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BIFURCATION FROM INFINITY WITH OSCILLATORY NONLINEARITY FOR NEUMANN PROBLEMS

MAYA CHHETRI, NSOKI MAVINGA, ROSA PARDO

Honoring the memory of Alan Lazer

Abstract. We consider a sublinear perturbation of an elliptic eigenvalue problem with Neumann boundary condition. We give sufficient conditions on the nonlinear perturbation which guarantee that the unbounded continuum, bifurcating from infinity at the first eigenvalue, contains an unbounded sequence of turning points as well as an unbounded sequence of resonant solutions. We prove our result by using bifurcation theory combined with a careful analysis of the oscillatory behavior of the continuum near the bifurcation point.

1. Introduction

We consider the nonlinear elliptic equation with Neumann boundary condition

$$\begin{gathered} -\Delta u = \lambda u + f(\lambda, x, u), \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$, $\partial/\partial\eta := \eta(x) \cdot \nabla$ denotes the outer normal derivative on $\partial\Omega$, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Here the nonlinear perturbation $f : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $f = f(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$.

Observe that problem (1.1) is a perturbation of the eigenvalue problem

$$\begin{gathered} -\Delta\varphi = \lambda\varphi\,, \quad \text{in } \Omega \\ \frac{\partial\varphi}{\partial\eta} = 0\,, \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.2}$$

It is well-known that the eigenvalue problem (1.2) has a sequence of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ with the property that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \cdots \to +\infty$ as $n \to \infty$. Each eigenvalue is of finite multiplicity whose corresponding eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ are orthogonal in $L^2(\Omega)$. The first eigenvalue $\lambda_1 = 0$ is simple and its corresponding eigenfunction $\varphi_1 \equiv$ const. in Ω and can be normalized so that $\varphi_1 \equiv 1$.

The behavior of a nonlinear perturbation f near zero and/or at infinity greatly influences the existence/multiplicity results for (1.1) with respect to the parameter

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λ. In this paper, we are focused on solutions *bifurcating from infinity*. Therefore, we assume that f satisfies the following assumptions for large arguments.

(H1) There exist $h \in L^r(\Omega)$ with $r > N/2$ and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$ and $U : \mathbb{R} \to \mathbb{R}^+$ satisfying
$$|f(\lambda, x, s)| \le \Lambda(\lambda) h(x) U(s), \quad \forall (\lambda, x, s) \in \mathbb{R} \times \Omega \times \mathbb{R}$$
with $\lim_{|s|\to\infty} \frac{U(s)}{s} = 0$.

(H2) There exist a function $B \in L^r(\Omega)$ with $r > N/2$, $\alpha < 1$ and $s_0 > 0$ such that for $s > s_0$, $\lambda \to 0$, and $x \in \Omega$, we have
$$\frac{|f(\lambda, x, s)|}{|s|^\alpha} \le B(x)\,.$$

(H3) $f(\lambda, x, s)$ is differentiable in s, and $\frac{\partial f}{\partial s}(\lambda, \cdot, \cdot) \in C(\Omega \times \mathbb{R})$ and
$$\sup_{|s|\ge M} \|\frac{\partial f}{\partial s}(\lambda, \cdot, s)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad \lambda \to 0 \text{ and } M \to +\infty\,. \tag{1.3}$$

(H4) For $x \in \Omega$,
$$\sup_{|s|\ge M} \frac{|f(\lambda, x, s) - f(0, x, s)|}{|s|^\alpha} \to 0 \quad \text{as} \quad \lambda \to 0 \text{ and } M \to +\infty\,.$$

Note that (H1) implies that f is sublinear at infinity in the variable s, that is,
$$\limsup_{|s|\to\infty} \frac{|f(\lambda, x, s)|}{|s|} = 0\,.$$
After the pioneering work of Rabinowitz [12], bifurcation from infinity for the sublinear perturbation of the linear eigenvalue problem is widely studied. The sublinearity assumption guarantees the existence of unbounded branches of solutions when λ approaches one of the eigenvalues of odd multiplicity. These branches bifurcate from infinity in the sense of Rabinowitz, see [11, 12]. For the existence of unbounded branches of solutions of Dirichlet and nonlinear boundary conditions, see [1, 3, 4, 10] and references therein.

The focus of this article is to study the weak solutions of (1.1) bifurcating from infinity. By a weak solution of (1.1), we mean a pair $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that
$$\int_\Omega \nabla u \nabla \psi + \int_\Omega u\psi = \lambda \int_\Omega u\psi + \int_\Omega f(\lambda, x, u)\psi, \quad \text{for all } \psi \in H^1(\Omega).$$
Note that by (H1), weak solutions of (1.1) lie in the space $W^{2,r}(\Omega)$, $r > N/2$, continuously embedded in $C(\overline{\Omega})$. Therefore, we consider $\mathbb{R} \times C(\overline{\Omega})$ as our underlying space.

The branch bifurcating from infinity at $\lambda_1 = 0$ forms a continuum (closed connected set) consisting of elements from the set
$$\{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : (\lambda, u) \text{ is a weak solution of (1.1)}\}\,.$$
The set of solutions bifurcating from infinity at $\lambda_1 = 0$ contains large positive solutions or large negative solutions (or both) of (1.1). Let $\mathscr{D}^+ \subset \mathbb{R} \times C(\overline{\Omega})$ (resp. $\mathscr{D}^- \subset \mathbb{R} \times C(\overline{\Omega})$) denote the continuum of positive, (resp. negative) solutions bifurcating at $\lambda_1 = 0$. It is known (see e.g. [12]) that the solutions in $\mathscr{D}^\pm$ can be expressed as
$$u = t + w, \quad \text{where } w = o(|t|) \text{ as } |t| \to \infty\,. \tag{1.4}$$

Our main focus is on the analysis of unbounded continuum $\mathscr{D}^+$ bifurcating at $\lambda_1 = 0$. In particular, we give sufficient conditions on f which guarantees that $\mathscr{D}^+$ is neither subcritical ($\lambda < 0$) nor supercritical ($\lambda > 0$). This leads to the existence of unbounded sequences of turning points and unbounded sequence of resonant solutions at $\lambda = 0$ on the continuum $\mathscr{D}^\pm$. We say that $(\lambda^*, u^*) \in \mathscr{D}^+$ is a *turning point* if there is a neighborhood of (λ^*, u^*) in $\mathbb{R} \times C(\overline{\Omega})$ such that there are no solutions (λ, u_λ) close to (λ^*, u^*) for $\lambda > \lambda^*$ or for $\lambda < \lambda^*$.

