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INFINITE DIMENSIONAL EXTENSIONS OF THE LANDESMAN-LAZER THEOREM

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Dedicated to the memory of Alan C. Lazer

ABSTRACT. We show that many of the results obtained for the Landesman-Lazer type problems can be extended to operators having unbounded essential spectra stretching from $-\infty$ to $+\infty$. The only requirement is that they have an isolated eigenvalue of finite multiplicity.

1. INTRODUCTION

In their pioneering work, Landesman and Lazer [14] proved the following theorem.

Theorem 1.1. *Let $g(s)$ be a continuous function on $\mathbb{R}$ such that*

$$g(s) \to g_{\pm} \quad \text{as } s \to \pm\infty. \tag{1.1}$$

Let λ_ℓ be a simple eigenvalue of the Dirichlet problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$. Then for each $h \in L^2(\Omega)$, a sufficient condition that there exist a weak solution of

$$-\Delta u - \lambda_\ell u = g(u) - h \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.3}$$

is

$$\int_\Omega hv\,dx < \int_\Omega (g_+ v^+ - g_- v^-)dx, \quad v \in V \setminus \{0\} \tag{1.4}$$

where $v^{\pm} = \max\{\pm v, 0\}$ and V is the eigenspace of (1.2) corresponding to the eigenvalue $\lambda = \lambda_\ell$.

It was also shown in [14] that (1.4) is also necessary if $g_- < g(s) < g_+$ for $s \in \mathbb{R}$. Hypothesis (1.4) excludes the possibility

$$g_+ = g_- \equiv g_0. \tag{1.5}$$

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On the other hand, if we weaken (1.4) to

$$\int_\Omega hv\,dx \le \int_\Omega (g_+v^+ - g_-v^-)\,dx, \quad v \in V, \tag{1.6}$$

it is no longer sufficient for a solution of (1.3) to exist. Various authors have given additional sets of hypotheses that, together with (1.6), will imply the existence of a solution of (1.3) (cf. the listings in the bibliography and the references quoted in them.)

In [19], we were able to replace the operator $-\Delta u - \lambda_\ell$ with a general self-adjoint operator $\mathcal{A}$ provided the essential spectrum of $\mathcal{A}$ is contained in the interval $(0,\infty)$. The reason for this restriction was the fact that we used the following theorem in the proof.

Theorem 1.2. *Let N be a closed subspace of a Hilbert space H and let $M = N^\perp$. Assume that at least one of the subspaces M, N is finite dimensional. Let G be a C^1 functional on H such that*

$$m_1 := \inf_{w\in M} \sup_{v\in N} G(v+w) < \infty,$$
$$m_0 := \sup_{v\in N} \inf_{w\in M} G(v+w) > -\infty.$$

Then there are a constant $c \in \mathbb{R}$ and a sequence $\{u_k\} \subset H$ such that

$$m_0 \le c \le m_1, \quad G(u_k) \to c, \quad G'(u_k) \to 0.$$

The restriction that "at least one of the subspaces M, N is finite dimensional" causes the restriction that the essential spectrum of $\mathcal{A}$ be bounded. In the present paper we are able to remove this restriction as long as the resolvent of $\mathcal{A}$ is not empty. In order to do this we must replace Theorem 1.2. We treat general boundary value problems for more general operators.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\mathcal{A}$ a selfadjoint operator on $L^2(\Omega)$. We assume that $\sigma_e(\mathcal{A})$ is not the whole of $\mathbb{R}$. For convenience, we assume there is an interval (a,b) containing 0 such that $(a,b)\cap\sigma(\mathcal{A}) = \{0\}$. We let $D = D([|\mathcal{A}|+1]^{(1/2)})$. With the scalar product $(u,v)_D = ([|\mathcal{A}|+1]^{(1/2)}u, [|\mathcal{A}|+1]^{(1/2)}v)$, it becomes a Hilbert space. We let

$$N = E(-\infty,a]\cap D, \quad M = E[b,\infty)\cap D, \quad Y = N(\mathcal{A}),$$

where $E(I)$ is the spectral projection of $\mathcal{A}$ over the interval I. Hence,

$$N = \{v\in D : (\mathcal{A}v,v) \le a\|v\|^2\},$$
$$M = \{w\in D : (\mathcal{A}w,w) \ge b\|w\|^2\},$$
$$Y = \{y\in D(\mathcal{A}) : \mathcal{A}y = 0\}$$

