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AN ASYMMETRIC PROBLEM AT RESONANCE WITH A ONE-SIDED AHMAD-LAZER-PAUL CONDITION

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Dedicated to the memory of Alan C. Lazer

Abstract. In this article, we study the semilinear elliptic boundary value problem

$$\begin{aligned} -\Delta u &= -\lambda_1 u^- + g(x,u), \quad \text{in } \Omega; \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned}$$

where u^- denotes the negative part of $u : \Omega \to \mathbb{R}$; λ_1 is the first eigenvalue of the N-dimensional Laplacian with Dirichlet boundary conditions in a connected, open, bounded set $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$; and $g{:}\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Assuming a one-sided Ahmad-Lazer-Paul condition, we establish conditions for existence and multiplicity of solutions by using variational methods and infinite-dimensional Morse Theory.

1. Introduction

Alan Lazer was one of the most influential contributors to critical point theory and its applications to differential equations starting in the 1960s. Castro [6] provided an overview of Lazer's main results and contributions in that field; among them, the work with Landesman on problems involving resonant conditions [14] and the one in collaboration with Ahmad and Paul [2] that inspired Rabinowitz to prove his well-known saddle point theorem [18]. These are the two results that motivated the work of this article.

Let Ω be a bounded, connected, open subset of $\mathbb{R}^N$, for $N \geq 2$, with smooth boundary $\partial\Omega$. Consider the Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda_k u + g(x,u), \quad x \in \Omega; \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where λ_k is an eigenvalue of the N-dimensional Laplacian $-\Delta$ in Ω with Dirichlet boundary conditions, and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and uniformly bounded; that is,

$$|g(x,s)| \leqslant M, \quad \text{for all } x \in \overline{\Omega}, \text{ and } s \in \mathbb{R}, \tag{1.2}$$

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and for some positive constant M. Set $f(x,s)=\lambda_k s+g(x,s)$, for $s\in\mathbb{R}$ and $x\in\overline{\Omega}$. It then follows, by (1.2), that,

$$\frac{f(x,s)}{s}\to\lambda_k \quad \text{as } |s|\to\infty, \text{ uniformly in } x.$$

We then say that problem (1.1) is a *resonant problem*. To study the existence of solutions of such problems, Ahmad, Lazer, and Paul [2] used a variational approach to problem (1.1), which we will describe next.

Denote by $X=H_0^1(\Omega)$ the Sobolev space obtained through completion of $C_c^\infty(\Omega)$ with respect to the metric induced by the norm

$$\|u\|=\Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}, \quad \text{for all } u\in X.$$

We can associate an energy functional $J:X\to\mathbb{R}$ to problem (1.1) by defining

$$J(u)=\frac{1}{2}\int_\Omega |\nabla u|^2\,dx-\frac{\lambda_k}{2}\int_\Omega u^2\,dx-\int_\Omega G(x,u)\,dx, \quad \text{for } u\in X, \tag{1.3}$$

where G is a primitive integral of g in the second variable satisfying $G(x,0)=0$, for $x\in\overline{\Omega}$; that is, $G(x,s)=\int_0^s g(x,\xi)\,d\xi$, for $(x,s)\in\overline{\Omega}\times\mathbb{R}$.

The Fréchet derivative of the functional J defined in (1.3) is given by

$$\langle J'(u),\varphi\rangle=\int_\Omega \nabla u\cdot\nabla\varphi\,dx-\lambda_k\int_\Omega u\varphi\,dx-\int_\Omega g(x,u)\varphi\,dx, \quad \text{for all } \varphi\in X. \tag{1.4}$$

A weak solution of problem (1.1) corresponds to a critical point of J; that is, a solution of the equation

$$\int_\Omega \nabla u\cdot\nabla\varphi\,dx-\lambda_k\int_\Omega u\varphi\,dx=\int_\Omega g(x,u)\varphi\,dx, \quad \text{for all } \varphi\in X. \tag{1.5}$$

The authors of [2] also imposed an additional assumption on G and the eigenspace corresponding to λ_k; namely,

$$\lim_{\|v\|\to\infty}\int_\Omega G(x,v(x))\,dx=+\infty, \tag{1.6}$$

or

$$\lim_{\|v\|\to\infty}\int_\Omega G(x,v(x))\,dx=-\infty, \tag{1.7}$$

where $v\in\ker(\Delta+\lambda_k I)$. Assuming this condition, Ahmad, Lazer and Paul obtained the existence result given in the following theorem.

Theorem 1.1 ([2, Theorem 1])**.** *Under conditions* (1.2) *and* (1.6) *or* (1.7)*, problem* (1.1) *has a weak solution in* $H_0^1(\Omega)$*.*

The role of the Ahmad-Lazer-Paul (ALP) condition (1.6), or (1.7), is apparent when placing problem (1.1) in the framework of the saddle point theorem of Rabinowitz (See [18] or [8]). The ALP condition plays a crucial part in establishing the Palais-Smale condition for the functional associated with problem (1.1); the condition is also instrumental in verifying the geometric conditions required by the saddle point theorem of Rabinowitz.

In this article, we study a variant of problem (1.1) in which $k=1$ and g is not bounded. We consider the problem

$$\begin{gathered}-\Delta u=-\lambda_1 u^-+g(x,u), \quad \text{in } \Omega;\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.8}$$

where $u^- = \max\{-u, 0\}$ denotes the negative part of $u : \Omega \to \mathbb{R}$ and λ_1 is the first eigenvalue of the N-dimensional Laplacian with Dirichlet boundary conditions in Ω. We will assume the nonlinearity g and its primitive G satisfy the following conditions:

(A1) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g(x, 0) = 0$ for all $x \in \overline{\Omega}$.

(A2) There exists a constant σ such that
$$1 \leqslant \sigma < \frac{N+2}{N-2}, \quad \text{for } N \geqslant 3, \tag{1.9}$$
or $1 \leqslant \sigma < \infty$ for $N = 2$, and
$$\lim_{s \to +\infty} \frac{g(x,s)}{s^\sigma} = 0,$$
uniformly for a.e $x \in \Omega$.

(A3) There are constants $\mu > 2$ and $s_0 > 0$ such that
$$0 < \mu G(x,s) \leqslant s g(x,s), \quad \text{for } s \geqslant s_0, \text{ and } x \in \overline{\Omega}.$$

(A4) $\lim_{s \to -\infty} g(x,s) = 0$, uniformly for a.e. $x \in \overline{\Omega}$.

(A5) (ALP condition)
$$\lim_{t \to -\infty} \Big| \int_\Omega G(x, t\varphi_1(x))\,dx \Big| = +\infty,$$
uniformly in x, where φ_1 is the positive eigenfunction associated with the first eigenvalue of $(-\Delta, H_0^1(\Omega))$, and $\|\varphi_1\| = 1$.

Problem (1.8) was studied in [21] for the case in which the limit
$$\lim_{t \to -\infty} \int_\Omega G(x, t\varphi_1(x))\,dx$$
assumed a finite value; that is, when g has a *strong resonance* behavior at infinity. A similar problem was studied in [20], where the condition (A5) was given by a Landesman-Lazer condition; namely,
$$\int_\Omega g_{-\infty}(x)\varphi_1(x)\,dx > 0, \tag{1.10}$$
with $g_{-\infty}(x) = \lim_{s \to -\infty}[g(x,s) - \lambda_1 s]$ uniformly for a.e. $x \in \Omega$ and $g_{-\infty} \in L^\infty(\Omega)$ such that $|g_{-\infty}(x)| \leqslant M$, for all $x \in \Omega$, and for some constant $M > 0$. It can be shown, using L'Hôpital's rule, that if (1.10) is satisfied then it implies the ALP condition (A_5). Therefore, the problem (1.8) with the conditions (A1)–(A5) is more general and we are interested in determining existence and multiplicity of solutions of that problem in subsequent sections. For further readings on the Landesman-Lazer and ALP conditions, we refer the reader to the article by Fonda and Garrione in [11].