We note that problem (1.1) is a perturbed eigenvalue problem. Therefore, to investigate the subcritical or supercritical nature of the continuum $\mathscr{D}^+$ bifurcating from infinity at $\lambda = 0$, one must analyze the lower order terms of $f(\lambda, x, s)$ as $\lambda \to 0$ and $s \to \infty$. To do this, one defines

$$\underline{\mathbf{F}}_+ := \int_\Omega \liminf_{(\lambda,s)\to(0,+\infty)} \frac{sf(\lambda,\cdot,s)}{|s|^{1+\alpha}}, \quad \overline{\mathbf{F}}_+ := \int_\Omega \limsup_{(\lambda,s)\to(0,+\infty)} \frac{sf(\lambda,\cdot,s)}{|s|^{1+\alpha}}\,. \tag{1.5}$$

It is known that if $\underline{\mathbf{F}}_+ > 0$, then $\mathscr{D}^+$ is subcritical, while if $\overline{\mathbf{F}}_+ < 0$, then $\mathscr{D}^+$ is supercritical, see [7, Thm. 2.1] and [10, Thm. 4.3]. Moreover, if all the unbounded branches are either subcritical or supercritical then, the resonant problem, that is when $\lambda = 0$, has at least one solution, see [7, Cor. 3.5] and [10, Thm. 5.1].

Therefore, in this article we consider nonlinearities satisfying

$$\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+\,. \tag{1.6}$$

This condition means that the bifurcating continuum $\mathscr{D}^+$ is neither subcritical nor supercritical, and hence Landesman-Lazer type conditions do not hold. The main purpose of this article is to establish the existence of infinitely many resonant solutions at $\lambda = 0$ in the absence of Landesman-Lazer type conditions. We note that the condition (1.6) reflects the oscillatory behavior of $\mathscr{D}^+$ near infinity around the bifurcation point $\lambda = 0$, yielding infinitely many resonant solutions. In particular, we prove the following result.

Theorem 1.1. *Let* (H1)–(H4) *hold. Suppose there exist two increasing sequences* $\{t_n\}$ *and* $\{t'_n\}$ *that tend to* $+\infty$ *and satisfy*

$$-\infty < \lim_{n\to+\infty} \int_\Omega t'_n \frac{f(0,\cdot,t'_n)}{|t'_n|^{1+\alpha}} < 0 < \lim_{n\to+\infty} \int_\Omega t_n \frac{f(0,\cdot,t_n)}{|t_n|^{1+\alpha}} < \infty\,. \tag{1.7}$$

Then, the following assertions hold.

(I) *There exist two sequences* $\{(\lambda_n, u_n)\}$ *and* $\{(\lambda'_n, u'_n)\}$ *in* $\mathscr{D}^+$ *approaching* $(0, \infty)$ *as* $n \to \infty$, *with* $\lambda_n < 0$ *(subcritical), and* $\lambda'_n > 0$ *(supercritical).*

(II) *There is a sequence of turning points* $\{(\lambda_n^*, u_n^*)\} \in \mathscr{D}^+$ *such that*

$$\lambda_n^* \to 0 \quad \text{and} \quad \|u_n^*\|_{C(\overline{\Omega})} \to \infty, \quad \text{as } n \to \infty\,.$$

Furthermore, one can choose two subsequences of turning points, one of them subcritical, $\lambda^*_{2n+1} < 0$, *and the other supercritical,* $\lambda^*_{2n} > 0$.

(III) *There is a sequence of resonant solutions, that is, there are infinite solutions* $\{(0, \hat{u}_n)\} \in \mathscr{D}^+$ *with* $\|\hat{u}_n\|_{C(\overline{\Omega})} \to \infty$ *as* $n \to \infty$.

The case for $\mathscr{D}^-$ can be established in a similar fashion.

We briefly describe how each of the hypotheses (H1)–(H4) and (1.7) play crucial role in proving Theorem 1.1.

- As discussed earlier, (H1) guarantees that $\mathscr{D}^+$ bifurcates from infinity at $\lambda = 0$ and for each $(\lambda, u) \in \mathscr{D}^+$, u is given by (1.4).

- Assumption (H2) helps establishing the estimates $|\lambda| = O(t^{\alpha-1})$ and $|w| = O(t^{\alpha})$ as $t \to \infty$ in Proposition 2.3.
- Assumption (H3) ensures that the sign of $\underline{\mathbf{F}}_+$ and $\overline{\mathbf{F}}_+$ can be determined in terms of integrals involving only the parameter t instead of the solution variable u in Lemma 2.5.
- The technical assumption (H4) helps in the determination of the location of λ relative to $\lambda_1 = 0$. See the end of the proof of part (I).
- The assumption (1.7) determines the oscillatory behavior of the continuum $\mathscr{D}^+$ across the hyperplane $\lambda = 0$.

Results such as Theorem 1.1 have been studied in [2, 5] in the case of nonlinear boundary conditions, for bifurcation from infinity or from zero respectively. In [6] one can find a similar result on the existence of unbounded sequences of stable solutions, unstable solutions, and turning points, even in the absence of resonant solutions, also for nonlinear boundary conditions. To the best of our knowledge, such results are not known in the case of Neumann boundary conditions. In [3, 4, 7, 10], the existence of resonant solutions was established when the nonlinearity satisfies some type of Landesman-Lazer conditions. We note that the now ubiquitous Landesman–Lazer condition that guarantees the existence of a resonant solution first appeared in a paper by Landesman and Lazer in [9]. We are indebted to their pioneering work and feel privileged to honor Professor Lazer in this paper.

A motivating example concerning Theorem 1.1 is the oscillatory nonlinearity function

$$f(s) := |s|^{\alpha}[\sin(|s|^{\beta}) + C] \quad \text{with } \beta \neq 0 \text{ and } \alpha < 1.$$

If $\beta \in R$ and $C > 1$, or if $\beta < 0$ and $C > 0$, then from definition of $\underline{\mathbf{F}}_+$, see (1.5), $\underline{\mathbf{F}}_+ > 0$ and the bifurcation from infinity is subcritical. On the other hand if $\beta \in R$ and $C < -1$, or if $\beta < 0$ and $C < 0$, then $\overline{\mathbf{F}}_+ < 0$ and the bifurcation from infinity is supercritical.

