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THE MEAN VALUE PROPERTY AND ZEROS OF HOLOMORPHIC FUNCTIONS (GAUSS, POISSON, BOLZANO, AND CAUCHY MEET IN THE COMPLEX PLANE)

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Dedicated to the living memory of Alan C. Lazer

Abstract. An existence condition for a zero of holomorphic functions in a disk is stated and proved in a very simple way using the mean value property. It contains as special cases Bolzano's theorem and Brouwer fixed point theorem in a disk for holomorphic functions, the fundamental theorem of algebra and an asymptotic condition for the existence of zeros of transcendental entire functions. An elementary proof of the used mean value property is given.

1. Introduction

Alan Lazer is remembered for his numerous, deep and stimulating contributions to ordinary differential equations, nonlinear analysis and critical point theory. It may be less known that he contributed in two papers to the vast literature devoted to proving the fundamental theorem of algebra, namely the existence of a complex root to any complex algebraic equation. Not surprisingly for those who know Alan and his work, his two approaches are both original and unusual.

Lazer [9] used a sufficient condition for a maximum of a function of two variables together with an identity for the Laplacian of the square of the modulus of a holomorphic function proved in an elementary way. In [10], the authors gave four proofs of the fundamental theorem of algebra based upon a version of the Fourier transform and inversion formula for continuous functions.

Good examples have to be followed and we propose in this paper to deduce the fundamental theorem of algebra and other existence results for the zeros of holomorphic functions from a new geometric condition for the existence of a zero of a holomorphic function f in a closed disk $\overline{D}_R$ of center 0 and radius R (Theorem 3.1 in Section 3). The proof of this result is based upon the mean value property for a holomorphic function, telling that $f(0)$ is equal to the average of f over any circle of center 0 and radius $r \leq R$, Proceeding by contradiction and assuming that f has no zero in $\overline{D}_R$, the integral over $[0, 2\pi]$ of some associated holomorphic function is computed in two ways, one being the mean value property, and provides different

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results. Section 2 proposes a proof of the used mean value property avoiding any use of complex function techniques.

Theorem 3.1 contains as easy special cases or immediate consequences Shi Mau-Hsiang's extension to holomorphic functions of Bolzano's intermediate value theorem [17] (Corollary 4.1 in Section 4), Brouwer fixed point theorem for holomorphic functions on a closed ball (Corollary 4.3 in Section 4), the fundamental theorem of algebra (Corollary 5.1 in Section 5) and a recent necessary and sufficient asymptotic condition of Bao Qin Li [11] for the existence of a zero of an entire function (Corollary 6.1 in Section 6). An Appendix gives an advanced calculus proof of an extension property of holomorphic functions that is only used in proving the necessary condition in Corollary 6.1. Examples are given in the various sections.

This is how Gauss, Poisson, Bolzano and Cauchy meet in the complex field.

A different and longer proof of the fundamental theorem of algebra (FTA) based on the mean value property was given some years ago by Vyborny [18] and a much shorter one recently by Schep [16]. The monographs [1, 4, 8] and the survey [15] illustrate the richness and diversity of the proofs of the FTA, a good competitor, with more than 250 papers, among the mathematical statements having received the largest number N of (more or less) different proofs. Lazer's correct estimate in [10] is $N > 80$.

2. An elementary proof of the mean value property (or Gauss mean value theorem)

Let $D_R \subset \mathbb{C}$ denote the open disk of center 0 and radius $R > 0$, $\overline{D}_R$ its closure, ∂D_R its boundary.

The function $g : D_R \to \mathbb{C}$ is said to be *holomorphic* in D_R if, for each $z \in D_R$, the limit

$$g'(z) := \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} \tag{2.1}$$

exists and the *complex derivative* $g' : D_R \to \mathbb{C}$ of g is continuous. A holomorphic function in D_R is continuous in D_R, and the usual rules of the calculus of functions of one real variable immediately extend to the complex derivative. For example, for any integer $n \geq 1$, the function $z \to z^n$ is holomorphic on $\mathbb{C}$, and the same is true for any polynomial.