are orthogonal invariant subspaces of $\mathcal{A}$ with $D = N\oplus Y\oplus M$. We assume that $C_0^\infty(\Omega) \subset D \subset H^{m,2}(\Omega)$ for some $m > 0$. In particular,

$$\|u\|_{m,2} \le C\|u\|_D\,, \quad u\in D.$$

In the more general setting, (1.3) takes on the form

$$\mathcal{A}u = f(x,u), \quad u\in D(\mathcal{A}) \tag{1.7}$$

where $f(x,t)$ is a Carathéodory function satisfying

$$f(x,t)\to f_\pm(x) \quad \text{as } t\to\pm\infty \text{ a.e. in } \Omega. \tag{1.8}$$

(In the case of Theorem 1.1, $f(x,t) = g(t) - h(x)$.) Hypothesis (1.4) becomes

$$\int_\Omega (f_+ v^+ - f_- v^-)dx > 0, \quad v \in N(\mathcal{A}) \setminus \{0\}. \tag{1.9}$$

It is a simple matter to show that (1.9) implies both (I) and (II):

(I) $\inf_{v\in N(\mathcal{A})} \int_\Omega F(x,v)dx > -\infty$, where

$$F(x,t) := \int_0^t f(x,s)\, ds\,. \tag{1.10}$$

(II) For each $v \in N(\mathcal{A}) \setminus \{0\}$ there is a $w \in N(\mathcal{A})$ such that

$$\int_{v>0} f_+ w\, dx + \int_{v<0} f_- w\, dx \neq 0. \tag{1.11}$$

In our first result we show that (I) and (II) are sufficient to ensure the existence of a solution of (1.7). One can also replace (I) by

(I') $\sup_{v\in N(\mathcal{A})} \int_\Omega F(x,v)dx < \infty$.

Our second step is to replace (1.9) with a hypothesis which will allow

$$f_+(x) = f_-(x) \equiv f(x) \in N(\mathcal{A})^\perp \tag{1.12}$$

or even

$$f(x,t) \to 0 \quad \text{as } |t| \to \infty. \tag{1.13}$$

In case of (1.13), a simple sufficient condition is: There are functions $F_0(x), F_1(x) \in L^1(\Omega)$ such that

$$F_0(x) \le F(x,t) \le F_1(x), \quad x \in \Omega,\ t \in \mathbb{R} \tag{1.14}$$

and either

$$F(x,t) \to F_0(x) \quad \text{as } |t| \to \infty \tag{1.15}$$

or

$$F(x,t) \to F_1(x) \quad \text{as } |t| \to \infty. \tag{1.16}$$

In the case of (1.12) we can replace (1.14)–(1.16) with

$$F_0(x) \le F(x,t) - tf(x) \le F_1(x), \tag{1.17}$$

$$F(x,t) - tf(x) \to F_0(x) \quad \text{as } |t| \to \infty, \tag{1.18}$$

$$F(x,t) - tf(x) \to F_1(x) \quad \text{as } |t| \to \infty, \tag{1.19}$$

respectively.

Our main results are stated in Section 2 and proved in Section 3.

2. Semilinear boundary value problems

Let Ω be a domain in $\mathbb{R}^n$, and let $\mathcal{A}$ be a selfadjoint operator on $L^2(\Omega)$ such that

(A) There are constants $a < 0 < b$ such that the essential spectrum $\sigma_e(\mathcal{A})$ of $\mathcal{A}$ does not intersect (a,b) and $\lambda = 0$ is the only point of (a,b) in the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$.

(B) There is a function $V_0(x) > 0$ such that multiplication by V_0 is a compact operator from $D := D(|A|^{1/2})$ to $L^p(\Omega)$ for some $p \ge 1$.

(C) $v \neq 0$ a.e. for all $v \in N(\mathcal{A}) \setminus \{0\}$.

Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ such that

(D) $|f(x,t)| \le V(x) \in L^2(\Omega), x \in \Omega, t \in \mathbb{R}$,

(E) $f(x,t) \to f_\pm(x)$ a.e. as $t \to \pm\infty$,

(F) either

$$\sup_{v\in N(\mathcal{A})}\int_\Omega F(x,v)dx<\infty \tag{2.1}$$

or

$$\inf_{v\in N(\mathcal{A})}\int_\Omega F(x,v)dx>-\infty, \tag{2.2}$$

where

$$F(x,t):=\int_0^t f(x,s)ds. \tag{2.3}$$

Theorem 2.1. *In addition to hypotheses* (A)–(F) *assume that for each* $v\in N(\mathcal{A})\setminus\{0\}$ *there is a* $w\in N(\mathcal{A})$ *such that*

$$\int_{v>0} f_+w+\int_{v<0} f_-w\neq 0. \tag{2.4}$$

Then the problem

$$\mathcal{A}u=f(x,u), u\in D(\mathcal{A}) \tag{2.5}$$

has a solution.