This article is organized as follows: In section 2 we present some preliminary results that will be used throughout this paper. In section 3, we prove that the functional J satisfies the Palais-Smale condition. The critical groups at infinity of the energy functional associated to problem (1.8) are computed in section 4. In section 5, we compute the critical groups at the origin and establish the first existence result as a consequence of a result due to Perera and Schechter and the Morse relation. Finally, in section 6, we establish a second existence and multiplicity result by using a cutoff-technique, Ekeland's variational principle, and a standard argument involving the Morse relation.

2. Preliminaries

In this section, we establish some notation and results that will be used in subsequent sections.

Denote by $X = H_0^1(\Omega)$ the Sobolev space obtained through completion of $C_c^\infty(\Omega)$ with respect to the metric induced by the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 \, dx \Big)^{1/2}, \quad \text{for all } u \in X.$$

The associated energy functional $J : X \to \mathbb{R}$ to problem (1.8) is

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda_1}{2} \int_\Omega (u^-)^2 \, dx - \int_\Omega G(x, u(x)) \, dx, \tag{2.1}$$

for $u \in X$, and $G(x,s) = \int_0^s g(x,\xi)\, d\xi$ is a primitive of g. It follows from condition (A2) that $J \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle J'(u), \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \lambda_1 \int_\Omega u^- \varphi \, dx - \int_\Omega g(x,u)\varphi \, dx, \quad \text{for all } \varphi \in X. \tag{2.2}$$

Critical points of (2.1) correspond to weak solutions of (1.8); that is, points $u \in X$ such that

$$\langle J'(u), \varphi \rangle = 0, \quad \text{for all } \varphi \in X.$$

To apply some of the known existence theorems in critical point theory, we will need to verify that the functional J given in (2.1) satisfies a compactness condition known as the Palais-Smale condition, which we present next.

Definition 2.1. We say that a functional J defined on a Banach space X satisfies the Palais-Smale condition, or PS condition, if any sequence $(u_m) \subset X$ for which $J(u_m)$ is bounded and $J'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence. We will say that (u_m) is a PS sequence for J if

$$|J(u_m)| \leqslant C \quad \text{for all } m \quad \text{and} \quad J'(u_m) \to 0 \quad \text{as } m \to \infty, \tag{2.3}$$

where C is a constant.

In the next sections, we need to use some estimates on g and its primitive G. First, it follows from condition (A4) that there exists $s_1 > 0$ such that

$$-1 \leqslant g(x,s) \leqslant 1, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}. \tag{2.4}$$

From this estimate we obtain

$$|g(x,s)| \leqslant C_0, \quad \text{for } s \leqslant 0, \text{ and all } x \in \overline{\Omega}, \tag{2.5}$$

and for some constant $C_0 > 0$.

It also follows from (2.4) that

$$-|s| \leq s g(x,s) \leqslant |s|, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}. \tag{2.6}$$

Integrating the inequality in (2.4) we obtain

$$-C_1 - |s| \leqslant G(x,s) \leqslant C_1 + |s|, \quad \text{for } s < -s_1, \text{ and all } x \in \overline{\Omega}, \tag{2.7}$$

and for some constant $C_1 > 0$.

Combining the estimates (2.6) and (2.7), we can show that there exist constants $C_2, C_3 > 0$ such that

$$-C_2|s| - C_3 \leqslant g(x,s) - 2G(x,s) \leqslant C_2|s| + C_3 \quad \text{for } s \leqslant 0, \text{ and } x \in \overline{\Omega}. \tag{2.8}$$

Similarly, we obtain from (A2) that there exists a positive constant C_5 such that

$$|g(x,s)| \leqslant C_5 + |s|^\sigma, \quad \text{for } s \geqslant 0 \text{ and } x \in \overline{\Omega}. \tag{2.9}$$

For more details on these calculations, see [20, Lemma 3.1].

Finally, by condition (A3), there exists $s_0 > 0$ and constants $C_6, C_7 > 0$ such that

$$G(x,s) \geqslant C_6 s^\mu - C_7, \quad \text{for } s > s_0 \text{ and all } x \in \overline{\Omega}. \tag{2.10}$$

We will apply Morse theory to study the behavior of J near a critical point u_0 with the aim of obtaining multiplicity results for problem (1.8). In what follows, we discuss the concept of critical groups, which will be used in subsequent sections.

Put $J^c = \{u \in X | J(u) \leqslant c\}$, the sub–level set of J at c, and set

$$\mathcal{K} = \{u \in X | J'(u) = 0\},$$

the critical set of J.

Let u_0 denote an isolated critical point of J. The q-critical groups of J at u_0, with coefficients in a field $\mathbb{F}$, are defined by

$$C_q(J,u_0) = H_q(J^{c_0} \cap U, J^{c_0} \cap U \backslash \{u_0\}), \quad \text{for all } q \in \mathbb{Z}, \tag{2.11}$$

where $c_0 = J(u_0)$, U is a neighborhood of u_0 that contains no critical points of J other than u_0, and H_* denotes the singular homology groups. For negative values of q, the homology groups are defined to be the trivial group.

The critical groups are independent of the choice of U by the excision property of homology (see Hatcher [13]). To be consistent with the results presented in section 5, we will choose the field $\mathbb{F}$ to be $\mathbb{Z}_2$. For more information on the definition of critical groups, we refer the reader to [7, 17, 16, 15].

If the set of critical values of J is bounded from below and J satisfies the PS condition, the global behavior of J can be described by the critical groups at infinity defined by Bartsch and Li [3] as follows:

$$C_q(J,\infty) = H_q(X, J^{a_0}), \quad \text{for all } q \in \mathbb{Z}, \tag{2.12}$$

where $a_0 < \inf J(\mathcal{K})$. These groups are well–defined as a consequence of the second deformation lemma (see [17, Lemma 1.1.2]).

Next, we present the Morse relation. Let $J : X \to \mathbb{R}$ be a C^1 functional that satisfies the PS condition. Let $\mathcal{K}_{a_1}^{b_1} = \{u_1, \dots, u_n\}$ be a finite set of critical points of J such that $J(u) \in [a_1, b_1]$, for all $u \in \mathcal{K}_{a_1}^{b_1}$. Then, we can define the Morse-type numbers of the pair (J^{b_1}, J^{a_1}) by

$$M_q := M_q(J^{b_1}, J^{a_1}) = \sum_{u \in \mathcal{K}_{a_1}^{b_1}} \dim C_q(J,u), \quad q = 0,1,2,\dots. \tag{2.13}$$

Applying the infinite dimensional Morse theory developed in [15, 7, 16], we can derive the Morse relation

$$\sum_{q=0}^\infty M_q t^q = \sum_{q=0}^\infty \beta_q t^q + (1+t) \sum_{q=0}^\infty a_q t^q, \tag{2.14}$$

where $\beta_q = \dim C_q(J,\infty)$, and a_q are non-negative numbers. The numbers β_q are also called the Betti numbers of the pair (X, J^{a_0}) and are important invariants in the study of topological spaces.

3. Compactness condition

In this section, we will verify that the functional $J{:}X \to \mathbb{R}$ given in (2.1) satisfies Palais-Smale condition.

In what follows, we will use the symbol C to represent any positive constant. Hence, C might represent different constants in various estimates, even in the same inequality.

