\lambda > \lambda_1^+$ or $\lambda < \lambda_1^-$. Then*

$$
-\infty < \inf_{u\in H} \tilde{\Phi}_\lambda(u) < 0 \tag{3.5}
$$

Proof. We first prove that the functional is coercive (this proves the first inequality). Indeed, we have

$$
\begin{aligned}
&\tilde{\Phi}_\lambda(u_n) \\
&\ge \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \int_\Omega G(u_n) - \frac{\lambda}{2}\int_\Omega a(x) u_n^2 \\
&\ge \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \int_{[u_n \ge C_\lambda]} G(u_n) - \frac{|\lambda|}{2}\|a\|_\infty \int_\Omega |u_n|^2 \\
&= \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{p}{p+1} C_\lambda^{p-1} \int_{[u_n \ge C_\lambda]} u_n^2 - \frac{p-1}{p+1} C_\lambda^{p} \int_{[u_n \ge C_\lambda]} u_n - \frac{|\lambda|}{2}\|a\|_\infty \int_\Omega u_n^2 \\
&\ge \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \big(\frac{p}{p+1} - \frac{1}{2}\big)|\lambda|\,\|a\|_\infty \int_\Omega u_n^2 - c\Big(\int_\Omega u_n^2\Big)^{1/2} \\
&\quad - \frac{p}{p+1} C_\lambda^{p-1} \int_{[0 < u_n < C_\lambda]} u_n^2 \\
&\ge \min\Big\{\frac{1}{2}, \frac{p-1}{p+1}|\lambda|\|a\|_\infty\Big\}\|u_n\|_{H^1}^2 - c\|u_n\|_{H^1} - c_1.
\end{aligned}
$$

For the second inequality we treat first the case $\int_\Omega a \ne 0$. Suppose $\int_\Omega a(x)dx < 0$ (the case $\int_\Omega a(x)dx > 0$ is similar). For $\lambda > \lambda_1^+ (> 0)$, it is sufficient to evaluate $\widetilde{\Phi}_\lambda$ on $u_\epsilon = \epsilon\phi_1^+$, where ϕ_1^+ is the eigenfunction associated to λ_1^+ and $\epsilon > 0$ is small.

$$
\begin{aligned}
\widetilde{\Phi}_\lambda(u_\epsilon) &= \frac{\epsilon^2}{2}\int_\Omega |\nabla\phi_1^+|^2 + \frac{\epsilon^{p+1}}{p+1}\int_\Omega |\phi_1^+|^{p+1} - \frac{\epsilon^2\lambda}{2}\int_\Omega a(x)|\phi_1^+|^2 \\
&= \frac{\epsilon^2}{2}\Big(1 - \frac{\lambda}{\lambda_1^+}\Big)\int_\Omega |\nabla\phi_1^+|^2 + o(\epsilon^2) < 0, \quad \text{for } \epsilon \text{ small.}
\end{aligned}
$$

For $\lambda < \lambda_1^- = 0$ it is sufficient to evaluate $\widetilde{\Phi}_\lambda$ on $u_\epsilon = \epsilon$, ϵ small.

The degenerate case $\int_\Omega a(x) = 0$ requires a special treatment. Recall that in this case $\lambda_1^- = \lambda_1^+ = 0$. It is not sufficient to evaluate the functional on the (constant) first eigenfunction to obtain the second inequality in (3.5). Indeed, we need to evaluate $\widetilde{\Phi}_\lambda$ on a more suitable function. Since a is continuous (and sign-changing), it follows that there exists a ball $B_r(\bar{x}) \subset \Omega$ with $a(x) > 0$ on $B_r(\bar{x})$. Let $\eta \in C_0^1(\Omega)$ a positive function with supp $\eta = \overline{B}_r(\bar{x})$.

First we consider the case $\lambda > 0$. We evaluate $\widetilde{\Phi}_\lambda$ on $v_\varepsilon = \varepsilon\big(1 + \varepsilon^{\frac{p-1}{2}}\eta\big)$ (with $\varepsilon > 0$ small).