Theorem 2.2. *In addition to hypotheses* (A)–(F)*, assume that there are functions* $F_{0\pm}(x),F_{1\pm}(x)$ *in* $L^1(\Omega)$ *such that*

$$F_{0\pm}(x)\le F(x,t)-tf_\pm(x)\le F_{1\pm}(x),\quad x\in\Omega,\ t\in\mathbb{R}. \tag{2.6}$$

Assume also that when (2.1) *holds we have*

$$F(x,t)-t\,f_\pm(x)\to F_{0\pm}(x) \text{ a.e. as } t\to\pm\infty$$

and when (2.2) *holds we have*

$$F(x,t)-tf_\pm(x)\to F_{1\pm}(x)\quad \text{a.e. as } t\to\pm\infty.$$

Then (2.5) *has at least one solution.*

Corollary 2.3. *In addition to hypotheses* (A)–(E) *assume that*

$$f_\pm(x)=f(x)\in N(\mathcal{A})^\perp.$$

Assume also that there are functions $F_0(x),F_1(x)$ *in* $L^1(\Omega)$ *such that*

$$F_0(x)\le F(x,t)-tf(x)\le F_1(x),\quad x\in\Omega,\ t\in\mathbb{R} \tag{2.7}$$

and either

$$F(x,t)-tf(x)\to F_0(x)\quad \text{a.e. as } |t|\to\infty \tag{2.8}$$

or

$$F(x,t)-tf(x)\to F_1(x) \text{ a.e. as } |t|\to\infty. \tag{2.9}$$

Then (2.5) *has at least one solution.*

Corollary 2.4. *Assume hypotheses* (A)–(E) *and*

$$f(x,t)\to 0\quad \text{a.e. as } |t|\to\infty. \tag{2.10}$$

Assume also that there are functions $F_{0\pm}(x),F_{1\pm}(x)$ *in* $L^1(\Omega)$ *such that*

$$F_{0\pm}(x)\le F(x,t)\le F_{1\pm}(x),\quad x\in\Omega,\ t\in\mathbb{R} \tag{2.11}$$

and either

$$F(x,t)\to F_{0\pm}(x)\quad \text{a.e. as } t\to\pm\infty \tag{2.12}$$

or

$$F(x,t)\to F_{1\pm}(x)\quad \text{a.e. as } t\to\pm\infty. \tag{2.13}$$

Then (2.5) *has at least one solution.*

3. Proof of main results

In this section we prove the Theorems of Section 2. We must replace Theorem 1.2. To do so, we introduce a sandwich theory that works in the infinite-dimensional case.

Let N be a closed, separable subspace of a Hilbert space E. We can define a new norm $|\cdot|_w$ satisfying $|v|_w \le \|v\|\ \forall v \in E$ and such that the topology induced by this norm is equivalent to the weak topology of N on bounded subsets of N. We construct the norm so that $v_j \to v$ weakly in N implies $|v_j - v|_w \to 0$. Conversely, if $\|v_j\|, \|v\| \le C$ for all $j > 0$ and $|v_j - v|_w \to 0$, then $v_j \to v$ weakly in N. This can be done as follows: Let $\{e_k\}$ be an orthonormal basis for N.Define

$$(u,v)_w = \sum_{k=1}^{\infty} \frac{(u,e_k)(v,e_k)}{2^k}, \quad u,v \in N.$$

This is a scalar product. The corresponding norm squared is

$$|v|_w^2 = \sum_{k=1}^{\infty} \frac{|(v,e_k)|^2}{2^k}, \quad v \in N.$$

Then $|v|_w$ satisfies $|v|_w \le \|v\|$, $v \in N$. For $u \in E$ and $Q \subset E$, we define

$$d_w(u,Q) = \inf_{v \in Q} |u - v|_w.$$

(Cf. [20]). We denote E equipped with this scalar product and norm by E_w. It is a scalar product space with the same elements as E. In particular, if $(u_n = v_n + w_n)$ is $\|\cdot\|$-bounded and $u_n \overset{|\cdot|_w}{\to} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E.