Lemma 3.1. *If conditions* (A1)–(A5) *are satisfied, then the functional J defined in* (2.1) *satisfies the PS condition.*

Proof. We follow a similar approach to that used in [20, Proposition 4.1] and [21, Proposition 2.1]. Let $(u_m) \subset X$ be a PS sequence for J; that is,

$$\Big|\|u_m\|^2 - \lambda_1\|u_m^-\|_{L^2}^2 - \int_\Omega 2G(x,u_m)\,dx\Big| \leqslant C, \tag{3.1}$$

for all m and

$$\Big|\int_\Omega \nabla u_m\cdot\nabla\varphi\,dx + \lambda_1\int_\Omega u_m^-\varphi\,dx - \int_\Omega g(x,u_m)\varphi\,dx\Big| \leqslant \varepsilon_m\|\varphi\|, \tag{3.2}$$

for all $\varphi \in X$, where $\varepsilon_m \to 0^+$ as $m \to \infty$. Since g has subcritical growth at infinity, by virtue (1.9) in condition (A2), it is enough to show that the sequence $(\|u_m\|)$ is bounded (see, for instance, [18, Proposition B.35]).

Choose $\varphi = u_m$ in (3.2) and use the fact that $u_m = u_m^+ - u_m^-$, where $u_m^\pm = \max\{0, \pm u_m\}$ denotes the positive (negative) parts of u_m, respectively, to obtain

$$\Big|\|u_m\|^2 - \lambda_1\|u_m^-\|_{L^2}^2 - \int_\Omega g(x,u_m)u_m\,dx\Big| \leqslant \varepsilon_m(\|u_m^+\| + \|u_m^-\|), \quad \text{for all } m. \tag{3.3}$$

Combining (3.3) and (3.1), we have that

$$\Big|\int_\Omega [g(x,u_m)u_m - 2G(x,u_m)]\,dx\Big| \leqslant C + \varepsilon_m(\|u_m^+\| + \|u_m^-\|), \quad \text{for all } m. \tag{3.4}$$

Put $T(x,s) = g(x,s)s - 2G(x,s)$, for $x \in \Omega$ and $s \in \mathbb{R}$. Then, the integral on the left side of (3.4) can be written as

$$\int_\Omega T(x,u_m)\,dx = \Big[\int_{u_m<0} + \int_{0\leqslant u_m\leqslant s_0} + \int_{u_m>s_0}\Big]T(x,u_m)\,dx, \tag{3.5}$$

for all m. The constant s_0 is given by (2.10).

We will find estimates for each integral in the right-hand side of (3.5).

By (2.8), we have

$$\Big|\int_{u_m<0} T(x,u_m)\,dx\Big| \leqslant C\|u_m^-\| + C, \quad \text{for all } m. \tag{3.6}$$

Note that the second integral term in (3.5) is bounded uniformly with respect to m; that is,

$$\Big|\int_{0\leqslant u_m\leqslant s_0} T(x,u_m)\,dx\Big| \leqslant C, \quad \text{for all } m. \tag{3.7}$$

For the third integral term in (3.5), we use condition (A3) to obtain

$$\int_{u_m>s_0} T(x,u_m)\,dx \geqslant (\mu-2)\int_{u_m>s_0} G(x,u_m)\,dx, \quad \text{for all } m. \tag{3.8}$$

Then, combining (3.8) with (3.4) and (3.6), we obtain

$$\Big|\int_{u_m>s_0} G(x,u_m)\,dx\Big| \leqslant \varepsilon_m\|u_m^+\| + (C+\varepsilon_m)\|u_m^-\| + C, \quad \text{for all } m. \tag{3.9}$$

Set $\varphi = u_m^+$ in (3.2) to obtain

$$\Big|\|u_m^+\|^2 - \int_{u_m\geqslant 0} g(x,u_m)u_m\,dx\Big| \leqslant \varepsilon_m\|u_m^+\|, \quad \text{for all } m. \tag{3.10}$$

It follows from (3.10), in combination with (3.4), and (3.9), that

$$\begin{aligned}
\|u_m^+\|^2 &\leqslant \varepsilon_m\|u_m^+\| + \Big|\int_{u_m\geqslant 0} g(x,u_m)u_m\,dx\Big| \\
&\leqslant \varepsilon_m\|u_m^+\| + \Big|\int_{u_m\geqslant 0} T(x,u_m)\,dx\Big| + \Big|\int_{u_m\geqslant 0} 2G(x,u_m)\,dx\Big|, \\
&\leqslant \varepsilon_m\|u_m^+\| + \Big|\int_{\Omega} T(x,u_m)\,dx\Big| + \Big|\int_{u_m\geqslant 0} 2G(x,u_m)\,dx\Big|;
\end{aligned}$$

so that

$$\|u_m^+\|^2 \leqslant C + 3\varepsilon_m\|u_m^+\| + (C+2\varepsilon_m)\|u_m^-\|, \tag{3.11}$$

for all m.

Completing the square in (3.11), we can show that

$$\|u_m^+\| \leqslant \frac{3}{2}\varepsilon_m + \sqrt{C + 2\varepsilon_m\|u_m^-\|}, \quad \text{for all } m. \tag{3.12}$$

If the sequence $(\|u_m^-\|)$ is bounded, then, by virtue of (3.12), $(\|u_m^+\|)$ would also be bounded. That would imply that $(\|u_m\|)$ is bounded and this would complete the proof of the lemma. Therefore, it suffices to prove that $(\|u_m^-)\|$ is bounded.

Arguing by contradiction, assume that

$$\|u_m^-\| \to \infty, \quad \text{as } m\to\infty, \tag{3.13}$$

where we have passed to a subsequence, which we also denote by (u_m), if needed. Setting $\varphi = -u_m^-$ in (3.2) we obtain

$$\Big|\int_\Omega |\nabla u_m^-|^2\,dx - \lambda_1\int_\Omega (u_m^-)^2\,dx + \int_\Omega g(x,u_m)u_m^-\,dx\Big| \leqslant \varepsilon_m\|u_m^-\|, \tag{3.14}$$

for all m. Next, set

$$v_m = -\frac{u_m^-}{\|u_m^-\|}, \quad \text{for all } m; \tag{3.15}$$

so that, $\|v_m\| = 1$, for all m. It then follows by the Banach-Alaoglu's Theorem ([4, Theorem 3.16]) that there exists $v\in X$ and a subsequence of (v_m), which we will still denote by (v_m), such that

$$\begin{gathered}
v_m \rightharpoonup v \quad \text{weakly in } X, \\
v_m \to v \quad \text{in } L^q(\Omega), \text{ for all } q\in \big[1, \frac{2N}{N-2}\big) \\
v_m(x)\to v(x) \quad \text{a.e. in } \Omega, \\
|v_m(x)| \leqslant b(x), \quad \text{a.e. in } \Omega \text{ with } b\in L^q(\Omega).
\end{gathered} \tag{3.16}$$

Using the estimate in (2.9) and similar calculations to those used in [21, Proposition 2.1], it can be shown that

$$\Big|\int_\Omega \nabla v_m\cdot\nabla\varphi\,dx-\lambda_1\int_\Omega v_m\varphi\,dx\Big|\leqslant C\Big(\frac{1}{\|u_m^-\|}+\frac{\|u_m^+\|}{\|u_m^-\|}+\frac{\|u_m^+\|^\sigma_{L^{p\sigma}}}{\|u_m^-\|}\Big)\|\varphi\|, \tag{3.17}$$

for all m and all $\varphi\in X$, where

$$\lim_{m\to\infty}\frac{\|u_m^+\|^\sigma_{L^{p\sigma}}}{\|u_m^-\|}=0. \tag{3.18}$$

It also follows from (3.12) that

$$\lim_{m\to\infty}\frac{\|u_m^+\|}{\|u_m^-\|}=0. \tag{3.19}$$

Then, by (3.19), (3.13), and (3.18), we obtain from (3.17) that

$$\lim_{m\to\infty}\Big|\int_\Omega \nabla v_m\cdot\nabla\varphi\,dx-\lambda_1\int_\Omega v_m\varphi\,dx\Big|=0,\quad\text{for all }\varphi\in X. \tag{3.20}$$