If we consider g as a function of the two real variables (x, y) with $x = \Re z$, the real part of z, and $y = \Im z$, the imaginary part of z, then, by taking respectively h real and h purely imaginary in (2.1), we obtain the *Cauchy-Riemann equations*

$$g'(z) = \frac{\partial g}{\partial x}(z) = \frac{1}{i}\frac{\partial g}{\partial y}(z). \tag{2.2}$$

The last equality in (2.2) is necessary and sufficient for a complex function f of class C^1 on D_R to be holomorphic in D_R.

In particular, if $e^{it} = \cos t + i \sin t$, the chain rule and the relations (2.2) imply that, for all $r \in [0, R)$ and $t \in \mathbb{R}$,

$$\begin{aligned} \frac{\partial}{\partial r}[g(a+re^{it})] &= \frac{\partial g}{\partial x}(a+re^{it})\cos t + \frac{\partial g}{\partial y}(a+re^{it})\sin t \\ &= g'(re^{it})e^{it}, \end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}[g(a+re^{it})] &= \frac{\partial g}{\partial x}(a+re^{it})(-r\sin t)+\frac{\partial g}{\partial y}(a+re^{it})(r\cos t)\\ &= irg'(re^{it})e^{it}\end{aligned}$$

so that

$$\frac{\partial}{\partial r}[g(a+re^{it})] = \frac{1}{ir}\frac{\partial}{\partial t}[g(a+re^{it})]. \tag{2.3}$$

For the reader not familiar with the theory of complex functions, we recall an elementary proof of the *mean value property* for a holomorphic function in a disk, also called *Gauss mean value theorem* because of Gauss' similar result for harmonic functions. It was first stated and proved in 1823 by Poisson [13] for the sum of a power series, and is a special case of Cauchy's integral formula (see e.g. [14, p. 203]).

Lemma 2.1. *If the continuous function $g : \overline{D}_R \to \mathbb{C}$ is holomorphic in $D_R \setminus \{0, z\}$ for some $z \in D_R$, then*

$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{it})\,dt \quad \textit{for all} \quad r \in [0, R].$$

Proof. We define $I : [0, R] \to \mathbb{C}$ by

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} g(re^{it})\,dt.$$

As the functions $t \mapsto g(re^{it})$ and $t \mapsto \frac{\partial}{\partial r}[g(re^{it})]$ are continuous on $[0, 2\pi]$ for all $r \in [0, R]$ and $r \in (0, |z|) \cup (|z|, R)$ respectively, I is continuous on $[0, R]$ and continuously differentiable on $(0, |z|) \cup (|z|, R)$. We can apply Leibniz' rule, formula (2.3), the fundamental theorem of calculus and the 2π-periodicity of the function $g(re^{it})$ to obtain, for all $r \in (0, |z|) \cup (|z|, R)$,

$$I'(r) = \int_0^{2\pi} \frac{\partial}{\partial r}[g(re^{it})]\,dt = \frac{1}{ir}\int_0^{2\pi} \frac{\partial}{\partial t}[g(re^{it})]\,dt = 0.$$

Consequently, $I(r)$ is constant on $(0, |z|)$, and hence on $[0, |z|]$ by continuity, and, for $r \in [0, |z|)$, $I(r) = I(0) = g(0)$. Similarly, $I(r)$ is constant on $(|z|, R)$, and hence on $[|z|, R]$ by continuity, and, for $r \in [|z|, R]$, $I(r) = I(|z|) = g(0)$. □

3. A geometric condition for the existence of a zero of a holomorphic function

Denoting by $\overline{z}$ the conjugate $\overline{z} = x - iy$ of $z = x + iy \in \mathbb{C}$, so that $\Re\overline{z} = \Re z$ and $\Im\overline{z} = -\Im z$, we state and prove the main theorem of this note.