Therefore, we consider here the range $\beta > 0$ and $-1 < C < 1$ and note that Theorem 1.1 applies if

$$\beta > 0, \quad \alpha + \beta < 1, \quad \text{and} \quad -1 < C < 1.$$

Therefore, in this range of parameters, there exist unbounded sequences of subcritical and supercritical solutions, subcritical and supercritical turning points and infinite resonant solutions.

The restriction $\alpha+\beta < 1$ on the size of β is needed in order to satisfy the condition (1.3). This restriction means that the "oscillating" nonlinearities f cannot oscillate very fast.

In Section 2, we discuss some preliminaries, functional framework and prove technical results associated with assumptions (H1)–(H3) that will be used in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1 using bifurcation theory combined with technical results of Section 2. We also state and prove a corollary that characterizes the λ-intervals from the bifurcation point to the turning points.

2. Preliminaries and auxiliary results

In this section, we discuss the functional framework and establish few auxiliary results needed in the proof of Theorem 1.1. Let us start by analyzing the behavior of a sequence of solutions when we know explicitly that the solutions blow up.

Proposition 2.1. *Let* (H1) *hold. Let* $\{(\lambda_n, u_n)\} \subset \mathscr{D}^+$ *where* $\lambda_n \to \lambda_0$, $u_n \geq 0$, *and* $\|u_n\|_{C(\overline{\Omega})} \to \infty$, *then* $\lambda_n \to 0$, *and there exists a subsequence, again denoted by* u_n, *such that*

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|_{C(\overline{\Omega})}} = 1, \quad \text{in } C^\mu(\overline{\Omega}) \text{ for some } \mu \in (0,1).$$

Proof. Let $v_n = u_n/\|u_n\|_{C(\overline{\Omega})}$. Since $u_n \in W^{2,r}(\Omega)$ (see [8, p. 162]) with $r > N/2$, by the compact embedding theorem, we obtain $u_n \in C^\gamma(\overline{\Omega})$ for some $\gamma \in (0,1)$. Then, since (H1) holds, we obtain that $\|v_n\|_{C^\gamma(\overline{\Omega})} \leq C$. Using the compact embedding $C^\gamma(\overline{\Omega}) \hookrightarrow C^{\gamma'}(\overline{\Omega})$ for $0 < \gamma' < \gamma$, we deduce that there exists a convergent subsequence (again denoted by v_n) such that $v_n \to \varphi$ in $C^{\gamma'}(\overline{\Omega})$. Since $v_n \geq 0$ and $\|v_n\|_{C(\overline{\Omega})} = 1$, it is easy to see that $0 \leq \varphi \not\equiv 0$. Moreover, v_n satisfies

$$\begin{gathered} -\Delta v_n = \lambda_n v_n + \frac{f(\lambda, x, u_n)}{\|u_n\|_{C(\overline{\Omega})}}, \quad \text{in } \Omega \\ \frac{\partial v_n}{\partial \eta} = 0, \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.1}$$

Passing to the limit in the weak formulation of (2.1) and using that $\frac{f(\lambda,x,u_n)}{\|u_n\|_{C(\overline{\Omega})}} \to 0$ in $L^r(\Omega)$, we obtain

$$\begin{gathered} -\Delta\varphi = \lambda_0\varphi, \quad \text{in } \Omega \\ \frac{\partial\varphi}{\partial\eta} = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

with $0 \leq \varphi \not\equiv 0$. Then necessarily $\varphi \equiv 1$ and $\lambda_0 = 0$. □

Next, we will prove that under hypothesis (H2), if $u = t + w$ is a solution as given in (1.4), then w satisfies

$$w = O(|t|^\alpha) \quad \text{as } |t| \to \infty\,.$$

We analyze first the linear problem. Let $\lambda \in (-\infty, \lambda_2)$ and $g(\lambda,\cdot) \in L^r(\Omega)$ with $r > N/2$, and consider the linear problem

$$\begin{gathered} -\Delta u = \lambda u + g(\lambda, x), \quad \text{in } \Omega \\ \frac{\partial u}{\partial\eta} = 0, \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.2}$$

Then, (2.2) has a unique solution $u \in W^{2,r}(\Omega)$ (see [8, p. 162]) if $\lambda \neq 0$. Moreover, since $r > N/2$, by the compact embedding Theorem $u \in C(\overline{\Omega})$. We observe that (2.2) is a linear perturbation of the eigenvalue problem. Therefore, to take advantage of this structure, we decompose

$$L^r(\Omega) = \operatorname{span}[\varphi_1] \oplus \operatorname{span}[\varphi_1]^\perp = \operatorname{span}[1] \oplus \big\{\phi \in L^r(\Omega) : \int_\Omega \phi = 0\big\}\,. \tag{2.3}$$

Then for $g(\lambda,\cdot) \in L^r(\Omega)$, with $r > N/2$ and $g(\lambda,\cdot) \not\equiv \text{const.}$, there exists a unique decomposition

$$g(\lambda,\cdot) = a_1(\lambda) + g_1(\lambda,\cdot),$$

where $a_1(\lambda)$ (the projection onto span[1]), and $g_1(\lambda,\cdot)$ (orthogonal to span[1]) are given by

$$a_1(\lambda) := \frac{1}{|\Omega|}\int_\Omega g(\lambda,\cdot) \quad \text{and} \quad \int_\Omega g_1(\lambda,\cdot) = 0. \tag{2.4}$$

By the Fredholm Alternative, the linear problem (2.2) has a unique solution if $\lambda \neq 0$ (recall $\lambda_1 = 0$) and does not have solution if $\lambda = 0$ and $a_1(0) \neq 0$. Hence, for $\lambda \neq 0$ the solution $u = u(\lambda)$ of (2.2) belongs to $W^{2,r}(\Omega)$, (see [8, p. 162]) and hence to $L^r(\Omega)$. Therefore, the solution u has a unique decomposition in $L^r(\Omega)$ given by

$$u = \frac{-a_1(\lambda)}{\lambda} + w\,, \quad \text{with } \int_\Omega w = 0\,. \tag{2.5}$$

Moreover, $w = w(\lambda)$ solves the problem

$$\begin{gathered} -\Delta w = \lambda w + g_1(\lambda, x), \quad \text{in } \Omega \\ \frac{\partial w}{\partial \eta} = 0, \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.6}$$

where g_1 is as defined by (2.4).