It then follows from the definition of v_m, (3.16), and (3.20) that

$$\int_\Omega \nabla v\cdot\nabla\varphi\,dx-\lambda_1\int_\Omega v\varphi\,dx=0,\quad\text{for all }\varphi\in X;$$

so that, v is a weak solution of the eigenvalue problem

$$\begin{gathered}-\Delta u=\lambda_1 u,\quad\text{in }\Omega;\\ u=0,\quad\text{on }\partial\Omega\,.\end{gathered} \tag{3.21}$$

We can also show that v is a nontrivial solution of (3.21) and that $v=-\varphi_1$, where φ_1 is the eigenfunction for the problem (3.21) associated with λ_1 with $\varphi_1>0$ in Ω and $\|\varphi_1\|=1$. To see this, divide both sides of the estimate in (3.14) by $\|u_m^-\|^2$ to get

$$\Big|1-\lambda_1\int_\Omega (v_m)^2\,dx-\int_\Omega\frac{g(x,u_m)}{\|u_m^-\|}v_m\,dx\Big|\leqslant\frac{\varepsilon_m}{\|u_m^-\|},\quad\text{for all }m, \tag{3.22}$$

where we have used the definition of v_m in (3.15).

Using the definition of v_m in (3.15), we can write

$$\int_\Omega\frac{g(x,u_m)}{\|u_m^-\|}v_m\,dx=\int_{\Omega_m^-}\frac{g(x,u_m)}{\|u_m^-\|}v_m\,dx,\quad\text{for all }m, \tag{3.23}$$

where

$$\Omega_m^-=\{x\in\Omega\mid u_m(x)\leqslant 0\},\quad\text{for all }m.$$

Thus, using the estimate in (2.5), the assumption in (3.13), the properties of the sequence (v_m) in (3.16), and the Lebesgue dominated convergence theorem, we obtain from (3.23) that

$$\lim_{m\to\infty}\int_\Omega\frac{g(x,u_m)}{\|u_m^-\|}v_m\,dx=0. \tag{3.24}$$

Combining the estimate in (3.22) with the result in (3.24), we then obtain that

$$1-\lambda_1\int_\Omega v^2\,dx=0,$$

from which we deduce that v is nontrivial.

It follows from Hopf's Lemma (see, for instance, [10, Theorem 4 on page 333]) that

$$v < 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} > 0 \text{ on } \partial\Omega, \tag{3.25}$$

where ν denotes the outward unit normal vector to $\partial\Omega$. Thus, using (3.25), (3.19), the definition of v_m in (3.15), and the properties of the sequence (v_m) in (3.16), we obtain

$$u_m(x) \to -\infty \quad \text{as } m \to \infty, \text{ and for a.e. } x \in \Omega. \tag{3.26}$$

Next, take $\varphi = u_m^+$ in (3.2) and divide by $\|u_m^+\|$ to obtain

$$\Big| \|u_m^+\|^2 - \int_\Omega g(x, u_m) u_m^+ \, dx \Big| \leqslant \varepsilon_m \|u_m^+\|, \quad \text{for all } m. \tag{3.27}$$

For $m = 1, 2, 3, \dots$, we set

$$y_m = \begin{cases} u_m^+/\|u_m^+\|, & \text{if } \|u_m^+\| \neq 0; \\ 0, & \text{if } \|u_m^+\| = 0\,. \end{cases} \tag{3.28}$$

Then, $\|y_m\| \leqslant 1$, for all m. Thus, we may assume, passing to subsequences if necessary, that there exists $y \in X$ such that

$$\begin{gathered} y_m \rightharpoonup y \quad \text{weakly in } X \text{ as } m \to \infty; \\ y_m \to y \quad \text{in } L^2(\Omega) \text{ as } m \to \infty; \\ y_m(x) \to y(x) \quad \text{a.e. in } \Omega \text{ as } m \to \infty; \\ |y_m(x)| \leqslant h(x), \quad \text{a.e. in } \Omega, \text{ for all } m, \text{ with } h \in L^2(\Omega). \end{gathered} \tag{3.29}$$

With the sequence (y_m) defined in (3.28), we may therefore rewrite (3.27) as

$$\Big| \|u_m^+\| - \int_\Omega g(x, u_m) y_m \, dx \Big| \leqslant \varepsilon_m, \quad \text{for all } m. \tag{3.30}$$

Now, it follows from the properties of (y_m) in (3.29), the assertion in (3.26), assumption (A_4), and the Lebesgue dominated convergence theorem that

$$\lim_{m\to\infty} \int_\Omega g(x, u_m) y_m \, dx = 0. \tag{3.31}$$

Hence, combining (3.30) and (3.31), we obtain

$$\lim_{m\to\infty} \|u_m^+\| = 0. \tag{3.32}$$

We may also assume, in view of (3.32), that

$$u_m^+(x) \to 0 \quad \text{as } m \to \infty, \text{ for a.e. } x \in \Omega. \tag{3.33}$$

Next, we consider the decomposition of X given by $X = V \oplus W$, where

$$V = \operatorname{span}\{\varphi_1\} \quad \text{and} \quad W = V^\perp.$$

Write $u_m^- = t_m\varphi_1 + w_m$ with $w_m \in W$ and (t_m) a sequence of real numbers. We then have that

$$u_m = u_m^+ - t_m\varphi_1 - w_m, \quad \text{for all } m. \tag{3.34}$$

Since X is a Hilbert space, it follows from (3.34) that

$$\|u_m\|^2 = \|u_m^+\|^2 + t_m^2\|\varphi_1^2\|^2 + \|w_m\|^2, \quad \text{for all } m, \tag{3.35}$$

and

$$\|u_m^-\|_{L^2}^2 = t_m^2\|\varphi_1\|_{L^2}^2 + \|w_m\|_{L^2}^2, \quad \text{for all } m. \tag{3.36}$$

Combining (3.35), (3.36), and using the fact that $\|\varphi_1\|^2 = \lambda_1\|\varphi_1\|^2_{L^2}$, we obtain

$$\frac{1}{2}\|u_m\|^2 - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx = \frac{1}{2}\|u_m^+\|^2 + \frac{1}{2}\|w_m\|^2 - \frac{\lambda_1}{2}\int_\Omega w_m^2\,dx, \tag{3.37}$$

for all m.

Since $w_m \in W$, it follows from Poincaré's inequality that

$$\lambda_2\int_\Omega (w_m)^2\,dx \leqslant \int_\Omega |\nabla w_m|^2\,dx, \tag{3.38}$$

where λ_2 is the second eigenvalue of the N-dimensional Laplacian in Ω with Dirichlet boundary conditions.

We claim that

$$\lim_{m\to\infty}\|w_m\| = 0. \tag{3.39}$$

To prove this assertion, we set $\varphi = w_m$ in (3.2) to obtain

$$\Big|\int_\Omega |\nabla w_m|^2\,dx - \lambda_1\int_\Omega (w_m)^2\,dx - \int_\Omega g(x,u_m)u_m\,dx\Big| \leqslant \varepsilon_m\|w_m\|, \tag{3.40}$$

for all m, where $\varepsilon_m \to 0^+$ as $m \to \infty$.