Theorem 3.1. *If the continuous function $f : \overline{D}_R \to \mathbb{C}$ is holomorphic in $D_R \setminus \{0\}$, and if there exists a continuous function $h : \overline{D}_R \to \mathbb{C}$ holomorphic in $D_R \setminus \{0\}$, such that $h(0) = 0$ and such that $\Re[\overline{h}f]$ or $\Im[\overline{h}f]$ does not change sign on ∂D_R, then f has at least one zero in $\overline{D}_R$.*

Proof. It suffices to prove the theorem under the stronger assumption that $\Re[\overline{h}f]$ or $\Im[\overline{h}f]$ keeps a strict sign on ∂D_R. Indeed, for $k = 1, 2, \dots$, if say the strict sign is positive, the functions f_k defined by $f_k(z) = k^{-1}h(z) + f(z)$ verify the stronger assumption when f satisfies the assumption of Theorem 3.1, and a sequence $(z_k)_{k\geq 1}$ in $\overline{D}_R$ verifying $f_k(z_k) = 0$ contains a subsequence converging to some

$z^* \in D_R$ such that $f(z^*) = 0$. For a negative strict sign, it suffices to take $f_k(z) = -k^{-1}h(z) + f(z)$.

Proceeding by contradiction, if $f(z) \neq 0$ for all $z \in \overline{D}_R$ and f satisfies the strong assumption, the function h/f is continuous on $\overline{D}_R$, holomorphic in $D_R \setminus \{0\}$, and hence, by Lemma 2.1 applied to $g = h/f$, we obtain

$$\int_0^{2\pi} \frac{h(Re^{it})}{f(Re^{it})}\,dt = 2\pi \frac{h(0)}{f(0)} = 0. \tag{3.1}$$

On the other hand,

$$\begin{aligned}\int_0^{2\pi} \frac{h(Re^{it})}{f(Re^{it})}\,dt &= \int_0^{2\pi} \frac{h(Re^{it})\overline{f(Re^{it})}}{|f(Re^{it})|^2}\,dt \\ &= \int_0^{2\pi} \Big\{ \frac{\Re[\overline{h(Re^{it})}f(Re^{it})]}{|f(Re^{it})|^2} - i\frac{\Im[\overline{h(Re^{it})}f(Re^{it})]}{|f(Re^{it})|^2} \Big\}\,dt \neq 0,\end{aligned}$$

because the real part or the imaginary part of the integrated function keeps a strict sign for all $t \in [0, 2\pi]$, a contradiction with (3.1). □

Remark 3.2. As

$$\begin{aligned}\Re[\overline{h(z)}f(z)] &= \Re h(x,y)\Re f(x,y) + \Im h(x,y)\Im h(x,y), \\ \Im[\overline{h(z)}f(z)] &= -\Im h(x,y)\Re f(x,y) + \Re h(x,y)\Im f(x,y)\end{aligned}$$

are respectively the inner product in $\mathbb{R}^2$ of the mappings $v(x,y) = (\Re h(x,y), \Im h(x,y))$ and $w(x,y) = (\Re f(x,y), \Im f(x,y))$, and of the mappings $Jv(x,y)$ and $w(x,y)$, with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the symplectic matrix, the assumptions of Theorem 3.1 are geometric conditions on those mappings on ∂D_R and are local (not asymptotic). Furthermore, Theorem 3.1 provides a localization $z \in \overline{D}_R$ for the obtained zeros.

Remark 3.3. In the frame of Brouwer's degree theory [5], Theorem 3.1 is the consequence of the fact that the (strict) assumption implies that v and w are non zero and homotopic on ∂D_R and that the mapping v associated to the holomorphic mapping h vanishes at 0 and hence has a positive Brouwer degree (see e.g. [5]). The simplicity and shortness of the proof given in this paper with respect to a proof based upon Brouwer degree is a consequence of using the supplementary product structure in $\mathbb{C}$ and replacing the estimation of the Brouwer degrees $d[w, D_R, 0]$ and $d[v, D_R, 0]$ by the much more easy evaluation of $\frac{1}{2\pi}\int_0^{2\pi} \frac{h(Re^{it})}{f(Re^{it})}\,dt$.

Remark 3.4. There is no uniqueness conclusion in Theorem 3.1. Indeed, the function $f(z) = z^n - 1$, with $n \geq 2$ an integer, is such that, for $|z| = R \geq 1$, $\Re[\overline{z}^n(z^n - 1)] = R^{2n} - \Re\overline{z}^n \geq R^n(R^n - 1) \geq 0$ and we know that $f(z)$ has n zeros in $\overline{D}_R$.