On the other hand, if $\lambda = 0$, by the Fredholm Alternative and by (2.4), there exists a function $v \in W^{2,r}(\Omega)$ such that $v + c$ solves (2.6) for any $c \in \mathbb{R}$. Let us choose $c_0 \in \mathbb{R}$ such that $\int_\Omega v + c_0 = 0$ and define $w(0) = v + c_0$. This implies that $w(\lambda) \in \operatorname{span}[1]^\perp$ is well defined for any $\lambda \in (-\infty, \lambda_2)$.

The lemma below estimates the $C(\overline{\Omega})$ norm of the solution of (2.6) if $g \in L^r(\Omega)$.

Lemma 2.2. *For each compact set $K \subset (-\infty, \lambda_2) \subset \mathbb{R}$, there exists a constant $C = C(K)$, independent of $\lambda \in K$, such that*

$$\|w(\lambda)\|_{C(\overline{\Omega})} \leq C\|g_1(\lambda, \cdot)\|_{L^r(\Omega)}\,,$$

where w satisfies $\int_\Omega w = 0$ and (2.6)*, and g_1 satisfies* (2.4)*.*

Proof. We observe that $w = w(\lambda)$ satisfying (2.5)-(2.6) is well defined for any $\lambda \in K$ by the discussion above.

We first show that $w(\lambda)$ is uniformly bounded for any λ in a neighborhood of $\lambda_1 = 0$. Assume to the contrary that there is a sequence $\lambda_n \to 0$ with $\|w(\lambda_n)\|_{C(\overline{\Omega})} \to \infty$. Then it follows from [7, 10, 11, 12] that

$$\frac{w(\lambda_n)}{\|w(\lambda_n)\|_{C(\overline{\Omega})}} \to \varphi_1 \equiv 1 \text{ uniformly (up to a subsequence) in } \overline{\Omega}\,.$$

This contradicts that $\int_\Omega w(\lambda_n) = 0$. Therefore, there exist $\delta > 0$ and $c > 0$ such that $\|w(\lambda)\|_{C(\overline{\Omega})} < c$ independent of λ for any $|\lambda| < \delta$.

Second, let $\lambda \in K \setminus (-\delta, \delta)$. By the Fredholm Alternative, $w(\lambda) \in W^{2,r}(\Omega)$ is the unique solution of (2.6). Using the L^r-estimate and the embedding of $W^{2,r}(\Omega)$ into $C(\overline{\Omega})$, we obtain

$$\|w(\lambda)\|_{C(\overline{\Omega})} \leq C\|w(\lambda)\|_{W^{2,r}(\Omega)} \leq C\|g_1(\lambda, \cdot)\|_{L^r(\Omega)} < \infty\,.$$

To conclude, let $\lambda \in K$ and

$$T(\lambda) : \big\{g_1 \in L^r(\Omega) : \int_\Omega g_1 = 0\big\} \to C(\overline{\Omega})$$

be a family of operators defined by $T(\lambda)g_1 := w(\lambda)$, where $w(\lambda)$ is the solution of (2.6). Then, $T(\lambda)$ is continuous for every $\lambda \in K$. Moreover, $\sup_{\lambda \in K} \|T(\lambda)g_1\|_{C(\overline{\Omega})} < \infty$ from the previous two paragraphs. Therefore, by the Uniform Boundedness Principle, there exists a constant $C = C(K)$ such that

$$\|w(\lambda)\|_{C(\overline{\Omega})} \leq C(K)\|g_1\|_{L^r(\Omega)} \quad \text{for any } \lambda \in K,$$

as desired. ∎

Proposition 2.3. *Let* (H1) *and* (H2) *hold. Then, there exists a neighborhood of* $(0,\infty)\subset\mathbb{R}\times C(\overline{\Omega})$ *given by*

$$\mathcal{O}:=\{(\lambda,u)\in\mathbb{R}\times C(\overline{\Omega}):|\lambda|<\delta_0,u(x)>0,\ \|u\|_{C(\overline{\Omega})}>M_0\},$$

for some small δ_0 *and large* M_0*, such that the following hold:*

(i) *There exist positive constants* C_1,C_2 *(independent of* λ*) such that if* $(\lambda,u)\in\mathscr{D}^+\cap\mathcal{O}$ *and* $(\lambda,u)\neq(0,\infty)$*, then*

$$u=t+w\quad\text{where } t>0,\ \int_\Omega w=0, \tag{2.7}$$

$$\|w\|_{C(\overline{\Omega})}\le C_1\|B\|_{L^r(\Omega)}t^\alpha\quad\text{as } t\to\infty, \tag{2.8}$$

$$|\lambda|\le C_2t^{\alpha-1}\quad\text{as } t\to\infty\,. \tag{2.9}$$

(ii) *There exists* $t_0>0$ *such that for all* $t\ge t_0$ *there exists* $(\lambda,u)\in\mathscr{D}^+\cap\mathcal{O}$ *satisfying* $u=t+w$ *with* $\int_\Omega w=0$.

Proof. Let $\mathcal{O}$ be as defined above for $\delta_0>0$ and $M_0>0$. Then, since $\mathscr{D}^+$ bifurcates from infinity at $\lambda=0$, there exist $\delta_0>0$ and $M_0>0$ such that $\mathscr{D}^+\cap\mathcal{O}\neq\emptyset$.