Next, use the inequality in (3.38) to obtain the estimate

$$\Big(1 - \frac{\lambda_1}{\lambda_2}\Big)\|w_m\|^2 \leqslant \int_\Omega |\nabla w_m\|^2\,dx - \lambda_1\int_\Omega (w_m)^2\,dx, \quad \text{for all } m. \tag{3.41}$$

Put $\alpha = 1 - \frac{\lambda_1}{\lambda_2}$ in (3.41) and combine (3.40) and (3.41) to obtain

$$\alpha\|w_m\|^2 \leqslant \varepsilon_m\|w_m\| + \Big|\int_\Omega g(x,u_m)w_m\,dx\Big|, \quad \text{for all } m. \tag{3.42}$$

For $m = 1, 2, 3, \dots$, we set

$$z_m = \begin{cases} w_m/\|w_m\|, & \text{if } \|w_m\| \neq 0;\\ 0, & \text{if } \|w_m\| = 0\,. \end{cases} \tag{3.43}$$

Then, $\|z_m\| \leqslant 1$, for all m. Thus, we may assume, passing to subsequences if necessary, that there exists $z \in X$ such that

$$\begin{gathered} z_m \rightharpoonup z \quad \text{weakly in } X \text{ as } m \to \infty;\\ z_m \to z \quad \text{in } L^2(\Omega) \text{ as } m \to \infty;\\ z_m(x) \to z(x) \quad \text{a.e. in } \Omega \text{ as } m \to \infty;\\ |z_m(x)| \leqslant k(x), \quad \text{a.e. in } \Omega, \ \text{ for all } m, \ \text{with } k \in L^2(\Omega). \end{gathered} \tag{3.44}$$

With the sequence (z_m) defined in (3.43), we may therefore rewrite (3.42) as

$$\alpha\|w_m\| \leqslant \varepsilon_m + \Big|\int_\Omega g(x,u_m)z_m\,dx\Big|, \quad \text{for all } m. \tag{3.45}$$

Now, it follows from (3.26), the properties of the sequence (z_m) in (3.44), assumption (A4), and the Lebesgue dominated convergence theorem that

$$\lim_{m\to\infty}\int_\Omega g(x,u_m)z_m\,dx = 0. \tag{3.46}$$

Consequently, combining (3.45) and (3.46), it follows that $\lim_{m\to\infty}\|w_m\| = 0$, which is the assertion in (3.39).

In view of (3.39), we may also assume that

$$w_m(x) \to 0 \quad \text{as } m \to \infty, \text{ for a.e. } x \in \Omega. \tag{3.47}$$

In view of (3.34), (3.26) and (3.33), we obtain from (3.47) that the sequence (t_m) must satisfy

$$t_m \to \infty \quad \text{as } m \to \infty. \tag{3.48}$$

Next, use the assertions in (3.32), (3.39) and (3.38) to obtain from (3.37) that

$$\lim_{m\to\infty} \Big(\frac{1}{2}\|u_m\|^2 - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big) = 0. \tag{3.49}$$

Now, it follows from the definition of the functional J in (2.1) that

$$\Big|\int_\Omega G(x,u_m)\,dx\Big| \leqslant |J(u_m)| + \Big|\frac{1}{2}\|u_m\|^2 - \frac{\lambda_1}{2}\int_\Omega (u_m^-)^2\,dx\Big|, \tag{3.50}$$

for all m.

Thus, using the assertions in (3.32), (3.39), (2.3), and (3.49), we obtain from (3.50) that

$$\Big|\int_\Omega G(x,u_m)\,dx\Big| \leqslant C + \bar{\varepsilon}, \tag{3.51}$$

for all m and some $\bar{\varepsilon} > 0$.

It follows from (3.51) that

$$-\infty < \liminf_{m\to\infty}\int_\Omega G(x,u_m)\,dx \leqslant \limsup_{m\to\infty}\int_\Omega G(x,u_m)\,dx < \infty. \tag{3.52}$$

Applying the mean value theorem we obtain

$$G(x,-t_m\varphi_1) - G(x,u_m) = g(x,-t_m\varphi_1 + \theta_m(u_m^+ - w_m))(u_m^+ - w_m), \quad \text{for all } m,$$

where (θ_m) is a sequence of real numbers in the interval $(0,1)$. Consequently,

$$\begin{aligned} &\int_\Omega G(x,-t_m\varphi_1)\,dx - \int_\Omega G(x,u_m)\,dx \\ &= \int_\Omega g(x,-t_m\varphi_1 + \theta_m(u_m^+ - w_m))(u_m^+ - w_m)\,dx, \end{aligned} \tag{3.53}$$

for all m.

Now, it follows from assumption (A_4), in conjunction with (3.32), (3.33), (3.39), (3.47), (3.48) and the Lebesgue dominated convergence theorem, that

$$\lim_{m\to\infty}\Big|\int_\Omega g(x,-t_m\varphi_1 + \theta_m(u_m^+ - w_m))(u_m^+ - w_m)\,dx\Big| = 0. \tag{3.54}$$

Next, combine (3.54) and (3.53) to obtain that

$$\limsup_{m\to\infty}\int_\Omega G(x,-t_m\varphi_1)\,dx \leqslant \limsup_{m\to\infty}\int_\Omega G(x,u_m)\,dx. \tag{3.55}$$

Similar calculations show that

$$\liminf_{m\to\infty}\int_\Omega G(x,u_m)\,dx \leqslant \liminf_{m\to\infty}\int_\Omega G(x,-t_m\varphi_1)\,dx. \tag{3.56}$$

Combining the assertions in (3.52), (3.55) and (3.56) we then obtain

$$-\infty < \liminf_{m\to\infty}\int_\Omega G(x,-t_m\varphi_1)\,dx \leqslant \limsup_{m\to\infty}\int_\Omega G(x,-t_m\varphi_1)\,dx < \infty,$$

which is in direct contradiction with the ALP condition (A5). Thus, the sequence $(\|u_m\|)$ is bounded. The proof of the lemma is now complete. □

4. Critical groups at infinity

In this section, we compute the critical groups of J at infinity under the following additional assumption on g and its primitive G:

(A6) There exists $s_- < 0$ such that
$$2G(x,s) - g(x,s)s \leqslant 0, \quad \text{for all } s < s_-. \tag{4.1}$$

Assume that the critical values of J are bounded from below and set $a_0 = \inf J(\mathcal{K})$. Let M be a real number satisfying $M > -a_0$. We will show that any compact set $A \subset J^{-M}$ is contractible in J^{-M}. This will allow us to compute the reduced homology groups $\widetilde{H}_q(J^{-M})$, for $q \in \mathbb{Z}$, and, by using an argument involving the exact sequence of reduced homology groups, we will be able to compute $C_q(J,\infty)$, for $q \in \mathbb{Z}$.

Combining condition (A3) and (A6), we can find a constant $K_1 > 0$ such that
$$2G(x,s) - g(x,s)s \leqslant K_1, \quad \text{for all } s \in \mathbb{R} \text{ and } x \in \overline{\Omega}. \tag{4.2}$$

Proposition 4.1. [19, Proposition 7.1] *Under the conditions* (A1)–(A6)*, any compact set A of the sublevel set $J^{-M} = \{u \in X : J(u) \leqslant -M\}$ is contractible in J^{-M} for*
$$M \geqslant \max\{K_1|\Omega|, -a_0\},$$
where K_1 is given by (4.2).

As a consequence of this proposition, and that singular chains are linear combinations of compact sets, the set J^{-M} has the same homology type of a point; that is,
$$\widetilde{H}_q(J^{-M}) \cong 0, \quad \text{for all } q \in \mathbb{Z}, \tag{4.3}$$
where $\widetilde{H}_*$ denotes the reduced singular homology.