Example 3.5. Let us define the holomorphic function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = z^n e^z + a\cosh z,$$

where $n \geq 1$ is an integer and $a \in \mathbb{C}$. When $z \in \partial D_R$, we have, if we take $h(z) = z^n e^z$, so that $\overline{h(z)} = \overline{z}^n e^{\overline{z}}$,

$$\Re[\overline{z}^n e^{\overline{z}}(z^n e^z + a\cosh z)] = |z|^{2n} e^{2\Re z} + \Re[a\overline{z}^n e^{\overline{z}}\cosh z]$$

$$\geq R^{2n}e^{-2R} - |a|R^n e^R \cosh z|$$
$$\geq R^n(R^n e^{-2R} - |a|e^R \cosh R) \geq 0$$

if

$$|a| \leq \frac{R^n e^{-3R}}{\cosh R} = \frac{2R^n e^{-4R}}{1+e^{-2R}}.$$

As $e^{-2R} \leq 1$ if $R > 0$, this is surely satisfied if $|a| \leq R^n e^{-4R}$, and the expression in the right-hand member has its maximum value for $R = \frac{n}{4}$. Therefore, f has a zero in $\overline{B}_{n/4}$ when

$$|a| \leq e^{n(\log n - 2\log 2 - 1)}.$$

This implies in particular that, given any $a \in \mathbb{C}$, the function $f(z) = z^n e^z + a\cosh z$ has a zero in $\overline{B}_{n/4}$ for all sufficiently large n.

4. Hadamard-Shi's existence theorem and Brouwer's fixed point theorem for a holomorphic function

The special case $h(z) = z$ of Theorem 3.1, proved in [12] by a similar argument, using Cauchy's integral theorem instead of the mean value property, generalizes a result that Mau-Hsiang Shi [17] had obtained under the stronger assumption $\Re[\overline{z}f(z)] > 0$ on ∂D_R, using Rouché's theorem in complex analysis (see e.g. [14, p. 390]).

Corollary 4.1. *If the continuous function $f : \overline{D}_R \to \mathbb{C}$ is holomorphic in D_R, and if $\Re[\overline{z}f(z)]$ or $\Im[\overline{z}f(z)]$ does not change sign on ∂D_R, then f has at least one zero in $\overline{D}_R$.*

Corollary 4.1 extends to holomorphic functions Bolzano's condition "$f(-R)$ and $f(R)$ have opposite signs" or, equivalently, "$-Rf(-R)$ and $Rf(R)$ have the same sign" for the existence of a zero in $[-R, R]$ of the continuous function $f : [-R, R] \to \mathbb{R}$. As

$$\Re[\overline{z}f(z)] = \langle (x, y), (\Re f(x, y), \Im f(x, y)) \rangle,$$

we have named it *Hadamard-Shi's theorem* in [12], by reference to the early use of this condition by Hadamard of [7] in his proof of the n-dimensional Brouwer fixed point theorem for a continuous self mapping on a closed n-ball (see e.g. [5, Chapter 8]).

Remark 4.2. According to Remark 3.2, the assumption on $\Re[\overline{z}f(z)]$ means that the vector field corresponding to f qualitatively behaves like the outer or the inner normal vector field on ∂B_R, and the assumption on $\Im[\overline{z}f(z)]$ means that the vector field corresponding to f qualitatively behaves like the tangent vector field on ∂D_R.

Hadamard's argument and Corollary 4.1 provide a version for holomorphic functions of a slightly *generalized Brouwer's fixed point theorem* first obtained by Birkhoff and Kellogg in 1922 [2].