$$\begin{aligned}
\widetilde{\Phi}_\lambda(v_\varepsilon) &= \frac{\varepsilon^2}{2}\int_\Omega |\nabla\big(1+\varepsilon^{\frac{p-1}{2}}\eta\big)|^2 + \frac{\varepsilon^{p+1}}{p+1}\int_\Omega |1+\varepsilon^{\frac{p-1}{2}}\eta|^{p+1} \\
&\quad - \frac{\varepsilon^2\lambda}{2}\int_\Omega a(x)(1+\varepsilon^{\frac{p-1}{2}}\eta)^2 \\
&= \frac{\varepsilon^{p+1}}{2}\int_\Omega |\nabla\eta|^2 + \frac{\varepsilon^{p+1}}{p+1}\int_\Omega |1+\varepsilon^{\frac{p-1}{2}}\eta|^{p+1} - \varepsilon^{2+\frac{p-1}{2}}\lambda\int_\Omega a(x)\eta \\
&\quad - \frac{\varepsilon^{p+1}\lambda}{2}\int_\Omega |\eta|^2 \\
&= -\varepsilon^{2+\frac{p-1}{2}}\lambda\int_\Omega a(x)\eta + O(\varepsilon^{p+1}) < 0
\end{aligned}$$

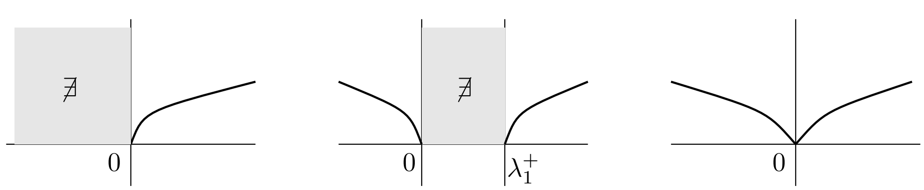
for ε small, since $\int_\Omega a(x)\eta > 0$.

For the case $\lambda < 0$, we change η with $-\eta$. □

The proof of Theorem 3.2 is now easily completed, observing that the infimum of $\tilde{\Phi}_\lambda < 0$ is attained, since $\tilde{\Phi}_\lambda$ is weakly lower semi-continuous. □

We can summarize the solution situation of Theorems 3.1 and 3.2 in the following bifurcation diagrams. We recall that variational methods do not yield continuous branches of solutions, so the figures are (possibly) a simplification.

The first plot on the left shows the standard bifurcation diagram when $a(x)$ is a positive weight. The plot in the middle gives the situation when $a(x)$ changes sign (with $\int_\Omega a < 0$). We see that there is a bounded interval with non-existence of positive solution, while there is existence everywhere else. Finally, the plot on the right illustrates that in the degenerate case, that is for a sign-changing weight $a(x)$ with $\int_\Omega a(x) = 0$, we have a positive solution for every $\lambda \neq 0$. It is interesting to note that from $0 = \lambda_1^- = \lambda_1^+$ emanate *two bifurcation branches*, albeit the corresponding eigenspace is one-dimensional, spanned by the constant 1.



$a(x)$ definite, $a(x) > 0$ $a(x)$ indefinite, $\int_\Omega a(x) < 0$ $a(x)$ indefinite, $\int_\Omega a(x) = 0$

FIGURE 1. Bifurcation diagram for equation (3.2)

3.2. **Existence and non-existence of solutions for problem** (1.3) **with** (+)**.** We now consider the problem

$$\begin{gathered} -\Delta u = \lambda a(x)u + u^p \quad \text{in } \Omega \\ u > 0 \quad \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a = a(x)$ is a sign changing continuous function, and $p > 1$. We first state the following existence result.

Theorem 3.5. *Assume that $a(x)$ changes sign, and that $1 < p < \frac{N+2}{N-2}$. If $\lambda \in (\lambda_1^-, \lambda_1^+) := I_a$, then problem* (3.6) *has a positive solution.*

Remark 3.6. Recall that

$$\begin{gathered} I_a = (0, \lambda_1^+) \quad \text{if } \int_\Omega a(x)dx < 0, \\ I_a = (\lambda_1^-, 0) \quad \text{if } \int_\Omega a(x)dx > 0, \\ I_a = \emptyset \quad \text{if } \int_\Omega a(x)dx = 0\,. \end{gathered}$$

Proof. We prove the existence result for $\int_\Omega a(x) < 0$ via a variational approach. The proof for $\int_\Omega a(x) > 0$ is similar.

Let us observe that weak solutions of (3.6) correspond to critical points of the functional

$$\Psi_\lambda : H^1(\Omega) \to \mathbb{R}, \quad \Psi_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega u^{p+1} - \frac{\lambda}{2}\int_\Omega a(x)u^2.$$

For $\lambda \in (0, \lambda_1^+)$ we can apply the classical Mountain Pass theorem of Ambrosetti-Rabinowitz, and we first need to prove some geometric estimates. Theorem 3.5 then follows in a standard way, since we have compactness due to the subcritical growth.