Based on the definition of the reduced homology groups (see [13, Page 110]), we can show that
$$H_q(J^{-M}) \cong \widetilde{H}_q(J^{-M}) \text{ for } q > 0 \quad \text{and} \quad H_0(J^{-M}) \cong \widetilde{H}_0(J^{-M}) \oplus \mathbb{Z}_2. \tag{4.4}$$

Hence, using an argument similar to that presented in [13, Example 2.18, page 118] with the sequence of reduced homology groups, (4.3), (4.4), and the fact that X is also contractible, we conclude that
$$C_q(J,\infty) \cong H_q(X, J^{-M}) \cong \widetilde{H}_q(X) \cong 0, \quad \text{for all } q \in \mathbb{Z}. \tag{4.5}$$

5. Critical groups at the origin and first existence result

Since we are assuming that $g(x,0) = 0$ for all $x \in \Omega$, according to assumption (A1), problem (1.8) has the trivial solution. Thus, the question of existence in this section refers to existence of nontrivial solutions of problem (1.8). In this section, we determine conditions on the nonlinearity g that will guarantee the existence of at least one nontrivial solution of problem (1.8). To do this, we will compute the critical groups of J at the origin. The computation of these critical groups will follow as a consequence of a result due to Perera and Schechter [17], which we will present below.

The Perera and Schechter result that we are about to discuss refers to the energy functional $J{:}X \to \mathbb{R}$ associated with the boundary value problem

$$\begin{gathered} -\Delta u = f(x,u), \quad x \in \Omega; \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{5.1}$$

where $f{:}\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies

(A7) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $f(x,0) = 0$, for all $x \in \Omega$; and
(A8) $\frac{\partial f}{\partial s}(x,0) = a$, for all $x \in \Omega$, where $a > 0$.

Namely, the functional J is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u(x))\,dx, \quad \text{for } u \in X, \tag{5.2}$$

where

$$F(x,s) = \int_0^s f(x,\xi)\,d\xi, \quad \text{for } (x,s) \in \overline{\Omega} \times \mathbb{R}. \tag{5.3}$$

We denote by λ_ℓ, $\ell \geqslant 1$, the isolated eigenvalues of the N-dimensional Laplacian over Ω with Dirichlet boundary conditions. These eigenvalues have finite multiplicities and satisfy

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_\ell < \dots$$

Set

$$N_\ell = \oplus_{j=1}^{\ell} \ker(-\Delta - \lambda_j I) \quad \text{and} \quad M_\ell = N_\ell^\perp, \tag{5.4}$$

and put

$$d_\ell = \dim N_\ell. \tag{5.5}$$

We then have the decomposition of the space X as $X = N_\ell \oplus M_\ell$.

In [17], Perera and Schechter proved the following result, which allows us to compute the critical groups of J at the origin.

Theorem 5.1. [17, Theorem 3.3.2] *Assume that*

$$|f(x,s)| \leqslant C(|s|^{r-1} + 1), \quad \textit{for all } (x,s) \in \overline{\Omega} \times \mathbb{R}, \tag{5.6}$$

for some $r \in [2, 2^*)$ *and a constant* $C > 0$*, where* 2^* *is the critical Sobolev exponent, holds and* 0 *is an isolated critical point of* J *given in* (5.2) *and* (5.3)*.*

(i) *If there is a* $\delta > 0$ *such that*

$$\frac{f(x,s)}{s} < \lambda_1 \quad \textit{for all } x \in \overline{\Omega}, \textit{ and } 0 < |s| \leqslant \delta, \tag{5.7}$$

then $C_q(J,0) \cong \delta_{q,0}\mathbb{Z}_2$*, for all* $q \in \mathbb{Z}$*.*

(ii) *If there is a* $\delta > 0$ *such that*

$$\lambda_\ell < \frac{f(x,s)}{s} < \lambda_{\ell+1}, \quad \textit{for all } x \in \overline{\Omega}, 0 < |s| \leqslant \delta,$$

then $C_q(J,0) \cong \delta_{q,d_\ell}\mathbb{Z}_2$*, for all* $q \in \mathbb{Z}$*.*

To apply the Perera-Schechter result to problem (1.8), let

$$f(x,s) = -\lambda_1 s^- + g(x,s), \quad \text{for } (x,s) \in \overline{\Omega} \times \mathbb{R}. \tag{5.8}$$

Observe that the function f defined in (5.8) satisfies the condition in (5.6) as a consequence of assumption (A2).

In addition to the assumptions (A1)–(A6) satisfied by g and its primitive G, we make further assumptions on g that guarantee that the function f defined in (5.8) satisfies (A7) and (A8). This amounts to assuming that

$$\lim_{s\to 0^+}\frac{g(x,s)}{s}=a,\quad \text{uniformly in } x\in\overline{\Omega}, \tag{5.9}$$

$$\lim_{s\to 0^-}\frac{g(x,s)}{s}=a-\lambda_1,\quad \text{uniformly in } x\in\overline{\Omega}. \tag{5.10}$$

We also need to assume that g is C^1 except at $s=0$ and that $\frac{\partial g}{\partial s}$ has a jump discontinuity at $s=0$, as specified by (5.9) and (5.10).

We consider two cases: [(i)] $a<\lambda_1$; and [(ii)] $\lambda_\ell<a<\lambda_{\ell+1}$, for some $\ell\geqslant 1$.

Since we are assuming that the function f given in (5.8) satisfies (A7) and (A8), in view of (5.9) and (5.10), we have that

$$\lim_{s\to 0}\frac{f(x,s)}{s}=a,\quad \text{uniformly in } x\in\overline{\Omega}. \tag{5.11}$$

Thus, in the case (i) $a<\lambda_1$, it follows from (5.11) that there exists $\delta>0$ such that

$$\frac{f(x,s)}{s}<\lambda_1,\quad \text{for } 0<|s|\leqslant\delta,$$

uniformly in $x\in\Omega$. It then follows that the condition on f in part (i) of the Perera-Schechter result in Theorem 5.1 is satisfied. Hence, the critical groups of J at the origin are given by

$$C_q(J,0)=\delta_{q,0}\mathbb{Z}_2\quad \text{for all } q\in\mathbb{Z}, \tag{5.12}$$

in the case (i) $a<\lambda_1$.

Next, consider the case (ii) $\lambda_\ell<a<\lambda_{\ell+1}$. In view of (5.11), taking $\varepsilon=\min\{\lambda_{\ell+1}-a,a-\lambda_\ell\}$, there exists $\delta>0$ small enough such that

$$a-\varepsilon<\frac{f(x,s)}{s}<a+\varepsilon,\quad \text{for } 0<|s|\leqslant\delta \text{ and } x\in\overline{\Omega};$$

so that

$$\lambda_\ell<\frac{f(x,s)}{s}<\lambda_{\ell+1},\quad \text{for } 0<|s|\leqslant\delta \text{ and } x\in\overline{\Omega}. \tag{5.13}$$

It follows from (5.13) that the conditions of Theorem 5.1 (ii) are satisfied. Hence, the critical groups of J at the origin in the case (ii) $\lambda_\ell<a<\lambda_{\ell+1}$ are given by

$$C_q(J,0)=\delta_{q,d_\ell}\mathbb{Z}_2\quad \text{for all } q\in\mathbb{Z}. \tag{5.14}$$

Therefore, the critical groups of J at the origin assume different values based on the value of a. We summarize the results below:

$$C_q(J,0)=\begin{cases}\delta_{q,0}\mathbb{Z}_2, & \text{for } a<\lambda_1,\\ \delta_{q,d_\ell}\mathbb{Z}_2, & \text{for } \lambda_\ell<a<\lambda_{\ell+1},\ \ell\geqslant 1.\end{cases} \tag{5.15}$$

We will use the information on the critical groups of J at the origin to prove that problem (1.8) has at least one nontrivial solution in the case $a\in(\lambda_\ell,\lambda_{\ell+1})$, for some $\ell\geqslant 1$. Indeed, arguing by contradiction, assume that the origin is the only critical point of J; that is, $\mathcal{K}=\{0\}$, and (5.14) is satisfied; then, by (4.5) and a standard argument with the Morse relation (2.14) with $t=-1$, we obtain that

$$M_{d_\ell}(-1)^{d_\ell}=\beta_0(-1)^0,$$

where $M_{d_\ell} = 1$, according to (5.14), and $\beta_0 = 0$, according to (4.5); so that, $(-1)^{d_\ell} = 0$, which is a contradiction.