We adjust our assumptions on G for the infinite dimensional case of $\dim N = \infty$. Our requirements on G are given by

Definition 3.1. Let N be a closed separable subspace of a Hilbert space E. A C' functional G on E is called an N-weak-to-weak continuously differentiable functional on E if $|v_n - v|_w \to 0$ implies that there is a renamed subsequence satisfying

$$|G'(v_n) - G'(v)|_w \to 0.$$

This means that

$$v_n = Pu_n \to v \text{ weakly in } E, \text{ and } w_n = (I-P)u_n \to w \text{ strongly in } E$$

imply that there is a renamed subsequence satisfying

$$G'(v_n + w_n) \to G'(v + w) \quad \text{weakly in } E,$$

where P is the projection of E onto N.

The replacement for Theorem 1.2 is as follows.

Theorem 3.2 (Sandwich Theorem). *Let N be a closed separable subspace of a Hilbert space E, and let $M = N^\perp$. For G an N-weak-to-weak continuously differentiable functional on E, assume that*

$$a_0 = \sup_N G < \infty, \quad b_0 = \inf_M G > -\infty.$$

Then there is a sequence $\{u_k\} \subset E$ *such that*

$$G(u_k) \to c, \quad b_0 \le c \le a_0, \quad (1 + d_w(u_k, M))\|G'(u_k)\| \to 0. \tag{3.1}$$

(Cf. [21]*)*

In proving Theorems 2.1 and 2.2, the approach is basically the same for both of them. We begin by letting

$$N' = \oplus_{\lambda<0} N(\mathcal{A}-\lambda), \quad N = N' \oplus N(\mathcal{A}), \quad M' = N^{\perp} \cap D, \quad M = M' \oplus N(\mathcal{A}). \tag{3.2}$$

By hypothesis (A), $N', N(\mathcal{A}), N$ are separable and

$$D = M \oplus N' = M' \oplus N. \tag{3.3}$$

It is easily verified that the functional

$$G(u) := (\mathcal{A}u, u) - 2\int_\Omega F(x, u)dx \tag{3.4}$$

is continuously differentiable on D. We take

$$(u, v)_D = ([|\mathcal{A}| + 1]^{(1/2)}u, [|\mathcal{A}| + 1]^{(1/2)}v), \tag{3.5}$$

as the scalar product on D. We have

$$(G'(u), v) = 2(\mathcal{A}u, v) - 2(f(x, u), v), \quad u, v \in D. \tag{3.6}$$

Consequently, (2.4) is equivalent to

$$G'(u) = 0, \quad u \in D. \tag{3.7}$$

To apply Theorem 3.2 we must verify that $G(u)$ is an N-weak-to-weak continuously differentiable functional on D.

Suppose $v_k = Pu_k \to v$, weakly in D, $g_k = (I - P)u_k \to g$ strongly in D, where P is the projection of D onto N. Since the u_k are bounded in D, there is a renamed subsequence converging to a limit u weakly in D, $Vu_k \to Vu$ in $L^2(\Omega)$ and a.e. in Ω. Let $u' \in D$ be given. Then $f(x, u_k(x))u'(x)$ converges to $f(x, u(x))u'(x)$ a.e. and is dominated by $V|u'|$ which is in $L^1(\Omega)$. Consequently, we have

$$\int_\Omega f(x, u_k(x))u'(x)dx \to \int_\Omega f(x, u(x))u'(x)dx$$

as $k \to \infty$. Thus,

$$\begin{aligned}(G'(u_k), u')/2 &= (\mathcal{A}u_k, u') - \int_\Omega f(x, u_k(x))u'(x) \\ &\to (\mathcal{A}u, u') - \int_\Omega f(x, u(x))u'(x) \\ &= (G'(u), u')/2.\end{aligned}$$

This gives $G'(v_k + g_k) \to G'(v + g)$ weakly in D. Hence $G(u)$ is an N-weak-to-weak continuously differentiable functional on D.