Corollary 4.3. *If the continuous function $c : \overline{D}_R \to \mathbb{C}$ is holomorphic in D_R and such that $c(\partial D_R) \subseteq \overline{D}_R$, then c has at least one fixed point in $\overline{D}_R$.*

Proof. Defining $f : \overline{D}_R \to \mathbb{C}$ by $f(z) = z - c(z)$, we have, for all $z \in \partial D_R$,

$$\Re[\overline{z}f(z)] = |z|^2 - \Re[\overline{z}c(z)] \geq R^2 - |\overline{z}c(z)| = R^2 - R|c(z)| \geq 0,$$

and the result follows from Corollary 4.1. □

Example 4.4. Given $a \in \mathbb{C}$ and $c(z) = a \cos z$, we have, if $|z| = R$,

$$\begin{aligned} |c(z)| = |a \cos z| &= |a| \Big| \frac{e^{iz} + e^{-iz}}{2} \Big| \\ &\le |a| \frac{e^{-\Im z} + e^{\Im z}}{2} = |a| \cosh \Im z \\ &\le |a| \cosh R \le R \end{aligned}$$

if $|a| \le R/\cosh R$. Now, the function $R \mapsto R/\cosh R$ reaches its maximum on $(0, +\infty)$ when $R = 1/\tanh R$, namely $R^* \approx 1.199\dots$. Hence, for $|a| \le R^*/\cosh R^* \approx 0.662\dots$, $c(\partial B_{R^*}) \subseteq \overline{B}_{R^*}$ and c has a fixed point in $\overline{B}_{R^*}$ by Corollary 4.3.

5. The fundamental theorem of algebra

The *fundamental theorem of algebra* is an easy consequence of Theorem 3.1.

Corollary 5.1. Any polynomial

$$p(z) = \sum_{k=0}^{n} a_k z^k \quad (a_n \neq 0)$$

of effective degree $n \ge 1$ with coefficients $a_k \in \mathbb{C}$ $(k = 0, 1, \dots, n)$ has at least one zero in $\mathbb{C}$.

Proof. Without loss of generality, we can assume that $a_n = 1$. We take $f(z) = p(z)$ and $h(z) = z^n$ in Theorem 3.1. If $z \in \partial D_r$,

$$\begin{aligned} \Re[\overline{z}^n p(z)] &= |z|^{2n} + \sum_{k=1}^{n-1} \Re[a_k z^k \overline{z}^n] \ge r^{2n} - \sum_{k=0}^{n-1} |\Re[a_k z^k \overline{z}^n]| \\ &\ge r^{2n} - \sum_{k=0}^{n-1} |a_k| r^{k+n} = r^{2n}\Big(1 - \sum_{k=0}^{n-1} |a_k| r^{k-n}\Big). \end{aligned}$$

Now, there exists $R > 0$ such that $\sum_{k=0}^{n-1} |a_k| R^{k-n} \le 1$, and p satisfies the assumptions of Theorem 3.1 on $\overline{D}_R$. □

Remark 5.2. The fundamental theorem of algebra can also be directly deduced from Lemma 2.1, as shown by Schep [16]. If p has no zero, $1/p$ is holomorphic on $\mathbb{C}$ and, by Lemma 2.1 with $g = 1/p$, we have

$$0 \neq \frac{1}{p(0)} = \frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{p(re^{it})} \quad \text{for all } r > 0.$$

As $p(z) \to \infty$ as $z \to \infty$, the right-hand member tends to 0 when $r \to \infty$ uniformly in $t \in [0, 2\pi]$, leading to a contradiction by going to the limit under the integral sign in the formula above.

If $n \ge 1$ is an integer, and $c : \overline{D}_R \to \mathbb{C}$, we say that $z \in \overline{D}_R$ is a *n-branch point of* c if $z^n = c(z)$. A 1-branch point of c is a fixed point of c. We have the following *n-branch point theorem.*

Corollary 5.3. *If the continuous function $c : \overline{D}_R \to \mathbb{C}$ is holomorphic in D_R and if there exists an integer $n \ge 1$ such that $c(\partial D_R) \subseteq \overline{D}_{R^n}$, then c has an n-branch point in $\overline{D}_R$.*

Proof. We apply Theorem 3.1 to $f(z) = z^n - c(z)$ and $h(z) = z^n$ and have, for all $z \in \partial D_R$,

$$\Re[\overline{z}^n (z^n - c(z))] = R^{2n} - \Re[\overline{z}^n c(z)] \geq R^{2n} - R^n |c(z)| \geq 0.$$

□

Remark 5.4. *The fundamental theorem of algebra is also a consequence of the n-branch point theorem applied to* $c(z) = -\sum_{k=0}^{n-1} a_k z^k$. *Indeed,*

$$|c(z)| \leq \sum_{k=0}^{n-1} |a_k| R^k \leq R^n \quad \text{for all } z \in \partial D_R,$$

if $R > 0$ *is taken so large that* $\sum_{k=0}^{n-1} |a_k| R^{k-n} \leq 1$.