(i) Let $(\lambda,u)\in\mathscr{D}^+\cap\mathcal{O}$. Because of (2.3), u can be written as $u=t+w$ with $\int_\Omega w=0$, hence (2.7) holds. Integrating by parts (1.1) and using the divergence theorem, we obtain

$$-\lambda\int_\Omega u=\int_\Omega f(\lambda,x,u)\,.$$

Since $u=t+w$ and $\int_\Omega w=0$, we obtain

$$-\lambda\,t\,|\Omega|=\int_\Omega f(\lambda,x,t+w)\,. \tag{2.10}$$

Now, using (H1) and that $w=o(|t|)$ as $|t|\to\infty$,

$$\frac{|f(\lambda,x,t+w)|}{|t|}=\frac{|f(\lambda,x,t+w)|}{|t+w|}\,\Big|1+\frac{w}{t}\Big|\to0\quad\text{as } t\to\infty\,.$$

Therefore, by the Lebesgue dominated convergence theorem and (2.10), we obtain $\lambda\to0$ as $t\to\infty$. We note that (H2) yields

$$|f(\lambda,x,t+w)|=|t|^\alpha\frac{|f(\lambda,x,t+w)|}{|t+w|^\alpha}|1+\frac{w}{t}|^\alpha\le|t|^\alpha\,B(x)|1+\frac{w}{t}|^\alpha\,. \tag{2.11}$$

Therefore, it follows from (2.10) that

$$|\lambda|\le\frac{|t|^{\alpha-1}}{|\Omega|}\int_\Omega\Big(B(x)|1+\frac{w}{t}|^\alpha\Big)\le C\|B\|_{L^r(\Omega)}|t|^{\alpha-1}\,.$$

This shows (2.9).

By (H1) $f(\lambda,.,u(.))\in L^r(\Omega)$, and hence there exists a unique decomposition

$$f(\lambda,x,s)=f_1(\lambda,x,s)+\int_\Omega f(\lambda,x,s)\,,$$

where $\int_\Omega f(\lambda,x,s)$ is the projection onto span[1] and f_1 is orthogonal to span[1], that is, $\int_\Omega f_1(\lambda,x,s)=0$. By Lemma 2.2, we have

$$\|w\|_{L^\infty(\Omega)}\le C\|f_1\|_{L^r(\Omega)}\le C\|f\|_{L^r(\Omega)}\,.$$

Hence, from (2.11) and that $w=o(|t|)$, we obtain the estimate (2.8),

$$\|w\|_{L^\infty(\Omega)}\le C\|B\|_{L^r(\Omega)}|t|^\alpha\quad\text{as } t\to\infty\,.$$

This completes part (i).

(ii) Since $\mathscr{D}^+$ bifurcates from infinity at $\lambda = 0$, one has that $\mathscr{D}^+ \cap \mathcal{O}$, although not necessarily connected, contains an unbounded connected component $\mathscr{S}$. Therefore, if $(\lambda, u) \in \mathscr{S} \subset \mathscr{D}^+ \cap \mathcal{O}$, we necessarily have

$$u = t + w \quad \text{with } \int_\Omega w = 0 \text{ and } t = \int_\Omega u\,. \tag{2.12}$$

Using the continuity of the projection $t = \int_\Omega u$, we infer that the set

$$\{t \in \mathbb{R} : (1.1) \text{ has a solution satisfying } (2.12)\}$$

contains an unbounded connected set. Therefore, part (ii) holds. $\square$

As an immediate consequence of the estimate for w given by (2.8) in Proposition 2.3, we have the following corollary:

Corollary 2.4. *Assume* (H1) *and* (H2) *hold. Let* $\{(\lambda_n, u_n)\} \subset \mathscr{D}^+ \cap \mathcal{O}$ *be such that* $\lambda_n \to 0$ *and* $u_n = t_n + w_n$ *with* $\int_\Omega w_n = 0$ *and* $t_n = \int_\Omega u_n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|_{C(\overline{\Omega})}} = 1 \quad \text{uniformly in } \overline{\Omega}\,,$$
$$\lim_{n\to\infty} \frac{u_n}{t_n} = 1 \quad \text{uniformly in } \overline{\Omega}\,,$$
$$\lim_{n\to\infty} \frac{\|u_n\|_{C(\overline{\Omega})}}{t_n} = 1, \quad \text{uniformly in } \overline{\Omega}.$$

We note that, with minor modification in the proof, the results of Corollary 2.4 remain valid when only (H1) is satisfied.

To guarantee that (1.7) is enough to conclude the existence of subcritical ($\lambda < 0$) and supercritical ($\lambda > 0$) solutions in the unbounded continuum $\mathscr{D}^+$, we will use the following result.

Lemma 2.5. *Let f satisfy* (H3). *Suppose there exist $\alpha < 1$ and a function $B_1 \in L^1(\Omega)$ such that for $x \in \Omega$, and for all (λ, s) close to the bifurcation point $(0, +\infty)$, we have*

$$\frac{f(\lambda, x, s)}{|s|^\alpha} \le B_1(x)\,. \tag{2.13}$$

Let $\lambda_n \to 0$, $t_n \uparrow \infty$ and $w_n \in L^\infty(\Omega)$, such that $\|w_n\|_{L^\infty(\Omega)} = O(|t_n|^\alpha)$ as $n \to \infty$. Then

$$\liminf_{n\to+\infty} \int_\Omega \frac{(t_n + w_n) f(\lambda_n, \cdot, t_n + w_n)}{|t_n + w_n|^{1+\alpha}} \ge \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n, \cdot, t_n)}{|t_n|^{1+\alpha}}\,, \tag{2.14}$$

and

$$\limsup_{n\to+\infty} \int_\Omega \frac{(t_n + w_n) f(\lambda_n, \cdot, t_n + w_n)}{|t_n + w_n|^{1+\alpha}} \le \limsup_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n, \cdot, t_n)}{|t_n|^{1+\alpha}}. \tag{2.15}$$

Proof. For any $w \in L^\infty(\Omega)$ and $t > 0$ such that $|w| < t/2$, using the Mean Value Theorem, we have (with a constant C that may change from line to line)

$$\begin{aligned} &\int_\Omega |f(\lambda,\cdot,t+w) - f(\lambda,\cdot,t)|\,dx \\ &\le C\|w\|_{L^\infty(\Omega)} \int_\Omega \int_0^1 |\frac{\partial f}{\partial s}(\lambda,\cdot,t+\tau w)|\,d\tau\,dx \\ &\le C\|w\|_{L^\infty(\Omega)} \sup_{\tau\in[0,1]} \big\|\frac{\partial f}{\partial s}(\lambda,\cdot,t+\tau w)\big\|_{C(\overline{\Omega})}. \end{aligned} \tag{2.16}$$

Then, whenever $\|w\|_{L^\infty(\Omega)} = O(|t|^\alpha)$, using (2.16) and (H3), we obtain

$$\begin{aligned} &\int_\Omega \frac{|f(\lambda,\cdot,t+w) - f(\lambda,\cdot,t)|}{|t|^\alpha}\,dx \\ &\le C \sup_{|s|\ge M} \big\|\frac{\partial f}{\partial s}(\lambda,\cdot,s)\big\|_{L^\infty(\Omega)} \frac{\|w\|_{L^\infty(\Omega)}}{|t|^\alpha} \to 0 \end{aligned} \tag{2.17}$$

as $\lambda \to 0$ and $M \to \infty$.