Let $\underline{\lambda}$ be the largest negative point in the spectrum $\sigma(A)$ of A, and let $\bar{\lambda}$ be the smallest positive point. Then we have

$$(\mathcal{A}v, v) \le \underline{\lambda}\|v\|^2, \quad v \in N', \tag{3.8}$$

$$\bar{\lambda}\|w\|^2 \le (\mathcal{A}w, w), \quad w \in M'. \tag{3.9}$$

Assume that (2.1) holds. By hypothesis (D) and (2.3), $G(v) \le \underline{\lambda}\|v\|^2 + 2\|V\|\cdot\|v\| \to -\infty$ as $\|v\| \to \infty, v \in N'$. For $w \in M$ we write $w = w_0 + w'$, where $w_0 \in N(\mathcal{A})$ and $w' \in M'$. Since

$$|F(x,w) - F(x,w_0)| \le V(x)|w'|$$

we have

$$G(w) \ge \bar{\lambda}\|w'\|^2 - 2\int_\Omega F(x,w_0)dx - 2\|V\|\cdot\|w'\|.$$

Consequently, (2.1) implies

$$b_0 = \inf_M G > -\infty, \quad a_0 = \sup_N G < \infty. \tag{3.10}$$

We can now apply Theorem 3.2 to conclude that there is a sequence $\{u_k\} \subset D$ such that

$$G(u_k) \to c, \quad b_0 \le c \le a_0, \quad (1 + d_w(u_k, M))\|G'(u_k)\| \to 0.$$

Let $u_k = v_k + w_k + \rho_k y_{0k}$ where $v_k \in N', w_k \in M', y_{0k} \in N(\mathcal{A})$ and $\|y_{0k}\| = 1$, $\rho_k \ge 0$. We claim that

$$\|u_k\|_D \le C. \tag{3.11}$$

To see this we note that (3.1) and (3.6) imply

$$(\mathcal{A}v_k, v_k) - (f(x,u_k), v_k) = o(\|v_k\|). \tag{3.12}$$

From this we see that $\|v_k\|^2 = 0(\|v_k\|)$, and consequently that $\|v_k\| \le C$. Similarly, we have

$$(\mathcal{A}w_k, w_k) - (f(x,u_k), v_k) = o(\|w_k\|) \tag{3.13}$$

from which we see that $\|w_k\|_D \le C$. Suppose $\rho_k \to \infty$. There is a renamed subsequence such that $y_{0k} \to y_0$ in $N(\mathcal{A})$. Clearly $\|y_0\| = 1$. Thus by hypothesis (C), $y_0 \ne 0$ a.e. This means that

$$|u_k| = |v_k + w_k + \rho_k y_{0k}| \to \infty \quad \text{a.e.}, \tag{3.14}$$

$$f(x,u_k) \to q(x) \quad \text{in } L^2(\Omega) \tag{3.15}$$

where

$$q(x) = \begin{cases} f_+(x), & y_0(x) > 0, \\ f_-(x), & y_0(x) < 0. \end{cases} \tag{3.16}$$

Let $u_k' = v_k + w_k$. Then $u_k' \in N(\mathcal{A})^\perp$ and $\|u_k'\|_D \le C$. Thus there is a renamed subsequence such that $u_k' \to u_1$ weakly in $N(\mathcal{A})^\perp$. Since

$$(\mathcal{A}u_k', h) - (f(x,u_k), h) = o(\|h\|_D), \tag{3.17}$$

in the limit we have

$$\mathcal{A}u_1 = q. \tag{3.18}$$

This implies that $q \in N(\mathcal{A})^\perp$, i.e., that

$$\int_{y_0>0} f_+ v + \int_{y_0<0} f_- v = 0, \quad v \in N(\mathcal{A}). \tag{3.19}$$

But this contradicts (2.4). Hence the ρ_k are uniformly bounded, and (3.11) holds. Thus there is a renamed subsequence such that $\rho_k y_{0k} \to w_0$ in $N(\mathcal{A})$. By hypothesis (B) there is a renamed subsequence of $V_0 u_k$ converging in $L^p(\Omega)$ and a renamed subsequence of that converging a.e. in Ω. If we put $u = u_1 + w_0$, we see that $u_k \to u$ a.e. in Ω. From (3.17) and hypothesis (D) we see that u is a solution of (2.5). This proves Theorem 2.1 when (2.1) holds. If (2.2) holds, we write $v = v_0 + v'$ for $v \in N$, where $v_0 \in N(\mathcal{A})$ and $v' \in N'$. We then have

$$G(v) \leq \underline{\lambda}\|v'\|^2 - 2\int_\Omega F(x, v_0)dx + 2\|V\|\|v'\|, v \in N. \tag{3.20}$$