6. Li's asymptotic condition for the existence of a zero for an entire function

Recall that an *entire* function $f : \mathbb{C} \to \mathbb{C}$ is a function which is holomorphic on $\mathbb{C}$. It is either a polynomial or a *transcendental entire* function, like $\exp z$, $\sin z$ or $\cosh z$ for example.

Theorem 3.1 provides a very simple proof of the sufficiency part of a simpler equivalent statement of a recent result of Bao Qin Li [11].

Corollary 6.1. *The entire function* $f : \mathbb{C} \to \mathbb{C}$ *has a zero if and only if there exists an entire function* $h : \mathbb{C} \to \mathbb{C}$ *such that* $h(0) = 0$ *and* $\lim_{z\to\infty} h(z)/f(z)$ *exists and is not zero.*

Proof. Sufficiency. Let $h : \mathbb{C} \to \mathbb{C}$ be an entire function such that $h(0) = 0$ and $\lim_{z\to\infty} \frac{h(z)}{f(z)} = b + ic \neq 0$. As

$$\lim_{r\to+\infty} \Re\Big[\frac{h(re^{it})}{f(re^{it})}\Big] = \lim_{r\to\infty} \frac{\Re[\overline{h(re^{it})} f(re^{it})]}{|f(re^{it})|^2} = b,$$

$$\lim_{r\to+\infty} \Im\Big[\frac{h(re^{it})}{f(re^{it})}\Big] = - \lim_{r\to\infty} \frac{\Im[\overline{h(r(e^{it})} f(re^{it})]}{|f(re^{it})|^2} = -c,$$

uniformly in $t \in [0, 2\pi]$, and $b^2 + c^2 \neq 0$, there exists $R > 0$ such that $\Re[\overline{h(Re^{it})} f(Re^{it})]$ or $\Im[\overline{h(Re^{it})} f(Re^{it})]$ keeps a constant sign for all $t \in [0, 2\pi]$. Using Theorem 3.1, f has a zero in $\overline{D}_R$.

Necessity. Let $a \in \mathbb{C}$ such that $f(a) = 0$. The function $\widetilde{h} : \mathbb{C} \to \mathbb{C}$ defined by

$$\widetilde{h}(z) = \frac{z+a}{z} f(z+a) \quad \text{if} \quad z \neq 0,\ h(0) = af'(a)$$

is entire (see e.g.' [14, p. 212] or see Corollary 7.4 in the Appendix). Hence the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) = \widetilde{h}(z-a) = \frac{z}{z-a} f(z) \quad \text{if} \quad z \neq a,\ h(a) = af'(a),$$

is entire and such that

$$\lim_{z\to\infty} \frac{h(z)}{f(z)} = \lim_{z\to\infty} \frac{z}{z-a} = 1.$$

□

Remark 6.2. As shown in [11], the fundamental theorem of algebra is also a consequence of Corollary 6.1 with the choice of $h(z) = z^n$.

7. Appendix: an elementary proof for the removability of an apparent singularity

In this appendix, for the reader not familiar with the techniques of the theory of complex functions, we give an elementary proof of the holomorphic continuation property used only in the necessity part of Corollary 6.1. We start with an easy consequence of Lemma 2.1.