First, we prove that the functional Ψ_λ has a mountain-pass geometry.

Proposition 3.7 (0 is a local minimum)**.** *Assume $\int_\Omega a(x)dx < 0$ and $\lambda \in (0, \lambda_1^+)$. Then there exist $\eta > 0$ and $\rho > 0$ such that*

$$\Psi_\lambda(u) \geq \eta > 0 \quad \forall u : \|u\| = \rho.$$

Proof. It is sufficient to prove that there exists $\delta > 0$ such that

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega a(x)u^2 \geq \delta > 0, \quad \forall u : \|u\| = 1. \tag{3.7}$$

Indeed, if (3.7) holds, then, thanks to the compact embedding $H^1 \subset\subset L^{p+1}$,

$$\Psi_\lambda(\rho u) \geq \delta\rho^2 - C\rho^{p+1} \geq \frac{\delta}{2}\rho^2 := \eta,$$

for a suitable ρ small.

Note first that $J(u) \geq 0$ for all $u \in H^1$. Indeed, if $\int_\Omega a(x)u^2dx \leq 0$ this is trivial. Otherwise we use the variational characterization of λ_1^+:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega a(x)u^2 \geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1^+}\Big)\int_\Omega |\nabla u|^2 \geq 0.$$

We now prove (3.7) by contradiction: suppose that there exists a sequence $\{u_n\}$ such that

$$\|u_n\| = 1 \quad \text{and} \quad J(u_n) \to 0^+ .$$

Split $u_n = w_n + \alpha_n$ where $\alpha_n = \frac{\int_\Omega a(x)u_n(x)dx}{\int_\Omega a(x)dx}$, so that $\int_\Omega a(x)w_n(x)dx = 0$. We have $\alpha_n \to 0$, since otherwise, up to subsequence, $|\alpha_n| \geq \delta > 0$, for some positive δ, and

$$J(u_n) = J(w_n) + \frac{\lambda\alpha_n^2}{2}\Big|\int_\Omega a(x)\Big| \geq \frac{\lambda\delta^2}{2}\Big|\int_\Omega a(x)\Big|$$

Thus we have $1 = \|u_n\|^2 = \|w_n\|^2 + o(1)$.

Then there is $\eta > 0$ such that $\int_\Omega |\nabla w_n|^2 \geq \eta > 0$. If not, up to subsequences, $\int_\Omega |\nabla w_n|^2 \to 0$ and (w_n is bounded in H^1) and $w_n \to w$ in $L^2(\Omega)$, and hence there exists w such that

$$w_n \rightharpoonup w \quad \text{weakly in } H^1 \text{ and strongly in } L^2$$

In particular $\|\nabla w\|_2 = 0$, so that w is a constant of norm 1. But this leads to a contradiction, since

$$0 = \int_\Omega a(x)w_n \to w\int_\Omega a(x) \ \neq 0 .$$

Now, since $\int_\Omega |\nabla w_n|^2 \geq \eta > 0$, we can use again the argument above to obtain

$$o(1) = J(u_n) = \frac{1}{2}\int_\Omega |\nabla w_n|^2 - \frac{\lambda}{2}\int_\Omega a(x)w_n^2 + \frac{\lambda\alpha_n^2}{2}\Big|\int_\Omega a(x)\Big| \geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1^+}\Big)\eta,$$

which is a contradiction. Hence, (3.7) holds. □

To complete the geometric requirements of the Mountain Pass Theorem, we need to find a function $\overline{u}$ such that $\Psi_\lambda(\overline{u}) < 0$. But this is trivial: it is sufficient to evaluate Ψ_λ on constant functions α:

$$\Psi_\lambda(\alpha) = -\frac{1}{p+1}\int_\Omega \alpha^{p+1} - \frac{\lambda}{2}\int_\Omega a(x)\alpha^2 \to -\infty, \quad |\alpha| \to +\infty.$$

Finally, we can apply the mountain-pass (MP) theorem of Ambrosetti & Rabinowitz and find a non trivial solution of problem (1.3). This completes the proof of Theorem 3.5. □

Remark 3.8. We note that by means of minimization arguments concerning the ground state level given by the MP-Theorem, the positivity of the solution is standard (see e.g. [21]).

Next, we turn to non-existence results for equation (3.6).

Theorem 3.9. *Suppose that $a(x)$ changes sign, that $1 < p < \frac{N+2}{N-2}$, and assume that $\lambda \notin (\lambda_1^-, \lambda_1^+)$. Then equation* (3.6) *has no positive solution.*

Proof. First we note that for $\lambda = 0$ there is no positive solution in any case.