Also

$$G(w) \geq \bar{\lambda}\|w\|^2 - 2\|V\|\|w\| \to \infty \quad \text{as } \|w\| \to \infty, w \in M' \tag{3.21}$$

Thus

$$\inf_{M'} G > -\infty, \quad \sup_N G < \infty. \tag{3.22}$$

Using the second break up in (3.3) we can apply Theorem 3.2 again to conclude that there is a sequence satisfying

$$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad (1 + d_w(u_k, M))\|G'(u_k)\| \to 0.$$

We now proceed as before to conclude that (2.5) has a solution. This completes the proof of Theorem 2.1.

Now we turn to the proof of Theorem 2.2. Assume that (2.1) and (2.7) hold. As before we get (3.10), and we can apply Theorem 3.2 to the first break up in (3.3) to obtain a sequence satisfying

$$G(u_k) \to c, \quad b_0 \leq c \leq a_0, \quad (1 + d_w(u_k, M))\|G'(u_k)\| \to 0.$$

Assume that (3.11) does not hold, i.e., $\rho_k \to \infty$. We use the same reasoning as before to show that (3.14) and (3.15) hold, where $q(x)$ satisfies (3.16). This in turn implies (3.18) and (3.19). Combining (3.17) and (3.18) we have

$$(\mathcal{A}[u_k' - u_1], h) - (f(x, u_k) - q, h) = o(\|h\|_D). \tag{3.23}$$

This implies

$$u_k' \to u_1 \text{ in } D. \tag{3.24}$$

Now (2.7) and (3.14) imply

$$F(x, u_k) - u_k q(x) \to Q(x) \tag{3.25}$$

where

$$Q(x) = \begin{cases} F_{0+}(x), & y_0(x) > 0, \\ F_{0-}(x), & y_0(x) < 0. \end{cases} \tag{3.26}$$

Let

$$a(u, v) = (\mathcal{A}u, v), a(u) = (\mathcal{A}u, u), u, v \in D. \tag{3.27}$$

By (2.6) we have

$$\begin{aligned} G(u) &= a(u) - 2\int [F(x, u) - uq]dx - 2(u, q) \\ &\leq a(u) - 2a(u, u_1) - 2\int Q(x)dx \\ &= a(u - u_1) - a(u_1) - 2\int Q(x)dx. \end{aligned} \tag{3.28}$$

Let P be the (orthogonal) projection onto N', and take $u = v + w$ with $v \in N'$ and $w \in M$. Then

$$G(v + w) \leq a(v - Pu_1) + a(w - (I - P)u_1) - 2\int Q(x)dx - a(u_1). \tag{3.29}$$

Hence for each fixed $w \in M$,

$$G(v+w) \to -\infty \text{ as } \|v\| \to \infty, v \in N'. \tag{3.30}$$

Consequently, for each $w \in M$ there is a $v_1 \in N'$ such that

$$G(v_1+w) = \max_{v\in N'} G(v+w). \tag{3.31}$$

Now by (3.25) and (3.28)

$$G(u_k) \to a(u_1) - 2(u_1, q) - 2\int Q(x)dx \equiv c. \tag{3.32}$$

Take $w = (I-P)u_1$. Then by (3.29) and (3.32),

$$G(v+w) \le a(v-Pu_1) - a(u_1) - 2\int Q(x)dx = a(v-Pu_1) + c \tag{3.33}$$

holds for all $v \in N'$. Now by Theorem 3.2 and (3.31), there is a $v_1 \in N'$ such that

$$c \le G(v_1+w). \tag{3.34}$$

In view of (3.32) and (3.33) this implies

$$c \le G(v_1+w) \le a(v_1-Pu_1) + c. \tag{3.35}$$

Since $a(v_1 - Pu_1) < 0$ unless $v_1 = Pu_1$, we must take this value for v_1 in (3.34). Since $w = (I-P)u_1$,

$$c \le G(u_1). \tag{3.36}$$

If we combine this with (3.32) we obtain

$$\int [F(x,u_1) - u_1 q]dx \le \int Q(x)dx. \tag{3.37}$$

By (2.6) and (3.26),

$$Q(x) \le F(x,t) - tq(x), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{3.38}$$