Corollary 7.1. *If the continuous function $g : \overline{D}_R \to \mathbb{C}$ is holomorphic in $D_R \setminus \{0\}$ then, for each $z \in D_R$,*

$$\int_0^{2\pi} [g(re^{it}) - g(z)] \frac{re^{it}}{re^{it} - z}\, dt = 0 \quad \text{for all } r \in [0, R].$$

Proof. We define the function $h : \overline{D}_R \to \mathbb{C}$ by

$$h(u) = [g(u) - g(z)] \frac{u}{u - z} \quad \text{if} \quad u \neq z, \; h(z) = g'(z)z.$$

Clearly h is continuous on $\overline{D}_R$, holomorphic in $D_R \setminus \{0, z\}$, and $h(0) = 0$. Lemma 2.1 applied to h gives the result. $\square$

Lemma 7.2. *Given $z \in \mathbb{C}$, one has, for each $r > |z|$,*

$$\int_0^{2\pi} \frac{re^{it}}{re^{it} - z}\, dt = 2\pi.$$

Proof. The statement of this lemma is equivalent to

$$\int_0^{2\pi} \frac{z}{re^{it} - z}\, dt = 0.$$

If we define $J : (|z|, +\infty) \to \mathbb{C}$ by

$$J(r) = \int_0^{2\pi} \frac{z}{re^{it} - z}\, dt$$

then we have

$$\frac{\partial}{\partial r}\Big(\frac{z}{re^{it} - z}\Big) = -\frac{z}{(re^{it} - z)^2} = \frac{1}{ri}\frac{\partial}{\partial t}\Big(\frac{z}{re^{it} - z}\Big),$$

and, using the easily justified Leibniz rule, the fundamental theorem of calculus and the 2π-periodicity of the integrated function,

$$J'(r) = \int_0^{2\pi} \frac{\partial}{\partial r}\Big(\frac{z}{re^{it} - z}\Big)\, dt = \frac{1}{ri}\int_0^{2\pi} \frac{\partial}{\partial t}\Big(\frac{z}{re^{it} - z}\Big)\, dt = 0.$$

Hence, $J(r)$ is constant on $(|z|, +\infty)$ and

$$J(r) = \lim_{r \to +\infty} I(r) = \int_0^{2\pi} \lim_{r \to +\infty} \Big(\frac{z}{re^{it} - z}\Big)\, dt = 0.$$

$\square$

We now prove a generalized version of the mean value property due to Cauchy (*Cauchy's integral formula on a disc*) [3], expressing, for each $z \in D_R$, $g(z)$ as the average of its values on ∂D_R with respect to the complex measure $[Re^{it}/(Re^{it} - z)]\, dt$. It reduces to (2.1) for $z = 0$.

Lemma 7.3. *If $g \in C(\overline{D}_R) \cap \mathcal{H}(D_R \setminus \{0\})$, then, for each $z \in D_R$ one has*
$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \frac{Re^{it}}{Re^{it} - z}\, dt.$$

Proof. By Corollary 7.1, we have
$$g(z) \int_0^{2\pi} \frac{Re^{it}}{Re^{it} - z}\, dt = \int_0^{2\pi} g(Re^{it}) \frac{Re^{it}}{Re^{it} - z}\, dt,$$
and the result follows from Lemma 7.2. □

Finally, we prove that a continuous function in $\overline{D}_R$ which is holomorphic in $D_R \setminus \{0\}$ is holomorphic in D_R.

Corollary 7.4. *If the continuous function $g : \overline{D}_R \to \mathbb{C}$ is holomorphic in $D_R \setminus \{0\}$, then g is holomorphic in D_R.*

Proof. From Lemma 7.3 and the Leibniz rule, we have, for all $z \in D_R$,
$$g'(z) = \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \frac{Re^{it}}{(Re^{it} - z)^2}\, dt$$
and, for $z \in D_R \setminus \{0\}$,
$$\begin{aligned} g(z) - g(0) &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \Big(\frac{Re^{it}}{Re^{it} - z} - 1 \Big)\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \Big(\frac{z}{Re^{it} - z} \Big)\, dt, \end{aligned}$$
so that
$$\begin{aligned} g'(0) &= \lim_{z \to 0} \frac{g(z) - g(0)}{z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \lim_{z \to 0} \Big(\frac{1}{Re^{it} - z} \Big)\, dt \\ &= \frac{1}{2\pi R} \int_0^{2\pi} g(Re^{it}) e^{-it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(Re^{it}) \lim_{z \to 0} \frac{Re^{it}}{(Re^{it} - z)^2}\, dt \\ &= \lim_{z \to 0} g'(z). \end{aligned}$$
Hence, g' exists and is continuous in D_R. □

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