Suppose that $\int_\Omega a(x)dx \leq 0$. We show that then the problem (3.6) has no positive solutions for $\lambda \geq \lambda_1^+$ or $\lambda < 0$.

(a) Consider first $\lambda > \lambda_1^+ \geq 0$: suppose by contradiction that u is a positive solution of (P^+). We may read u as a positive eigenfunction (associated with the eigenvalue λ) of the problem

$$\begin{aligned} -\Delta\psi &= \lambda b(x)\psi, \quad \text{in } \Omega \\ \frac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

where $b(x) = a(x) + \frac{u^{p-1}}{\lambda} > a(x)$ in Ω.

If $\int_\Omega b(x)\,dx \geq 0$ (this is the case, for instance, when $\int_\Omega a(x)dx = 0$) we are done: from property (e), the unique positive eigenfunctions are related to $\lambda_1^-(b)$ and $\lambda_1^+(b)$, where $\lambda_1^-(b) \leq \lambda_1^+(b) = 0$.

If $\int_\Omega b(x)dx < 0$, the unique positive eigenfunctions are related to $0 = \lambda_1^-$ or $\lambda_1^+(b)$, so it must be $\lambda = \lambda_1^+(b)$. But since $b(x) > a(x)$, we have the inclusion $B^+(a) \subset B^+(b)$, where

$$B^+(a) = \{v \in H^1 : \int_\Omega a(x)v^2 > 0\}, \quad B^+(b) = \{v \in H^1 : \int_\Omega b(x)v^2 > 0\}.$$

Therefore,

$$\lambda = \lambda_1^+(b) = \inf_{B^+(b)} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega b(x)v^2} \leq \inf_{B^+(a)} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega a(x)v^2} = \lambda_1^+\,,$$

so that $\lambda \leq \lambda_1^+$. To exclude the case $\lambda = \lambda_1^+$, observe that both infima are actually minima, and hence $\lambda_1^+(b) < \lambda_1^+$.

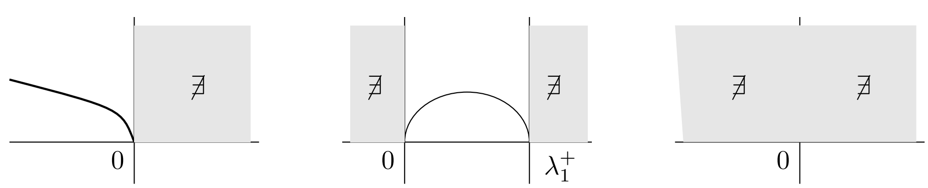
(b) Consider $\lambda < 0$: suppose by contradiction that u is a positive solution of (P^+). Again we may read u as a positive eigenfunction (associated to the eigenvalue λ) of the problem

$$\begin{aligned} -\Delta\psi &= \lambda b(x)\psi, \quad \text{in } \Omega \\ \frac{\partial\psi}{\partial\nu} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.9}$$

where $b(x) = a(x) + \frac{u^{p-1}}{\lambda} \leq a(x)$ in Ω, with $\int_\Omega b(x)dx < \int_\Omega a(x)dx \leq 0$.

Since $\int_\Omega b(x)dx < 0$, the unique positive eigenfunctions are related to $\lambda_1^- = 0$ or $\lambda_1^+(b) > 0$, so it must be $\lambda \geq 0$. The case $\int_\Omega a(x)dx > 0$ is handled similarly. $\square$

Again we can summarize Theorems 3.5 and 3.9 in the following bifurcation diagrams.



$a(x)$ definite, $a(x) > 0$ $a(x)$ indefinite, $\int_\Omega a(x) < 0$ $a(x)$ indefinite, $\int_\Omega a(x) = 0$

Figure 2. Bifurcation diagram for equation (3.6)