Now, let $\lambda_n \to 0$, $t_n \uparrow \infty$ and $w_n \in L^\infty(\Omega)$, such that $\|w_n\|_{L^\infty(\Omega)} = O(|t_n|^\alpha)$ as $n \to \infty$. Then, (2.17) yields

$$\begin{aligned} &\liminf_{n\to+\infty} \int_\Omega \frac{t_n\, f(\lambda_n,\cdot,t_n+w_n)}{|t_n|^{1+\alpha}} \\ &\ge \lim_{\substack{\lambda\to 0\\ n\to+\infty}} \int_\Omega \frac{t_n f(\lambda,\cdot,t_n+w_n) - t_n f(\lambda,\cdot,t_n)}{|t_n|^{1+\alpha}} + \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n,\cdot,t_n)}{|t_n|^{1+\alpha}} \\ &= \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n,\cdot,t_n)}{|t_n|^{1+\alpha}}\,. \end{aligned} \tag{2.18}$$

To establish (2.14), we estimate the left hand side of (2.18) from below. For this, we note that

$$\frac{t_n f(\lambda_n,\cdot,t_n+w_n)}{|t_n|^{1+\alpha}} = \frac{(t_n+w_n) f(\lambda_n,\cdot,t_n+w_n)}{|t_n+w_n|^{1+\alpha}} \big|1 + \frac{w_n}{t_n}\big|^\alpha .$$

Then, using that $1 + w_n/t_n \to 1$ in $L^\infty(\Omega)$ and (2.18), we obtain

$$\begin{aligned} \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n,\cdot,t_n)}{|t_n|^{1+\alpha}} &\le \liminf_{n\to+\infty} \int_\Omega \frac{t_n\, f(\lambda_n,\cdot,t_n+w_n)}{|t_n|^{1+\alpha}} \\ &= \liminf_{n\to+\infty} \int_\Omega \frac{(t_n+w_n) f(\lambda_n,\cdot,t_n+w_n)}{|t_n+w_n|^{1+\alpha}} \big|1 + \frac{w_n}{t_n}\big|^\alpha \\ &\le \liminf_{n\to+\infty} \int_\Omega \frac{(t_n+w_n) f(\lambda_n,\cdot,t_n+w_n)}{|t_n+w_n|^{1+\alpha}}\,. \end{aligned}$$

The integral on the right-hand side above is well defined by (2.13), hence (2.14) holds. Similar arguments will establish (2.15). Thus the proof is complete. □

3. Proof of Theorem 1.1

Roughly speaking, if there exist an unbounded sequence of subcritical solutions and another unbounded sequence of supercritical solutions in the continuum of solutions, then the connectedness of the continuum guarantees that there are infinite turning points and hence infinite resonant solutions.

Proof of Theorem 1.1. (I) We observe that conclusions (i)–(iii) of Proposition 2.3 hold for some neighborhood $\mathcal{O}$ of the bifurcation point $(0,+\infty) \in \mathbb{R} \times C(\overline{\Omega})$. Let $(\lambda_n, u_n) \to (0,+\infty)$ and $(\lambda'_n, u'_n) \to (0,+\infty)$ in $\mathscr{D}^+ \cap \mathcal{O}$ be two sequences. Then, using (2.12), we have

$$u_n = t_n + w_n \quad \text{and} \quad u'_n = t'_n + w'_n$$

with

$$\int_\Omega w_n = 0 = \int_\Omega w'_n, \quad t_n := \int_\Omega u_n, \quad t'_n := \int_\Omega u'_n\,.$$

Integrating by parts (1.1) for $(\lambda, u) = (\lambda_n, u_n)$ and thanks to the divergence Theorem we obtain

$$-\lambda_n t_n = \int_\Omega f(\lambda_n, x, u_n)\,.$$

Dividing by $t_n \|u_n\|^{\alpha-1}_{C(\overline{\Omega})}$ and using Corollary 2.4 yields

$$\liminf_{n\to\infty}\ -\frac{\lambda_n}{\|u_n\|^{\alpha-1}_{C(\overline{\Omega})}} = \liminf_{n\to\infty} \int_\Omega \frac{f(\lambda_n, x, u_n)}{\|u_n\|^{\alpha}_{C(\overline{\Omega})}}\,.$$

Moreover,

$$\begin{aligned}
\int_\Omega \frac{f(\lambda_n, x, u_n)}{\|u_n\|^{\alpha}_{C(\overline{\Omega})}} &= \int_\Omega \frac{f(\lambda_n, x, u_n)}{u_n^\alpha}\Big(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}\Big)^\alpha \\
&= \int_\Omega \frac{f(\lambda_n, x, u_n)}{u_n^\alpha}\Big[\Big(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}\Big)^\alpha - 1\Big] + \int_\Omega \frac{f(\lambda_n, x, u_n)}{u_n^\alpha}\,.
\end{aligned}$$

Furthermore, by Corollary 2.4,

$$\int_\Omega \Big|\frac{f(\lambda_n, x, u_n)}{u_n^\alpha}\Big[\Big(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}\Big)^\alpha - 1\Big]\Big| \le \int_\Omega B(x)\Big|\Big[\Big(\frac{u_n}{\|u_n\|_{C(\overline{\Omega})}}\Big)^\alpha - 1\Big]\Big| \to 0,$$

as $n \to \infty$, consequently

$$\liminf_{n\to\infty}\ -\frac{\lambda_n}{\|u_n\|^{\alpha-1}_{C(\overline{\Omega})}} \ge \liminf_{n\to\infty} \int_\Omega \frac{f(\lambda_n, x, u_n)}{u_n^\alpha}.$$