Hence, the assumption that $\mathcal{K}$ consisted only of the origin leads to a contradiction. Consequently, problem (1.8) has at least one nontrivial solution in the case $\lambda_\ell < a < \lambda_{\ell+1}$, for $\ell \geqslant 1$.

Similarly, we can use the same argument to show that the in the $a < \lambda_1$, problem (1.8) has a nontrivial solution. In fact, if we assume that $\mathcal{K} = \{0\}$, then, it follows from the Morse relation (2.14) with $t = -1$ that

$$M_0(-1)^0 = \beta_0(-1)^0,$$

where $M_0 = 1$, according to (5.12), and $\beta_0 = 0$, according to (4.5); so that, $(-1)^0 = 0$, which is a contradiction. We have therefore proved the following theorem.

Theorem 5.2. *Suppose that the conditions* (A1)–(A6) *hold for the function g in problem* (1.8) *and its primitive, G. In addition, assume that $g(x,s)$ is piecewise C^1 for $s \neq 0$, and that $\frac{\partial g}{\partial s}$ has a jump discontinuity at $s = 0$ determined by the conditions*

$$\lim_{s\to 0^+} \frac{g(x,s)}{s} = a \quad \textit{and} \quad \lim_{s\to 0^-} \frac{g(x,s)}{s} = a - \lambda_1, \tag{5.16}$$

uniformly in $x \in \Omega$, where $a > 0$. If $a < \lambda_1$ or $a \in (\lambda_\ell, \lambda_{\ell+1})$, for some $\ell \geqslant 1$, then problem (1.8) *has at least one nontrivial solution.*

We present here two examples of nonlinearities, g, that satisfy the conditions in the statement of Theorem 5.2. We assume that g depends only on $s \in \mathbb{R}$; so that, $g(x,s) = g(s)$, for all $x \in \overline{\Omega}$ and $s \in \mathbb{R}$, where we are using the same symbol, g, to denote the function of a single variable used to define $g(x,s)$. In these examples, we also assume that $N \geqslant 3$.

Example 5.3. For the case $a < \lambda_1$, define $g{:}\mathbb{R} \to \mathbb{R}$ to be

$$g(s) = \begin{cases} \frac{(a-\lambda_1)s}{1+s^2}, & \text{if } s < 0; \\ as + s^{p-1}, & \text{if } s \geqslant 0, \end{cases} \tag{5.17}$$

with $2 < p < 2^*$, where 2^* is the critical Sobolev exponent, $2^* = \frac{2N}{N-2}$.

Conditions (A1)–(A6) can be verified for the function g defined in (5.17). We note that the value of σ in condition (A2) can be taken to be in the range

$$p - 1 < \sigma < \frac{N+2}{N-2},$$

and the value of μ in condition $A3$ can be taken in the range $2 < \mu < p$. The conditions in (5.16) in the statement of Theorem 5.2 can also be verified for the function g given in (5.17).

Example 5.4. For the case when $a > \lambda_1$, define $g : \mathbb{R} \to \mathbb{R}$ to be

$$g(s) = \begin{cases} \frac{(\lambda_1-a)(s-\lambda_1+a)}{1+(s-\lambda_1+a)^2}, & \text{if } s < \lambda_1 - a; \\ s(a-\lambda_1+s), & \text{if } \lambda_1 - a \leqslant s < 0; \\ as + s^{p-1}, & \text{if } s \geqslant 0, \end{cases} \tag{5.18}$$

where p is as in Example 1.

As in Example 1, the function g defined in (5.18) can be shown to satisfy conditions (A1)–(A6) for the case $a > \lambda_1$. In addition, conditions in (5.16) in the statement of Theorem 5.2 are also satisfied by the function g given in (5.18).

6. Existence of a at least two non-trivial solutions

In this section, we continue the study of problem (1.8) under the assumptions (A_1)–(A_6) on g and G, as well as assumptions (A7) and (A8) on the function

$$f(x,s) = -\lambda_1 s^- + g(x,s), \quad \text{for } (x,s) \in \overline{\Omega} \times \mathbb{R}, \tag{6.1}$$

introduced in the previous section. Thus, we assume that $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is piecewise C^1 for $s \neq 0$, and that $\frac{\partial g}{\partial s}$ has a jump discontinuity at $s = 0$ determined by the conditions

$$\lim_{s \to 0^+} \frac{g(x,s)}{s} = a, \tag{6.2}$$

$$\lim_{s \to 0^-} \frac{g(x,s)}{s} = a - \lambda_1, \tag{6.3}$$

uniformly in $x \in \overline{\Omega}$, where $a > 0$.

Our goal is to prove the existence of at least two nontrivial solutions of problem (1.8) in the case in which $a \in (\lambda_\ell, \lambda_{\ell+1})$, for some $\ell \geqslant 1$, under the following additional assumption on g:

(A9) There exists $s_2 > 0$ such that $g(x, s_2) = 0$, for all $x \in \overline{\Omega}$.

We will also assume that d_ℓ is even, where d_ℓ is given in (5.4) and (5.5).

We first prove that, under assumptions (A1)–(A5) and (A9), problem (1.8) has a positive solution that is a local minimizer of the functional J defined in (2.1).

Theorem 6.1. *Suppose that the conditions* (A1)–(A5) *hold for the function g in problem* (1.8) *and its primitive, G. In addition, assume the jump conditions in* (6.2) *and* (6.3)*, and that (A9) holds. Then, problem* (1.8) *has a positive solution, u_1, that is a local minimizer of the function J defined in* (2.1)*. Furthermore,*

$$C_q(J, u_1) \cong \delta_{q,0}\mathbb{Z}_2, \quad \textit{for all } q \in \mathbb{Z}.$$

Proof. Define a function $\widetilde{f} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{f}(x,s) = \begin{cases} 0, & \text{for } s < 0; \\ g(x,s), & \text{for } s \in [0, s_2]; \\ 0, & \text{for } s > s_2, \end{cases} \tag{6.4}$$

and for all $x \in \overline{\Omega}$.

It follows from (6.4) that $\widetilde{F}(x,s) = \int_0^s \widetilde{f}(x,\xi)\, d\xi$, for $(x,s) \in \overline{\Omega} \times \mathbb{R}$, is given by

$$\widetilde{F}(x,s) = \begin{cases} 0, & \text{for } s < 0; \\ G(x,s), & \text{for } s \in [0, s_2]; \\ G(x,s_2), & \text{for } s > s_2, \end{cases} \tag{6.5}$$

and for all $x \in \overline{\Omega}$. Hence, the function $\widetilde{F}$ defined in (6.5) is bounded in $\overline{\Omega} \times \mathbb{R}$; so that, there exists a positive constant $\widetilde{M}$ such that

$$|\widetilde{F}(x,s)| \leqslant \widetilde{M}, \quad \text{for all } (x,s) \in \overline{\Omega} \times \mathbb{R}. \tag{6.6}$$

Next, we define the functional $\widetilde{J} : X \to \mathbb{R}$ by

$$\widetilde{J}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \int_\Omega \widetilde{F}(x,u)\, dx, \quad \text{for } u \in X, \tag{6.7}$$

where $\widetilde{F}$ is given in (6.5).