In the first plot on the left we show the situation for weights $a(x) > 0$. We see that it is complementary to the situation in Figure 1: the branch covers now the negative half-line of the λ parameters. The plot in the middle shows the situation for sign-changing weights $a(x)$, with $\int_\Omega a(x)dx < 0$. Now there exist solutions for every λ between the two first eigenvalues $\lambda_1^- = 0$ and λ_1^+, and no solution for all other λ's. Again, we see that the situation is complementary to the situation in Theorems 3.1 and 3.2. We have drawn the branch as a curve connecting λ_1^- and λ_1^+. This is justified for $1 < p < \frac{N}{N-2}$ by Theorem 3.10 which gives an *a priori* bound for all positive solutions for λ in a bounded interval. For $\frac{N}{N-2} \le p < \frac{N+2}{N-2}$ we do not have currently a proof of such a bound, we refer however to the proofs of such bounds for related equations with Dirichlet boundary conditions by de Figueiredo-Lions-Nussbaum [8] and Gidas-Spruck [10]. Finally, for the plot on the right we have the surprising result that for the degenerate case $\int_\Omega a(x)dx = 0$ we have *no positive solution*, for any $\lambda \in \mathbb{R}$; again, this is complementary to the situation of Theorems 3.1 and 3.2, where we have existence of a positive solution for all $\lambda \neq 0$.

3.3. **A priori bound for positive solutions of equation** (3.6)**.** In Figure 2 we have drawn a solution curve connecting the first eigenvalues λ_1^- and λ_1^+. This is justified by the following *a priori* bounds for positive solutions of equation (1.3), in the case that $1 < p < N/(N-2)$.

Theorem 3.10. *Let $1 < p < \frac{N}{N-2}$. Then for every $\Lambda > 0$ there exists a constant $c_0 = c_0(\Lambda)$ such that for $|\lambda| \le \Lambda$ it holds $\|u_\lambda\| \le c_0$, for every positive solution u_λ of equation*

$$\begin{gathered} -\Delta u = \lambda a(x)u + u^p \quad \text{in } \Omega \\ u > 0 \quad \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.10}$$

Proof. (a) First we integrate equation (3.10) over Ω and obtain

$$0 = \int_\Omega -\Delta u \, dx = \lambda \int_\Omega a(x)u(x)\, dx + \int_\Omega u^p \, dx$$

It follows that

$$\|u\|_p^p \le |\lambda| \, \|a\|_\infty \int_\Omega u(x)\, dx \le d \, \|u\|_p$$

and hence

$$\|u\|_p \le c \tag{3.11}$$

for all positive solutions.

(b) Now multiply equation (3.10) by u and integrate,

$$\int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega a(x)u^2(x)\, dx + \int_\Omega u^{p+1} dx \le c\|u\|_{p+1}^{p+1} \tag{3.12}$$

Now we use the well-known Gagliardo-Nirenberg inequality, see Nirenberg [17] which reads: suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with the cone-property. Then there exist constants c_1 and c_2 such that for all $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$

$$\|D^j u\|_p \le c_1 \|D^m u\|_r^a \|u\|_q^{1-a} + c_2\|u\|_q$$

where

$$\frac{1}{p} = \frac{j}{N} + a\big(\frac{1}{r} - \frac{m}{N}\big) + (1-a)\frac{1}{q}$$

Applying this inequality for $j = 0$, $p \to p+1$, $r = 2$, $m = 1$, $q \to p$, we obtain

$$\|u\|_{p+1} \le c\|\nabla u\|_2^a \|u\|_p^{1-a} + c\|u\|_p \tag{3.13}$$

where

$$\frac{1}{p+1} = a\big(\frac{1}{2} - \frac{1}{N}\big) + (1-a)\frac{1}{p}$$

This condition implies

$$a\Big(\frac{1}{p}+\frac{1}{N}-\frac{1}{2}\Big)=\frac{1}{p}-\frac{1}{p+1}=\frac{1}{p(p+1)}$$

and hence

$$a=\frac{1}{p+1}\,\frac{2N}{2N-(N-2)p}$$

By (3.13) and (3.11) we now have

$$\|u\|_{p+1}^{p+1}\le c\,\|\nabla u\|_2^{a\,(p+1)}+c$$

and hence by (3.12)

$$\|\nabla u\|_2^2\le c\,\|\nabla u\|_2^{\frac{2N}{2N-(N-2)p}}+c$$

We want that $\frac{2N}{2N-(N-2)p}<2$, which is the case if

$$1<p<\frac{N}{N-2}$$

Then $\|\nabla u\|_2\le c$, from which we obtain that $\|u\|\le c$. □

For $p\in\big[\frac{N}{N-2},\frac{N+2}{N-2}\big)$ we have no *a priori* bound for positive solutions readily available, and so we cannot exclude that the solution branches explode when $\lambda\to 0=\lambda_1^-$ or $\lambda\to\lambda_1^+$. However, in view of the a priori bounds for the Dirichlet problem by de Figueiredo-Lions-Nussbaum [8] and Gidas-Spruck [10] we tend to believe that this does not occur.

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