Then, utilizing $u_n = t_n + w_n$, we obtain

$$\begin{aligned}
&\liminf_{n\to\infty} \frac{0 - \lambda_n}{\|u_n\|^{\alpha-1}_{C(\overline{\Omega})}} \\
&\ge \liminf_{n\to\infty} \int_\Omega \frac{(t_n + w_n) f(\lambda_n, \cdot, t_n + w_n)}{|t_n + w_n|^{1+\alpha}} \\
&\ge \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(\lambda_n, \cdot, t_n)}{|t_n|^{1+\alpha}} \quad \text{(by Lemma 2.5)} \\
&= \liminf_{n\to+\infty} \int_\Omega \frac{t_n[f(\lambda_n, \cdot, t_n) - f(0, \cdot, t_n) + f(0, \cdot, t_n)]}{|t_n|^{1+\alpha}} \\
&\ge \liminf_{n\to+\infty} \int_\Omega \frac{t_n[f(\lambda_n, \cdot, t_n) - f(0, \cdot, t_n)]}{|t_n|^{1+\alpha}} + \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(0, \cdot, t_n)}{|t_n|^{1+\alpha}} \\
&= \liminf_{n\to+\infty} \int_\Omega \frac{t_n f(0, \cdot, t_n)}{|t_n|^{1+\alpha}} > 0 \quad \text{(by (H4) and (1.7))},
\end{aligned}$$

yielding $\lambda_n < 0$ for n sufficiently large. Analogously, we obtain $\lambda'_n > 0$ for n sufficiently large. This completes part (I).

(II) Let $\{t_n\}$ and $\{t'_n\}$ be two sequences of positive real numbers such that $t_n, t'_n \to +\infty$ as $n \to \infty$. Then, up to a subsequence, $t_n < t'_n < t_{n+1}$ for all $n \geq 1$ and t_n, $t'_n \geq t_0$, where t_0 is as defined in Proposition 2.3 (iii). Then, for t_n, $t'_n \geq t_0$, Proposition 2.3 (iii) guarantees $(\lambda_n, u_n), (\lambda'_n, u'_n) \in \mathscr{D}^+ \cap \mathcal{O}$ such that

$$u_n = t_n + w_n \text{ with } \int_\Omega w_n = 0 \quad \text{and} \quad u'_n = t'_n + w'_n \text{ with } \int_\Omega w'_n = 0\,.$$

We note that $\lambda_n < 0$ (subcritical) and $\lambda'_n > 0$ (supercritical) for n sufficiently large, by part (I).

It follows from Proposition 2.3 (i)-(ii) that if $(\lambda, u) \in \mathscr{D}^+ \cap \mathcal{O}$ and $\int_\Omega u = t > t_0$ then for t_0 sufficiently large, we obtain

$$\|u\|_{C(\overline{\Omega})} = \|t + w\|_{C(\overline{\Omega})} \leq (1 + C_1\|B\|_{L^r(\Omega)}|t_0|^{\alpha-1})t \leq 2t\,. \tag{3.1}$$

Let

$$K_n := \{(\lambda, u) \in \mathscr{D}^+ \cap \mathcal{O} : \int_\Omega u = t, \text{ and } t_n \leq t \leq t_{n+1}\}\,. \tag{3.2}$$

We claim that, for each $n \in \mathbb{N}$, K_n is a compact set in $\mathbb{R} \times C(\overline{\Omega})$. For this, let (μ_k, v_k) be a sequence in K_n. Obviously $t_n \leq \int_\Omega v_k \leq t_{n+1}$ for all k, hence (3.1) implies that $\|v_k\|_{C(\overline{\Omega})} \leq 2t_{n+1}$ for all k. Moreover, by Proposition 2.3 (i) we have that $|\lambda| \leq C_1 t^{\alpha-1} \leq C_1 t_0^{\alpha-1}$. Then, by [10, Thm. 2.4], there exists a constant C, independent of k, such that

$$\|v_k\|_{C^\alpha(\overline{\Omega})} \leq C_1\big(1 + \|v_k\|_{C(\overline{\Omega})}\big) \leq C\,.$$

Using the compact embedding $C^\alpha(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ for some $\beta \in (0, \alpha)$, we infer that there exists $u^* \in C^\beta(\overline{\Omega})$ such that $v_k \to u^*$ in $C^\beta(\bar{\Omega})$, up to a subsequence. Since (μ_k, v_k) satisfies

$$\begin{gathered} -\Delta v_k = \mu_k v_k + f(\mu_k, x, v_k), \quad \text{in } \Omega \\ \frac{\partial v_k}{\partial \eta} = 0, \quad \text{on } \partial\Omega \end{gathered}$$

and f is Carathéodory, $f(\mu_k, \cdot, v_k) \to f(\mu^*, \cdot, u^*)$ pointwise. Then, (H1) and the Lebesgue dominated convergence theorem imply $f(\mu_k, \cdot, v_k) \to f(\mu^*, \cdot, u^*)$ in $L^r(\Omega)$ as $k \to \infty$. Further, passing to the limit in the weak formulation of the above equation, we see that u^* is a weak solution of

$$\begin{gathered} -\Delta u^* = \mu^* u^* + f(\lambda^*, x, u^*), \quad \text{in } \Omega \\ \frac{\partial u^*}{\partial \eta} = 0, \quad \text{on } \partial\Omega\,. \end{gathered}$$

The convergence of $(\mu_k, v_k) \in K_n$, and the continuity of the projection P implies $t_0 \leq t_n \leq t^* = \int_\Omega u^* \leq t_{n+1}$. Hence, $(\mu^*, u^*) \in K_n$ establishing the compactness of K_n.

Since $t_n < t'_n < t_{n+1}$, there exists $(\lambda'_n, u'_n) \in K_n$ with $u'_n = t'_n + w'_n$ with $\int_\Omega w'_n = 0$ and $\lambda'_n > 0$ by part (I). Define

$$\lambda^*_n := \sup\{\lambda : (\lambda, u) \in K_n\}\,. \tag{3.3}$$

Then $\lambda^*_n \geq \lambda'_n > 0$. By repeating the limiting argument above combined with the compactness of K_n, we deduce that there exists u^*_n such that $(\lambda^*_n, u^*_n) \in K_n$.