It follows from the continuity of $\widetilde{f}$ and the estimate in (6.6) that the functional $\widetilde{J}$ given in (6.7) is well-defined and $\widetilde{J} \in C^1(X, \mathbb{R})$, with Fréchet derivative at $u \in X$ given by

$$\langle \widetilde{J}'(u), \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_\Omega \widetilde{f}(x,u)\varphi \, dx, \quad \text{for all } \varphi \in X. \tag{6.8}$$

It follows from the estimate in (6.6) and the definition of $\widetilde{J}$ in (6.7) that

$$\widetilde{J}(u) \geqslant \frac{1}{2}\|u\|^2 - \widetilde{M}|\Omega|, \quad \text{for all } u \in X. \tag{6.9}$$

Consequently, $\widetilde{J}$ is bounded from below in X.

It also follows from the estimate in (6.9) that $\widetilde{J}$ satisfies the PS condition. Indeed, let (u_m) denote a PS sequence for $\widetilde{J}$; so that,

$$|\widetilde{J}(u_m)| \leqslant C, \quad \text{for all } m,$$

and some positive constant C. Thus, using the estimate in (6.9),

$$\frac{1}{2}\|u_m\|^2 - \widetilde{M}|\Omega| \leqslant C, \quad \text{for all } m,$$

from which we deduce that (u_m) is a bounded sequence in X.

Hence, $\widetilde{J}$ is bounded from below and it satisfies the PS condition. Therefore, $\widetilde{J}$ has a global minimizer u_1 in X. This result can be obtained by applying Ekeland's variational principle (see [9]) or the deformation lemma (see [18]). Put

$$c_1 = J(u_1) = \min_{u \in X} \widetilde{J}(u). \tag{6.10}$$

Then, since $\widetilde{J}(0) = 0$, it follows from (6.10) that $c_1 \leqslant 0$.

In view of (6.8), we see that the global minimizer, u_1, of $\widetilde{J}$ is a weak solution of the boundary value problem

$$\begin{gathered} -\Delta u = \widetilde{f}(x,u), \quad \text{in } \Omega; \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{6.11}$$

Since $\widetilde{f}$ locally Lipschitz continuous in $\overline{\Omega} \times \mathbb{R}$, it follows from elliptic regularity theory (see [1] or [12]) that u_1 is also a classical solution of problem (6.11).

Now we show that

$$0 < u_1(x) < s_2, \quad \text{for } x \in \Omega, \text{ and } \frac{\partial u_1}{\partial \nu} < 0 \text{ on } \partial\Omega, \tag{6.12}$$

where ν is the outward normal unit vector at $\partial\Omega$.

To establish the assertion in (6.12), we first show that u_1 is not the trivial solution of problem (6.11).

Let $\overline{\varepsilon} > 0$ be such that $a - \overline{\varepsilon} > \lambda_1$. It then follows from (6.2) and the definition of $\widetilde{f}$ in (6.4) that there exists $\delta > 0$ such that $\delta < s_2$ and

$$0 < s < \delta \implies \widetilde{f}(x,s) > (a - \overline{\varepsilon})s, \quad \text{for all } x \in \overline{\Omega}.$$

Consequently,

$$\widetilde{F}(x,s) > \frac{1}{2}(a - \overline{\varepsilon})s^2, \quad \text{for } 0 < s < \delta \text{ and } x \in \overline{\Omega}. \tag{6.13}$$

Pick $\bar{t} > 0$ such that

$$0 < \bar{t} < \frac{\delta}{\max_{x\in\overline{\Omega}} \varphi_1(x)}, \tag{6.14}$$

and compute

$$\widetilde{J}(\bar{t}\varphi_1) = \frac{\bar{t}^2}{2} - \int_\Omega \widetilde{F}(x, \bar{t}\varphi_1(x))\, dx. \tag{6.15}$$

Then, in view of (6.14) and (6.13), we obtain from (6.15) that

$$\widetilde{J}(\bar{t}\varphi_1) < \frac{\bar{t}^2}{2} - \frac{\bar{t}^2}{2}(a - \bar{\varepsilon}) \int_\Omega (\varphi_1(x))^2\, dx,$$

from which we obtain

$$\widetilde{J}(\bar{t}\varphi_1) < \frac{\bar{t}^2}{2}\Big(1 - \frac{a - \bar{\varepsilon}}{\lambda_1}\Big). \tag{6.16}$$

Then, since we are assuming that $a - \bar{\varepsilon} > \lambda_1$, if follows from (6.16) that $\widetilde{J}(\bar{t}\varphi_1) < 0$, from which we get, in view of (6.10), that $c_1 < 0$. Consequently, $u_1 \neq 0$.

Next, an argument involving the use of Hopf's maximum principle (see, for instance, [10, Theorem 4, p. 333]), in conjunction with he fact that u_1 is not the trivial solution of problem (6.11), can be used to establish the assertion in (6.12).

It follows from the assertion in (6.12) that u_1 is a local minimizer of functional J in the $C_0^1(\overline{\Omega})$ topology. Thus, using a result due to Brézis and Nirenberg in [5, Theorem 1], we deduce that u_1 is also a local minimizer of J in the $H_0^1(\Omega)$ topology.

Hence, the critical groups of J at u_1 have the same homology type of a point (see for instance [7, Example 1, page 33]); that is,

$$C_q(J, u_1) \cong \delta_{q,0}\mathbb{Z}_2, \quad \text{for all } q \in \mathbb{Z}. \tag{6.17}$$

The proof of the theorem is now complete. □

Here is the main result of this section.

Theorem 6.2. *Suppose that conditions* (A1)–(A6) *hold for the function g in problem* (1.8) *and its primitive, G. In addition, assume that $g(x, s)$ is piecewise C^1 for $s \neq 0$, and that $\frac{\partial g}{\partial s}$ has a jump discontinuity at $s = 0$ determined by the conditions in* (6.2) *and* (6.3)*, where $a \in (\lambda_\ell, \lambda_{\ell+1})$, for some $\ell \geqslant 1$. Assume also that d_ℓ, as given in* (5.4) *and* (5.5)*, is even. Then, if* (A9) *also holds, problem* (1.8) *has at least two nontrivial solutions, where at least one of them is a positive solution.*

Proof. We will use a standard argument involving the Morse relation (2.14). Let u_1 be the positive solution of problem (1.8) given by Theorem 6.1. Then, the assertion in (6.17) holds.

Assume, by way of contradiction, that the critical set of J consists only of two critical points; namely, $\mathcal{K} = \{0, u_1\}$. Then, by (5.15), (6.17), and (4.5), we have

$$M_0 = 1, \quad M_1 = 1, \text{ and } \beta_0 = 0. \tag{6.18}$$

Therefore, substituting (6.18) into (2.14), with $t = -1$, we obtain

$$(-1)^{d_\ell} + (-1)^0 = 0. \tag{6.19}$$

Since d_ℓ is even in (6.19), we obtain that $1 + 1 = 0$, which is a contradiction. Therefore, J must have at least a second nontrivial critical point. This concludes the proof of the theorem. □

We end this section by presenting a concrete example of a nonlinearity g for which the conditions in the statement of Theorem 6.2 are satisfied.

Example 6.3. As in examples 1 and 2 in the previous section, assume that $N \geqslant 3$. For $a > \max\{1, \lambda_1\}$, define $g : \mathbb{R} \to \mathbb{R}$ to be

$$g(s) = \begin{cases} \frac{(\lambda_1 - a)(s - \lambda_1 + a)}{1 + (s - \lambda_1 + a)^2}, & \text{if } s < \lambda_1 - a; \\ s(a - \lambda_1 + s), & \text{if } \lambda_1 - a \leqslant s < 0; \\ s(s-1)(s-a), & \text{if } 0 \leqslant s < a; \\ a(a-1)(s-a) + (s-a)^{p-1}, & \text{if } s \geqslant a; \end{cases} \tag{6.20}$$

where $2 < p < 2^*$. The function g defined by (6.20) satisfies the hypotheses of Theorem 6.2 with $s_2 = 1$ in the statement of condition (A9).

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