Hence

$$Q(x) \le F(x,u) - uq, \quad x \in \Omega,\ u \in D. \tag{3.39}$$

If we combine this with (3.37) we see that

$$Q(x) \equiv F(x,u_1) - u_1 q. \tag{3.40}$$

Let

$$\Phi(u) = \int_\Omega [F(x,u) - uq]dx, \quad u \in D. \tag{3.41}$$

Then

$$(\Phi'(u), h) = (f(x,u) - q, h), \quad h \in D. \tag{3.42}$$

By (3.39) and (3.40),

$$\Phi(u_1) \le \Phi(u), \quad u \in D. \tag{3.43}$$

In view of (3.18) and (3.42), this implies

$$f(x,u_1) = q = Au_1. \tag{3.44}$$

Thus u_1 is a solution of (2.5). On the other hand, if (3.11) holds, we can use the same reasoning as before to conclude that (2.5) has a solution. This proves Theorem 2.2 for the case when (2.1) and (2.7) hold. If (2.2) and (2.8) hold, then we use the second break up in (3.3), and we prove (3.22) by means of (3.20) and (3.21). Again we apply Theorem 3.2 to conclude that (3.1) holds for some sequence. If (3.11) does not hold, this leads to (3.12)–(3.19) as before. We also obtain (3.23)–(3.25) with (3.26) replaced by

$$Q(x) = \begin{cases} F_{1+}(x), & y_0(x) > 0, \\ F_{1-}(x), & y_0(x) < 0. \end{cases} \tag{3.45}$$

In place of (3.28) we have, in view of (2.8),

$$G(u) \geq a(u - u_1) - a(u_1) - 2\int Q(x)dx, \quad u \in D. \tag{3.46}$$

This time we let P be the projection onto N (instead of N') and we make the break up $u = v + w, v \in N, w \in M'$. Then

$$G(v + w) \geq a(v - Pu_1) + a(w - (I - P)u_1) - 2\int Q(x)dx - a(u_1). \tag{3.47}$$

Thus $G(v + w) \to \infty$ as $\|w\| \to \infty$ for each fixed $v \in N$. Since G is weakly lower semicontinuous, for each $v \in N$ there is a $w_1 \in M$ such that

$$G(v + w_1) = \min_{w \in M'} G(v + w). \tag{3.48}$$

Again we see that (3.32) holds. We take $v = Pu_1$. Then by (3.47) and (3.32) we have

$$G(v+w) \geq a(w-(I-P)u_1) - a(u_1) - 2\int Q(x)dx = a(w-(I-P)u_1) + c \tag{3.49}$$

for all $w \in M'$. Now by Theorem 3.2, $c \geq G(v + w_1)$ for w, satisfying (3.48). Thus by (3.32) and (3.49)

$$c \geq G(v + w_1) \geq a(w_1 - (I - P)u_1) + c. \tag{3.50}$$

This is impossible unless $w_1 = (I - P)u_1$. But then

$$c \geq G(u_1). \tag{3.51}$$

If we combine this with (3.32), we obtain

$$\int [F(x, u_1) - u_1 q]dx \geq \int Q(x)dx. \tag{3.52}$$

By (2.6) and (3.45),

$$F(x, t) - tq(x) \leq Q(x), \quad x \in \Omega,\ t \in \mathbb{R} \tag{3.53}$$

which implies

$$F(x, u) - uq \leq Q(x), \quad x \in \Omega,\ u \in D. \tag{3.54}$$

It now follows form (3.52) and (3.54) that (3.40) holds. If we define $\Phi(u)$ by (3.41) we see that

$$\Phi(u) \leq \Phi(u_1), \quad u \in D, \tag{3.55}$$

from which (3.43) and (3.44) follow. Again u_1 is a solution of (2.5). If (3.11) holds, we obtain a solution in the same way as before. This completes the proof of Theorems 2.1 and 2.2.

Corollaries 2.3 and 2.4 are immediate consequences of Theorem 2.2. In them we can dispense with hypothesis (F) since it is implied by (2.9) and (2.7). In fact we have

$$\int_\Omega F(x, v)dx \leq \int_\Omega F_1(x)dx + \int_\Omega vf(x)dx.$$

If $v \in N(A)$ the last integral vanishes, and (2.1) holds. Similarly, (2.1) implies both (2.1) and (2.2).

Editor's note. We lament reporting that Professor Martin Schechter passed away on June 7, 2021, a couple of months after this article was accepted for publication. Professor Schechter was a driving force in Nonlinear Analysis for more than six decades.

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