Using that $\lambda^*_n > 0$ (supercritical) and t_n and t_{n+1} are associated with $\lambda_n < 0$ and $\lambda_{n+1} < 0$, respectively, we have that $t_n < \int_\Omega u^*_n < t_{n+1}$. We can deduce that there is no solution (λ, u) nearby (λ_n^*, u_n^*) with $\lambda > \lambda_n^*$. Otherwise, by the continuity of the projection, we have $t_n < \int_\Omega u < t_{n+1}$. This means $(\lambda, u) \in K_n$, contradicting the definition of λ_n^* in (3.3). Hence (λ_n^*, u_n^*) is a supercritical turning point.

Similarly, letting

$$K_n' := \big\{(\lambda, u) \in \mathscr{D}^+ \cap \mathcal{O} : \int_\Omega u = t' \text{ and } t_n' \le t' \le t_{n+1}'\big\}, \tag{3.4}$$

$$\lambda_{*,n} := \inf\{\lambda : (\lambda, u) \in K_n'\} \tag{3.5}$$

we can show the existence of $u_{*,n}$ such that $(\lambda_{*,n}, u_{*,n}) \in K_n'$ is a subcritical turning point, that is, $\lambda_{*,n} < 0$. Finally, combining the sequences $\{\lambda_{*,n}\}$ and $\{\lambda_n^*\}$ and relabeling, one can choose two subsequences of turning points, one of them subcritical, $\lambda_{2n+1}^* < 0$, and the other supercritical, $\lambda_{2n}^* > 0$. This completes the proof of part (II).

(III) Here we prove the existence of a sequence of resonant solutions, that is solutions u corresponding to $\lambda = 0$. It suffices to show that there exists $n_0 \in \mathbb{N}$ large enough such that for each $n \ge n_0$, both sets K_n and K_n' contain resonant solutions, that is, solutions of the form $(0, u)$.

We give the proof for the sets K_n. Suppose to the contrary that there exists a sequence of integers numbers $n_j \to +\infty$ such that K_{n_j} does not contain any resonant solutions. In that case, the compact sets $K_{n_j}^+ := \{(\lambda, u) \in K_{n_j} : \lambda \ge 0\}$ can be written as $K_{n_j}^+ := (\mathscr{D}^+ \cap \mathcal{O}) \cap \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) : \lambda > 0, t_{n_j} < \int_\Omega u < t_{n_j+1}\}$. Therefore $K_{n_j}^+$ contains at least one connected component of $\mathscr{D}^+$. This connected component is nonempty since there exists at least one solution (λ', u') with $\int_\Omega u' = t'$ with $t' \in (t_{n_j}, t_{n_j+1})$ and therefore $\lambda' > 0$. By construction, since $(t_{n_j}, t_{n_j+1}) \cap (t_{n_{j+1}}, t_{n_j+2}) = \emptyset$, we have that $K_{n_j}^+ \cap K_{n_j+1}^+ = \emptyset$ for $j \in \mathbb{N}$. We recall that a continuum (a closed connected set) cannot contain two nonempty disjoint connected components. Therefore, the fact that we constructed a sequence of nonempty, pairwise disjoint connected components of $\mathscr{D}^+$ contradicts that $\mathscr{D}^+$ is a continuum in $\mathbb{R} \times C(\overline{\Omega})$. Hence, there exists a sequence of resonant solutions, that is a solution u corresponding to $\lambda = 0$.

A similar argument applied to the sets K_n' also results in a sequence of resonant solutions. This completes the proof of (III), and hence of Theorem 1.1. □

Let K_n, K_n', λ_n^* and $\lambda_{*,n}$ be as defined in (3.2), (3.4), (3.3) and (3.5), respectively. Define the sets

$$M_n := \{\lambda : \lambda \ge 0 \text{ and } \exists u \text{ with } (\lambda, u) \in K_n\},$$
$$M_n' := \{\lambda : \lambda \le 0 \text{ and } \exists u' \text{ with } (\lambda, u') \in K_n'\}.$$

Then one can prove the following result.

Corollary 3.1. *For n sufficiently large, we have*

$$M_n = [0, \lambda_n^*], \tag{3.6}$$

$$M_n' = [\lambda_{*,n}, 0]. \tag{3.7}$$

Proof. First, we establish (3.6). By the definition of K_n and λ_n^*, one has

$$M_n \subseteq [0, \lambda_n^*].$$

Now, suppose to the contrary that $[0, \lambda_n^*] \subseteq M_n$ is not true for n sufficiently large. Then there exists a sequence $n_j \to +\infty$ such that $[0, \lambda_{n_j}^*] \not\subseteq M_{n_j}$. So, there exists

$\lambda_{n_j} \in [0, \lambda^*_{n_j}]$ but $\lambda_{n_j} \notin M_{n_j}$. Therefore, there is no function $u_{n_j} \in C(\overline{\Omega})$ with $(\lambda_{n_j}, u_{n_j}) \in K_{n_j}$. From the proof of part (II) of Theorem 1.1 above, we know that $(\lambda^*_{n_j}, u^*_{n_j}) \in K_{n_j}$, and so $\lambda^*_{n_j} \in M_{n_j}$. Hence necessarily $0 \le \lambda_{n_j} < \lambda^*_{n_j}$.

Let $\tilde{K}_{n_j} := \{(\lambda, u) \in K_{n_j}, \lambda > \lambda_{n_j}\}$. Then $\tilde{K}_{n_j} \neq \emptyset$ since $(\lambda^*_{n_j}, u^*_{n_j}) \in \tilde{K}_{n_j}$. Now, proceeding as in the proof of part (III) of Theorem 1.1 above, we can show that $\tilde{K}_{n_j}$ contains at least one nonempty connected component of $\mathscr{D}^+$. As in part (III) above, we can construct a sequence of nonempty, pairwise disjoint connected components of $\mathscr{D}^+$ for n_j large, a contradiction to the fact that $\mathscr{D}^+$ is a continuum. Hence (3.6) holds.

A similar argument establishes (3.7), completing the proof. □

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Maya Chhetri
UNC Greensboro, Greensboro, NC, USA
Email address: `m_chhetr@uncg.edu`

Nsoki Mavinga
Swarthmore College, Swarthmore, PA, USA
Email address: nmaving1@swarthmore.edu

Rosa Pardo
Universidad Complutense de Madrid, Madrid, Spain
Email address: rpardo@ucm.es